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Horn sentences with (W) and weak Mal'cev conditions

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1. Introduction

In [14, Problem 2.18] Jónsson asks whether for any universal lattice Horn sentence χ there exists a countably weak Mal'cev condition which holds in a variety $\mathbf{V}$ of algebras iff χ is satisfied in congruence lattices of members of $\mathbf{V}$. We adopt the definition of Mal'cev conditions from Taylor's paper [18] (i.e., any disjunction of strong Mal'cev conditions U_n, $n = 1, 2, \ldots$, where U_n implies U_{n+1} for all n, is called a Mal'cev condition). By countably (continuously, resp.) weak Mal'cev condition we will mean a conjunction of countably (at most continuously, resp.) many Mal'cev conditions. (Note that, up to equivalence, there are at most c = continuously many Mal'cev conditions, whence any irredundant conjunction of them consists of at most c members.) We say that a (universal lattice) Horn sentence χ satisfies (W), the Whitman condition, if the lattice finitely presented by χ (i.e., which is freely generated by the variables subjected to the premise equations of χ) satisfies (W). This condition on χ will be reduced to the Whitman condition on a suitable finite partial lattice associated with χ.

Our aim is to give an algorithm which associates appropriate continuously weak Mal'cev conditions with Horn sentences satisfying (W). The algorithm combines the ideas of McKenzie's limit table concept [15], the algorithm of Wille [21] and Pixley [17], and the methods of [1] and [2]. Note that another algorithm yields Mal'cev conditions for any lattice Horn sentence in case of n-permutable varieties (cf. [3]).

2. Horn sentences satisfying (W)

By a (universal lattice) Horn sentence we mean a first order sentence χ:

$$(\forall \alpha_0, \ldots, \alpha_t)\left(\left(\bigwedge_{i=1}^{k} p_i = q_i\right) \Rightarrow p \leq q\right)$$

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where p_i, q_i, p, and q are lattice terms of the variables $\alpha_0, \ldots, \alpha_t$ and $k \geq 0$. (Note that having $p \leq q$ instead of $p = q$ does not hurt the generality.) This Horn sentence is said to satisfy (W) if the finitely presented lattice $\mathrm{FL}(\{\alpha_0, \ldots, \alpha_t\}; p_1(\alpha_0, \ldots, \alpha_t) = q_1(\alpha_0, \ldots, \alpha_t), \ldots, p_k(\alpha_0, \ldots, \alpha_t) = q_k(\alpha_0, \ldots, \alpha_t))$ satisfies (W). (Observe that whether χ satisfies (W) depends only on its premises.) For example, the semidistributive laws, $\mathrm{SD}_\wedge : \alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \alpha \wedge (\beta \vee \gamma) \leq \alpha \wedge \beta$ and its dual, are Horn sentences satisfying (W).

Since χ in the above form is not appropriate to our algorithm, we consider a (seemingly) restricted class of Horn sentences. A Horn sentence will be called *normal* if it is of the form $(\forall \alpha_0, \ldots, \alpha_t)\ (\bigwedge \{\alpha_i \circ \alpha_j = \alpha_k : \circ \in \{\wedge, \vee\}, \alpha_i \circ \alpha_j$ is defined and equal to α_k in $L\} \Rightarrow p \leq q)$, shortly $L \Rightarrow p \leq q$ or $L \Rightarrow p(L) \leq q(L)$, where $L = (\{\alpha_0, \ldots, \alpha_t\}, \wedge, \vee)$ is a partial lattice. The concept of partial lattices can be found in Gratzer [8, pages 40–44] (cf. also Funayama [7]), but what we only need from it is the following: by defining $\alpha \leq \beta$ by $\alpha \vee \beta = \beta$ L becomes a poset with partially defined monotone operations $\wedge$, $\vee$. Furthermore, $\alpha \vee \beta = \sup\{\alpha, \beta\}$ and $\alpha \wedge \beta = \inf\{\alpha, \beta\}$ hold whenever the left-hand sides are defined.

Now we observe that any lattice Horn sentence

$$\chi : (\forall \alpha_0, \ldots, \alpha_t)\left(\bigwedge_{i=1}^{m} p_i = q_i \Rightarrow p \leq q\right)$$

is equivalent (modulo lattice theory) to a normal one. Let $B = \{\beta_1, \ldots, \beta_s\}$ be the set of all subterms occurring in the premises $p_i = q_i$ and put $\Phi = \{\beta_i \circ \beta_j = \beta_k :$ either $\beta_i = \beta_j = p_n$ and $\beta_k = q_n$ for some $n \leq m$ or the term β_k is $\beta_i \circ \beta_j\}$. Let $L = L_\chi$ be the set of free generators in the free lattice $\mathrm{FL}(\{\beta_1, \ldots, \beta_s\}, \Phi)$, then L, as a subset of a lattice, is a partial lattice by its own right. (Note that generally $|L| < s$, because free generators can collapse.) Now it is easy to see that χ and $L \Rightarrow p \leq q$ are equivalent.

A partial lattice L is said to satisfy (W), the Whitman condition (cf. Whitman [19]), if whenever $\alpha, \beta, \gamma, \delta \in L$, $\alpha \wedge \beta$, $\gamma \vee \delta$ are defined, and $\alpha \wedge \beta \leq \gamma \vee \delta$ then $\alpha \leq \gamma \vee \delta$ or $\beta \leq \gamma \vee \delta$ or $\alpha \vee \beta \leq \gamma$ or $\alpha \vee \beta \leq \delta$. A normal Horn sentence $L \Rightarrow p \leq q$ is said to satisfy (W) if L, as a partial lattice, satisfies (W). The following assertion ensures that to accomplish our aim it is sufficient to deal with normal Horn sentences satisfying (W).

PROPOSITION 2.1. *A Horn sentence*

$$\chi : \left(\bigwedge_{i=1}^{k} p_i = q_i\right) \Rightarrow p \leq q$$

satisfies (W) *if and only if the corresponding (and therefore equivalent) normal Horn sentence* $L_\chi \Rightarrow p \leq q$ *satisfies* (W).

Proof. By definitions we have to show that a finite partial lattice A satisfies (W) if and only if $\mathrm{FL}(A)$, the free lattice generated by A, satisfies (W). If $\mathrm{FL}(A)$ satisfies (W) then obviously so does A. To prove the converse we need the solution for the word problem of $\mathrm{FL}(A)$, due to Dean [6], and a result from Grätzer, Huhn, and Lakser's paper [10], but we cite only as much as very necessary. Suppose $a \wedge b \leq c \vee d$ holds for $a, b, c, d \in \mathrm{FL}(A)$ and we need to show that at least one of a, b, c, d can be omitted from $a \wedge b \leq c \vee d$ (preserving $\leq$, of course). If $a \wedge b \leq \varepsilon \leq c \vee d$ holds for no $\varepsilon \in A$ then this follows from [10, Proposition 2(iii)]. Otherwise, by [10, Proposition 1] and the description of joins of (dual) ideals of A, $\bigwedge^* (\alpha_i \wedge \beta_i) \leq \bigvee^* \gamma_j \vee \delta_j)$ holds for some α_i β_i, γ_j, $\delta_j \in A$, $a \leq \alpha_i$, $b \leq \beta_i$, $\gamma_j \leq c$, $\delta_i \leq d$. Here $\bigwedge^* (\varepsilon_i \in A$ stands for an appropriately bracketted meet of finitely many $\varepsilon_i = \alpha_i \wedge \beta_i \in A$, while $\bigvee^* (\gamma_j \vee \delta_j) \in A$ is understood dually. (Note that a finitary meet need not be defined *within* A at any possible bracketing.) by (W) one side of the inequality $\bigwedge^* (\alpha_i \wedge \beta_i) \leq \bigvee^* (\gamma_j \vee \delta_j)$ can be shortened, and repeating this shortening as many times as possible either the left-hand side or the right-hand side will be a single variable. If we get, say, $\beta_4 \leq \bigvee^* (\gamma_i \vee \delta_i)$ then $b \leq \beta_4 \leq \bigvee^* (\gamma_i \vee \delta_i) \leq \bigvee (c \vee d) = c \vee d$, while the other cases can be handled similarly. Q.E.D.

In the following statement, motivated by the limit table concept of McKenzie [15], a lattice property (equivalent to (W) for finite lattices) will be presented. This statement will be the key to exploit (W). For a set (or system) $A = \{\alpha_0, \alpha_1, \ldots, \alpha_t\}$ and a $(t+1)$-ary lattice term d we will often use the abbreviations $d(\alpha : \alpha \in A)$, $d(\alpha_i : i \leq t)$, and $d(A)$ for $d(\alpha_0, \alpha_1, \ldots, \alpha_t)$. (Of course, a fixed linear ordering of A will be supposed.)

PROPOSITION 2.2. *A finite partial lattice* A *satisfies* (W) *if and only if there exist* $|A|$*-ary lattice terms* α_i, $i < \omega$, $\alpha \in A$, *such that*

(a) *for* $\alpha \in A$ $\alpha_0(A) = \alpha$;

(b) *for* $\alpha \in A$ *and* $i < \omega$, $\alpha_{i+1} = \alpha_i \vee \mu_{\alpha,i}$ *with a suitable* $|A|$*-ary term* $\mu_{\alpha,i}$;

(c) *whenever* $\phi : A \to E(X)$ *is a join-homomorphism (i.e., a map preserving all the existing joins) into an arbitrary equivalence lattice* $E(X)$ *then the map* $\psi : A \to E(X)$, $\psi(\alpha) = \bigcup \{\alpha_i(\phi(\beta) : \beta \in A) : i < \omega\}$ *is a (both meet and join) homomorphism; and*

(d) *if* ϕ *happens to be a homomorphism in the previous condition then* ϕ *and* ψ *coincide.*

Proof. Suppose A satisfies (W) and let us define suitable terms α_i $(i > \omega$,

$\alpha \in A$) via induction. Put $\alpha_0 = \alpha$ (projection),

$$\mu_{\alpha,i} = \bigvee \{\beta_i \wedge \gamma_i : \beta \wedge \gamma \leq \alpha \quad \text{in} \quad A\},$$

and $\alpha_{i+1} = \alpha_i \vee \mu_{\alpha,i}$. (It is worth emphasizing that α_i can be given by a suitable algorithm.) Now (a), (b), and (d) trivially hold. Consider a join-homomorphism $\phi : A \to E(X)$ as in (c) and let α^i, α^ω, and $\mu^{\alpha,i}$ stand for $\alpha_i(\phi(\beta) : \beta \in A)$, $\psi(\alpha)$, and $\mu_{\alpha,i}(\phi(\beta) : \beta \in A)$, respectively. An easy induction shows that if $\alpha \leq \beta$ in A then $\alpha^i \leq \beta^i$ and $\alpha^\omega \leq \beta^\omega$ in $E(X)$. We claim that if $\varepsilon \vee \zeta = \eta$ in A then $\varepsilon^i \vee \zeta^i = \eta^i$, whence $\varepsilon^\omega \vee \zeta^\omega = \eta^\omega$, in $E(X)$. Really, by making use of (W), we have the following induction step:

$$\begin{aligned}\eta^{i+1} &= \eta^i \vee \mu^{\eta,i} = \eta^i \vee \bigvee \{\beta^i \wedge \gamma^i : \beta \wedge \gamma \leq \eta = \varepsilon \vee \zeta\} = \\ &= \eta^i \vee \bigvee \{\beta^i : \beta \leq \eta\} \vee \bigvee \{\gamma^i : \gamma \leq \eta\} \vee \\ &\quad \vee \bigvee \{\beta^i \wedge \gamma^i : \beta \wedge \gamma \leq \varepsilon\} \vee \bigvee \{\beta^i \wedge \gamma^i : \beta \wedge \gamma \leq \zeta\} = \\ &= \eta^i \vee \mu^{\varepsilon,i} \vee \mu^{\zeta,i} = \varepsilon^i \vee \zeta^i \vee \mu^{\varepsilon,i} \vee \mu^{\zeta,i} = \\ &= \varepsilon^{i+1} \vee \zeta^{i+1}.\end{aligned}$$

Therefore ψ is join-preserving and, consequently, monotone. Now consider a meet $\alpha \wedge \beta = \gamma$ in A. Since ψ is monotone, only $\alpha^\omega \wedge \beta^\omega \leq \gamma^\omega$ has to be checked. If $(x, y) \in \alpha^\omega \wedge \beta^\omega$ then $(x, y) \in \alpha^i \wedge \beta^j$ for some $i, j < \omega$, and we can assume that $i = j$. Hence we have $(x, y) \in \mu^{\gamma,i} \leq \gamma^{i+1} \leq \gamma^\omega$, indeed.

Conversely, suppose that although A fails to satisfy (W) via $\beta \wedge \gamma \leq \delta \vee \varepsilon$, there are lattice terms α_i satisfying (a), (b), (c), and (d). Put $K = \mathrm{FL}(A)$, the free lattice generated by A, and (without loss of generality, cf. Whitman [20]) let K be a sublattice of an equivalence lattice $E(X)$. We also need a lattice construction found in [4, 5]. Let I denote the interval $[\beta \wedge \gamma, \delta \vee \varepsilon]$ in K. Then a new lattice $K[I]$ can be constructed by letting $K[I] = (K \backslash I) \dot\cup (I \times \mathbf{2})$ and defining $\mu \leq \nu$ in $K[I]$ iff one of the following hold:

(1) $\mu, \nu \in K \backslash I$ and $\mu \leq \nu$ in K,

(2) $\mu = (\alpha, i)$, $\nu \in K \backslash I$ and $\alpha \leq \nu$ in K,

(3) $\mu \in K \backslash I$, $\nu = (\zeta, j)$ and $\mu \leq \zeta$ in K,

(4) $\mu = (\alpha, i)$, $\nu = (\zeta, j)$ and $\alpha \leq \zeta$ in K and $i \leq j$ in $\mathbf{2}$. There is a natural epimorphism $\kappa : K[I] \to K$ with $\kappa(\mu) = \mu$ for $\mu \in K \backslash I$ and $\kappa(\alpha, i) = \alpha$ for $\alpha \in I$. Moreover, the map $\phi : K \to K[I]$, $\phi(\mu) = \mu$ for $\mu \in K \backslash I$ and $\phi(\alpha) = (\alpha, 0)$ for $\alpha \in I$, is known to be a join-homomorphism.

First observe that $\alpha_i(\mu : \mu \in A) = \alpha$ holds in K for all $\alpha \in A$, $i < \omega$ (apply (d) to the identity map of K). Let $K[I]$ be considered as a sublattice of an equivalence lattice $E(Y)$, then the restriction of ϕ to A is a join-homomorphism $A \to E(Y)$.

Consider the homomorphism $\psi: A \to E(Y)$, $\psi(\mu) = \bigcup \{\mu_i(\phi(\alpha): \alpha \in A): i < \omega\}$. We claim that, for any $\mu \in K \setminus I$, $\mu_i(\phi(\alpha): \alpha \in A) = \phi(\mu) = \mu$, whence $\psi(\mu) = \mu \in K[I] \leq E(Y)$. Since $\kappa \circ \phi$ is the identity map on K we obtain $\kappa(\mu_i(\phi(\alpha): \alpha \in A)) = \mu_i(\kappa(\phi(\alpha)): \alpha \in A) = \mu_i(\alpha : \alpha \in A) = \mu$. But Ker κ restricted to $K \setminus I$ is the equality relation, whence $\mu_i(\phi(\alpha): \alpha \in A) = \mu$, indeed. Since β, γ, δ, $\varepsilon \in K \setminus I$ and ψ is a homomorphism, we obtain $\beta \wedge \gamma = \psi(\beta) \wedge \psi(\gamma) = \psi(\beta \wedge \gamma) \leq \psi(\delta \vee \varepsilon) = \psi(\delta) \vee \psi(\varepsilon) = \delta \vee \varepsilon$, i.e. $\beta \wedge \gamma \leq \delta \vee \varepsilon$ holds in $K[I]$. But an easy calculation in $K[I]$ shows that $\beta \wedge \gamma = (\beta \wedge \gamma, 1) \nleq (\delta \vee \varepsilon, 0) = \delta \vee \varepsilon$, a contradiction. The proof is complete.

Note that an easy modification of the above proof shows that Proposition 2.2 is true for countable partial lattices as well.

3. Mal'cev type conditions

Our Mal'cev type conditions will be given by certain graphs. First for any lattice term $p = p(\alpha : \alpha \in L)$ and integer $k \geq 2$ we define a graph $G_k(p)$ associated with p. The edges of $G_k(p)$ will be coloured by the variables $\alpha \in L$, and two distinguished vertices, the so-called left and right endpoints, will have special roles. In figures these endpoints will be always placed on the left-hand side and on the right-hand side, respectively. An α-coloured edge connecting the vertices x and y will be often denoted by (x, α, y). Before defining $G_k(p)$ we introduce two kinds of operations for graphs. We obtain the *parallel connection* of graphs G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying their left (right, resp.) endpoints (Figure 1). By taking disjoint graphs $H_1, H_2, \ldots, H_k$ $(k \geq 2)$ such that $H_i \cong G_1$ for i odd and $H_i \cong G_2$ for i even, and identifying the right endpoint of H_i and the left endpoint of H_{i+1} for $i = 1, 2, \ldots, k-1$ we obtain the *serial connection* of length k of the graphs G_1 and G_2. (The left endpoint of H_1 and the right one of H_k are the endpoints of the serial connection, cf. Figure 2.)

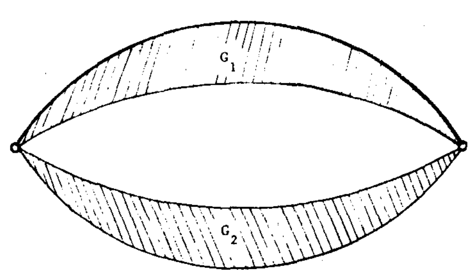


Figure 1

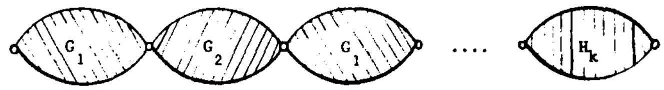
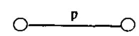


Figure 2

Now, if p is a variable then, for any $k \geq 2$, let $G_k(p)$ be the following graph



which consists of a single edge coloured by p. Let $G_k(p_1 \wedge p_2)$ ($G_k(p_1 \vee p_2)$, resp.) be the parallel connection (serial connection of length k) of the graphs $G_k(p_1)$ and $G_k(p_2)$.

For an algebra A, a lattice term $p(\alpha : \alpha \in L)$, congruences $\hat{\alpha}$ of A ($\alpha \in L$), a_0, $a_1 \in A$, and $k \geq 2$ we say that a_0 and a_1 *can be connected by* $G_k(p)$ *in* A if there is a map ϕ (referred to as *connecting map*) from the vertex set of $G_k(p)$ into A such that a_0 and a_1 are the images of the left and right endpoint, respectively, and for any $\alpha \in L$ and α-coloured edge (x, α, y) we have $(\phi(x), \phi(y)) \in \hat{\alpha}$. If it is necessary, we can emphasize that the colour α is represented by the congruence $\hat{\alpha}$.

The following statement follows by a straightforward induction from definitions, therefore its proof will be omitted (cf. also [2]).

PROPOSITION 3.1. *Consider an algebra* A, a_0, $a_1 \in A$, *a lattice term* $p(\alpha : \alpha \in L)$, *and congruences* $\hat{\alpha}$ *of* A *for* $\alpha \in L$. *Then* $(a_0, a_1) \in p(\hat{\alpha} : \alpha \in L)$ *iff* a_0 *and* a_1 *can be connected by* $G_k(p)$ *in* A *for some* $k \geq 2$ *iff there exists* $k_0 \geq 2$ *such that* a_0 *and* a_1 *can be connected by* $G_k(p)$ *in* A *for all* $k \geq k_0$.

Now, motivated by Proposition 3.2, we intend to find a stronger version of Proposition 3.1 for the case when $L \to \mathrm{Con}(A)$, $\alpha \mapsto \hat{\alpha}$ is a join-homomorphism. Let us consider a finite partial lattice L, a lattice term $p = p(L) = p(\alpha : \alpha \in L)$, a non-negative integer m and a sequence of integers $\mathbf{s} = (s_0, s_1, s_2, \ldots)$ where $s_i \geq 2$ for $i < \omega$. We define the graphs $G(p, L, s_0, s_1, \ldots, s_m)$ and $G(p, L, \mathbf{s})$ via the following induction. Let $G(p, L, s_0)$ be $G_{s_0}(p)$. Assume that $G_m = G(p, L, s_0, s_1, \ldots, s_m)$ has already been defined and to obtain G_{m+1} we "ferment" G_m in the following way. First for any proper join $\alpha \vee \beta = \gamma$ in L (proper means $|\{\alpha, \beta, \gamma\}| = 3$) for which α precedes β (according to a fixed linear ordering of L) and for any γ-coloured edge (x, γ, y) whose vertices x and y cannot be connected by a path (in G_m) coloured by α and β only we add $s_{m+1} - 1$ new vertices, say $z_1, z_2, \ldots, z_{s_{m+1}-1}$, and s_{m+1} new edges, namely the edges (x, z_1), $(z_1, z_2), \ldots, (z_{s_{m+1}-2}, z_{s_{m+1}-1})$, $(z_{s_{m+1}-1}, y)$, to G_m. The first, third, fifth, ... edge in the above sequence of new edges are coloured by α while the others are coloured by β (cf. Figure 3).

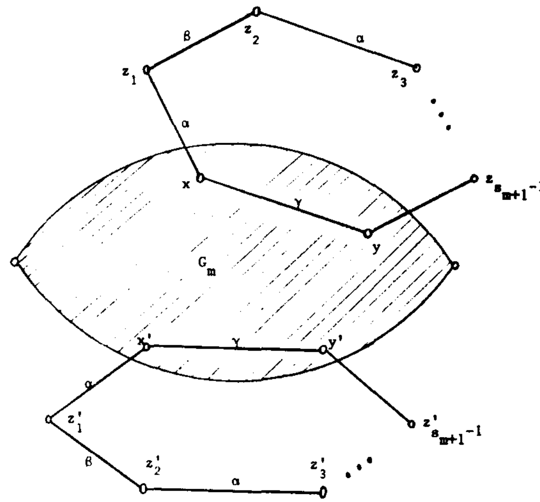


Figure 3

(All the newly adopted edges and vertices are pairwise distinct.) Let $G_{m+1/2}$ denote the graph we have obtained. For any pair of elements $\delta < \varepsilon$ in L and for any δ-coloured edge of $G_{m+1/2}$ if the two vertices of this edge are not connected with an ε-coloured edge yet, then let us add a new ε-coloured edge connecting them. Thus we have obtained the graph $G_{m+1} = G(p, L, s_0, s_1, \ldots, s_{m+1})$. Finally let $G(p, L, \mathbf{s})$ be the (directed) union of $G(p, L, s_0, s_1, \ldots, s_m)$, $m < \omega$.

For example, if $L = (\{\alpha, \beta, \gamma, \delta\}, \alpha < \delta, \beta < \delta, \gamma < \delta, \alpha \vee \beta = \beta \vee \alpha = \delta)$ (where only the proper meets and joins are indicated) and $p = \gamma$ then $G(p, L, 8, 4)$ and $G(p, L, 8, 4, 3)$ are given in Figures 4 and 5. Connectivity of two elements in an algebra by $G(L, p, \mathbf{s})$ is defined similarly as in case of $G_k(p)$. (The only difference is that now the vertex set can be infinite.)

PROPOSITION 3.2. *Suppose $\rho: L \to \mathrm{Con}(A)$ is a join-homomorphism from a finite partial lattice L into the congruence lattice of an algebra A, $p = p(\alpha: \alpha \in L)$ is a lattice term and $a_0, a_1 \in A$. Then $(a_0, a_1) \in p(\rho(\alpha): \alpha \in L)$ if and only if there is*

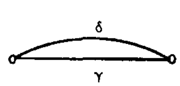


Figure 4

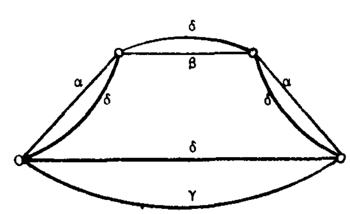


Figure 5

an $\mathbf{s}=(s_0, s_1, s_2, \ldots)$ $(s_i \geq 2)$ *such that* a_0 *and* a_1 *can be connected by* $G(p, L, \mathbf{s})$ *in* A, *representing the colours* α *by the congruences* $\rho(\alpha)$.

Proof. If a_0 and a_1 can be connected by $G(p, L, \mathbf{s})$ then they can be connected by $G_{s_0}(p)$, which is a subgraph of $G(p, L, \mathbf{s})$, as well. Hence $(a_0, a_1) \in p(\rho(\alpha)$: $\alpha \in L)$ follows from Proposition 3.1. Conversely, by Proposition 3.1 again, if $(a_0, a_1) \in p(\rho(\alpha): \alpha \in L)$ then a_0 and a_1 can be connected by $G_k(p) = G(p, L, k)$ for some $k \geq 2$. Choose s_0 to be the smallest possible k and let ϕ_0 be the connecting map. If $\alpha \vee \beta = \gamma$ in L and (x, γ, y) is an edge in $G(p, L, s_0)$ then from $(\phi_0(x), \phi_0(y)) \in \rho(\gamma)$ we conclude $(\phi_0(x), \phi_0(y)) \in \rho(\alpha) \circ \rho(\beta) \circ \rho(\alpha) \circ \cdots$ (n factors) for some $n \geq 2$. By finiteness we can choose a smallest n which is appropriate for any choice of α, β, γ, x, y. By letting s_1 be equal to this smallest common n ϕ_0 can be extended to a map ϕ_1 under which $G(p, L, s_0, s_1)$ connects a_0 and a_1. And so on, finally $\phi = \bigcup_{i<\omega} \phi_i$ will be the required connecting map. Q.E.D.

Now with any pair of finite graphs G_1 and G_2 we associate a strong Mal'cev condition $U(G_1 \leq G_2)$ in the following way. Let L be the set of all colours that occur on edges of G_1 and G_2, and let X and F be the vertex sets of G_1 and G_2, respectively, with x_0, $x_1 \in X$ and $f_0, f_1 \in F$ the endpoints. For $\alpha \in L$ let $\hat{\alpha}$ denote the smallest equivalence relation of X which contains $\{(x, y): (x, \alpha, y)$ is an edge of $G_1\}$. Choose a linear ordering of X and let x_α denote the smallest (according to the linear ordering) member of X for which $(x_\alpha, x) \in \hat{\alpha}$. (In other words, x_α is the first vertex of G_1 which can be connected with x by an α-coloured path.) Now $U(G_1 \leq G_2)$ is defined to be the following condition:

"There exist $|X|$-ary terms $f(x: x \in X)$, $f \in F$, which satisfy (1) the "endpoint" identities $f_i(x: x \in X) = x_i$ $(i = 0, 1)$ and (2) for any edge, say (f, α, g), in G_2 the corresponding identity $f(x_\alpha: x \in X) = g(x_\alpha: x \in X)$."

We should note that $U(G_1 \leq G_2)$ depends on the *fixed* orderings of L and X, but arbitrary other linear orderings would yield equivalent conditions. Although it would be possible to define unique orderings of, say, the vertex sets of our graphs $G_k(p)$, it is not worth doing so (cf. [1], where only the simplest graphs are handled).

For example, if G_1 and G_2 are the graphs given in Figures 6 and 7 then

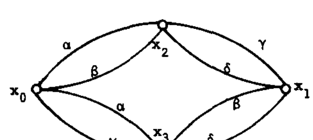


Figure 6

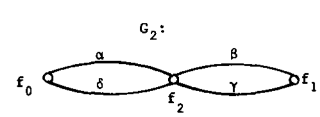


Figure 7

$U(G_1 \leq G_2)$ is the following condition:

"There are quaternary terms $f_i(x_0, x_1, x_2, x_3)$, $0 \leq i \leq 2$, which satisfy the identities

(1) $f_i(x_0, x_1, x_2, x_3) = x_i$ $(i = 0, 1)$ (endpoint identities) and

(2) $f_0(x_0, x_1, x_0, x_0) = f_2(x_0, x_1, x_0, x_0)$,
$f_0(x_0, x_1, x_1, x_1) = f_2(x_0, x_1, x_1, x_1)$,
$f_2(x_0, x_1, x_0, x_1) = f_1(x_0, x_1, x_0, x_1)$,
$f_2(x_0, x_1, x_1, x_0) = f_1(x_0, x_1, x_1, x_0)$."

Note that this condition is clearly equivalent to the following one:

"There is a quaternary term f_2 which satisfies the identities

$$f_2(x, y, x, x) = f_2(x, y, y, y) = f_2(y, x, y, x) = f_2(y, x, x, y) = x."$$

Now we can formulate our main result.

THEOREM. *Let $L \Rightarrow p \leq q$ be an arbitrary normal Horn sentence satisfying the Whitman condition (i.e., L is a finite partial lattice with (W), $p = p(L)$, $q = q(L)$ are arbitrary). Let α_i $(i < \omega, \alpha \in L)$ denote the lattice terms provided by Proposition* 2.2, *and let $q(L_i)$ stand for $q(\alpha_i(L)$: $\alpha \in L)$. Then for any variety* $\mathbf{V}$ *of algebras the following three conditions are equivalent.*

(i) *$L \Rightarrow p(L) \leq q(L)$ holds in the congruence lattice of any member of* $\mathbf{V}$;

(ii) *For any sequence $\mathbf{s} = (s_0, s_1, s_2, \ldots)$, $s_i \geq 2$, there exist integers $m \geq 1$ and $n \geq 2$ such that $U(m, n) = U(G(p, L, s_0, s_1, \ldots, s_m) \leq G_n(q(L_m)))$ holds in* $\mathbf{V}$;

(iii) *For any $\mathbf{s} = (s_0, s_1, \ldots)$, $s_i \geq 2$, there exists an integer $n \geq 2$ such that $U(n, n)$ from (ii) holds in* $\mathbf{V}$.

Further, for a fixed $\mathbf{s}$ "$(\exists n) U(n, n)$" *is a Mal'cev condition in Taylor's sense* [18].

4. The proof of Theorem

(i) *implies* (ii) *and* (iii). (Although (iii) implies (ii) trivially, for the sake of a later reference we handle (ii) separately.) Assume (i) and let $\mathbf{s} = (s_0, s_1, \ldots)$ be an arbitrary sequence of integers greater than 1. Let X be the vertex set of $G(p, L, \mathbf{s})$, with x_0 and x_1 the endpoints, and for $\alpha \in L$ let $\hat{\alpha}$ be the equivalence of X generated by $\{(x, y)$: (x, α, y) is an edge of $G(p, L, \mathbf{s})\}$. Let $A = F_{\mathbf{V}}(X)$ denote the free $\mathbf{V}$-algebra generated by X, and let $\mathrm{con}(\hat{\alpha})$ be the congruence of A generated by the relation $\hat{\alpha}$. We claim that the map $L \to \mathrm{Con}(A)$, $\alpha \mapsto \mathrm{con}(\hat{\alpha})$ is a join-homomorphism into the congruence lattice of A. Since $G(p, L, \mathbf{s})$ has been defined in such a way that the map $L \to E(X)$, $\alpha \mapsto \hat{\alpha}$ is a join-homomorphism, it is sufficient to show that $\operatorname{con}(\alpha \vee \beta) = \operatorname{con}(\alpha) \vee \operatorname{con}(\beta)$ for any equivalences α, $\beta \in E(X)$. But this follows easily from Mal'cev's lemma describing congruences generated by relations (cf. Grätzer [9, p. 55]).

Now consider the map ψ: $L \to \operatorname{Con}(A)$, $\psi(\alpha) = \bigcup \{\alpha_i (\operatorname{con}(\hat{\beta}) : \beta \in L) : i < \omega\}$. It is a homomorphism by Proposition 2.2, whence $p(\psi(\alpha): \alpha \in L) \leq q(\psi(\alpha): \alpha \in L)$. From Proposition 3.2 (by letting the connecting map be the identity map of X) we obtain $(x_0, x_1) \in p(\operatorname{con}(\hat{\alpha}): \alpha \in L) \subseteq p(\psi(\alpha): \alpha \in L) \subseteq q(\psi(\alpha): \alpha \leq L)$. Therefore, by Proposition 3.1, x_0 and x_1 can be connected by $G_n(q)$ in A for some n, representing a colour α by the congruence $\psi(\alpha)$ and using a connecting map ϕ. If (f, α, g) is an edge of $G_n(q)$ then $(\phi(f), \phi(g)) \in \psi(\alpha)$, whence $(\phi(f), \phi(g)) \in \alpha_m(\operatorname{con}(\hat{\beta}): \beta \in L)$ for some $0 < m < \omega$. Thus $\phi(f)$ and $\phi(g)$ can be connected by $G_k(\alpha_m)$ in A (representing a colour β by $\operatorname{con}(\hat{\beta})$) for some $k \geq 2$. Since any of n, m, k can be enlarged (and, by reflexivity, n can be enlarged without enlarging the codomain of ϕ) we can assume that m does not depend on (f, α, g) and $k = n$ for all (f, α, g). ($m = n$ is also available for (iii).) What we have obtained is that x_0 and x_1 can be connected by $G_n(q(\alpha_m(\beta : \beta \in L) : \alpha \in L) = G_n(q(L_m))$ in A, representing a colour β by $\operatorname{con}(\hat{\beta})$. Let X_m be the vertex set of $G(p, L, s_0, \ldots, s_m)$. Then $X_m \subseteq X$. For $\alpha \in L$ let $\hat{\alpha}(m)$ be the equivalance of X_m generated by $\{(x, y): (x, \alpha, y)$ is an edge of $G(p, L, s_0, \ldots, s_m)\}$. We can assume (via enlarging m if necessary) that all the elements of A associated with the vertices of $G(m, n) = G_n(q(L_m))$ belong to the (free) subalgebra generated by X_m. At this point is is worth stating two assertions separately.

CLAIM 4.1. For $\alpha \in L$ $\hat{\alpha}(m)$ and the restriction of $\hat{\alpha}$ to X_m coincide.

Proof. Suppose the claim is false. Choose x, y, α, and m for the smallest possible k so that $(x, y) \notin \hat{\alpha}(m)$ but x and y can be connected by an α-coloured path of length k in $G(p, L, \mathbf{s})$. Then any inner vertex of this path lies in $G(p, L, s_0, \ldots, s_{m+1}) \backslash G(p, L, s_0, \ldots, s_m)$. By the construction of G_{m+1} from G_m this path goes through the same vertices as some path adopted to G_m for a proper join $\delta \vee \beta = \gamma$ in L. We obtain $\alpha \geq \delta$ and $\alpha \geq \beta$ (in L). Hence $\alpha \geq \gamma$, which is impossible by the construction of G_m from G_{m-1}.

Let us fix a well-ordering of X such that $x_0, x_1 \in X_m = \{x \in X : x \text{ precedes } z\}$ for some $z \in X$. For $x \in X_m$ and $\alpha \in L$ $x_{\alpha(m)}$ will denote the first member of X_m for which $(x_{\alpha(m)}, x) \in \hat{\alpha}(m)$. (For $x \in X$ x_α has been defined similarly.) Let $A_m = F_{\mathbf{V}}(X_m)$ be the free $\mathbf{V}$-algebra generated by X_m.

CLAIM 4.2 (cf. also Wille [21] and Pixley [17]). Consider f, $g \in A_m$, i.e., $f = f(x : x \in X_m)$ and $g = g(x : x \in X_m)$ for some terms denoted by the same way. Then, for $\alpha \in L$, $(f, g) \in \operatorname{con}(\hat{\alpha}) \in \operatorname{Con}(A)$ implies that the identity $f(x_{\alpha(m)} : x \in X_m) = g(x_{\alpha(m)} : x \in X_m)$ holds in $\mathbf{V}$.

Proof. Consider the map $X \to X$, $x \mapsto x_\alpha$ and extent it to an endomorphism τ of A. Since $\operatorname{con}(\hat{\alpha}) \subseteq \operatorname{Ker} \tau$ and, by Claim 4.1, $x_\alpha = x_{\alpha(m)}$ for $x \in X_m$, we obtain $f(x_{\alpha(m)}: x \in X_m) = f(x_\alpha : x \in X_m) = f(\tau(x): x \in X_m) = \tau(f(x: x \in X_m)) = \tau(g(x: x \in X_m)) = g(\tau(x): x \in X_m) = g(x_\alpha: x \in X_m) = g(x_{\alpha(m)}: x \in X_m)$, from which the assertion follows.

Now, returning to the proof, there are $|X_m|$-ary terms $f(x: x \in X_m)$ associated with the vertices $f \in G(n, m)$, where f_0 and f_1 are the endpoints. Thus $f_i(x: x \in X_m) = x_i$ (in A), $i = 0, 1$, whence the endpoint identities hold in $\mathbf{V}$. The satisfaction of the remaining identities follows from Claim 4.2.

(ii) *implies* (i). Assume (ii) and let $\phi: L \to \operatorname{Con}(A)$ be an arbitrary homomorphism where $A \in \mathbf{V}$. Suppose $(a_0, a_1) \in p(\phi(\alpha): \alpha \in L)$. We have to show that $(a_0, a_1) \in q(\phi(\alpha): \alpha \in L)$. By Proposition 3.2 there exists a sequence $\mathbf{s} = (s_0, s_1, \ldots)$ such that a_0 and a_1 can be connected by $G(p, L, \mathbf{s})$ in A, using a connecting map $\rho: X \to A$ and representing the colours α by $\phi(\alpha)$. Consider the integers m and n for this $\mathbf{s}$ provided by (ii) and with any vertex f of $G(n, m) = G_n(q(L_m))$ let us associate $f(\rho(x): x \in X_m) \in A$ (where f also denotes the corresponding term from $U(m, n)$). We claim that this way a_0 and a_1 are connected by $G(n, m)$. From the endpoint identities we have $f_i(\rho(x): x \in L) = \rho(x_i) = a_i$, $i = 0, 1$. If (f, α, g) is an edge of $G(n, m)$ then, by making use of the corresponding identity from $U(m, n)$ and $(\rho(x), \rho(x_\alpha)) \in \phi(\alpha)$ $(\alpha \in L)$, we obtain $f(\rho(x): x \in X_m)\phi(\alpha)f(\rho(x_\alpha): x \in X_m) = g(\rho(x_\alpha): x \in X_m)\phi(\alpha)g(\rho(x): x \in X_m)$.

Now we have $(a_0, a_1) \in q(\alpha_m(\phi(x): x \in L)$ by Proposition 3.1, whence Proposition 2.2 yields $(a_0, a_1) \in q(\psi(\alpha): \alpha \in L) = q(\phi(\alpha): \alpha \in L)$.

Now suppose that, for a fixed $\mathbf{s}$, $U(n, n)$ holds in a variety $\mathbf{V}$ via the terms $f(x: x \in X_m)$. As it follows from Claim 4.1, by adding irrelevant variables to these $f(x: x \in X_m)$ $U(G(p, L, s_0, \ldots, s_{n+1}) \leq G_n(q(L_n)))$ also holds in $\mathbf{V}$ via the same terms. Since $\alpha_{i+1} = \alpha_i \vee \mu_{\alpha,i}$, $G_n(q(L_n))$ can be found (though it is a little bit scattered about) in $G_n(q(L_{n+1}))$. Hence $U(G(p, L, s_0, \ldots, s_{n+1}) \leqq G_n(q(L_{n+1})))$ holds in $\mathbf{V}$ via assigning the earlier terms to the vertices of $G_n(q(L_{n+1}))$ appropriately. (E.g., to any vertex of a particular copy of $G_n(\mu_{\alpha,i})$ we assign the same term.) Similarly, assigning the earlier terms only, $U(n+1, n+1) = U(G(p, L, s_0, \ldots, s_{n+1}) \leqq G_{n+1}(q(L_{n+1}))$ holds in $\mathbf{V}$ as well. Our theorem has been proved.

5. Concluding remarks and corollaries

First we mention some examples when our Theorem applies. If χ is a lattice identity, i.e. a Horn sentence without premise, then L_χ is an antichain, $G(p, L, s_0, s_1, \ldots, s_m)$ coincides with $G(p, L, s_0)$, and $\alpha_n(L_\chi) = \alpha$ (projection).

Hence we obtain:

COROLLARY 5.1 (Wille [21], Pixley [17]). *The congruence lattices of members of a variety* $\mathbf{V}$ *satisfy a given lattice identity* $\chi: p(\alpha: \alpha \in L_\chi) \leq q(\alpha: \alpha \in L_\chi)$ *if and only if for all* $s_0 \geq 2$ *there exists* $n \geq 2$ *such that* $U(G(p, L_\chi, s_0) \leq G_n(q))$ *holds in* $\mathbf{V}$.

Note that $U(G(p, L_\chi, s_0) \leq G_n(q))$ is the same strong Mal'cev condition which occurs in [21] and [17]. Similarly, for a Horn sentence χ with join-free premise $G(p, L_\chi, s_0, \ldots, s_n)$ coincides with $G(p, L_\chi, s_0, 2)$, whence our algorithm yields a countably weak Mal'cev condition. An example for this case is $SD_\wedge$ (cf. also [2]). Our algorithm applies for $SD_\vee$, the dual of $SD_\wedge$, too, but yields a continuously weak Mal'cev condition only (cf. also [1]). Further, Herrmann's paper [11] indicates that the case of

$$\chi: (\alpha \wedge \beta = \gamma \wedge \delta \;\&\; \alpha \vee \beta = \gamma \vee \delta) \Rightarrow p \leq q,$$

which satisfies (W), might be of some interest. We also mention that if L is a finite subdirectly irreducible lattice satisfying (W) and α/β is a critical quotient in L then $L \Rightarrow \alpha \leq \beta$, a Horn sentence with (W), is equivalent to "L is not embeddable".

A variety $\mathbf{V}$ is said to be congruence n-permutable if $\alpha \vee \beta = \alpha \circ \beta \circ \alpha \circ \beta \circ \cdots$ (n factors) holds for any α, $\beta \in \mathrm{Con}(A)$, $A \in \mathbf{V}$. Congruence 2-permutable varieties seem to be hopeful candidates for our algorithm to be applied. (Compare, e.g., with [13], where the Wille-Pixley algorithm is applied for lattice identities. Moreover, as a result of Hutchinson [12] states, the classification [13] of rings via lattice identities holding in the congruence lattices of the corresponding module varieties can be refined by using Horn sentences instead of lattice identities.) Thus it is worth formulating the following.

COROLLARY 5.2. *Let* $L \Rightarrow p \leq q$ *be a Horn sentence with* (W) *and let* $\mathbf{V}$ *be an* n*-permutable variety,* $n \geq 2$. *Then* $L \Rightarrow p \leq q$ *holds in the congruence lattices of members of* $\mathbf{V}$ *if and only if* $U(G(p, L, n, \ldots, n) \leq G_n(q(L_m)))$, *where* $n, n, \ldots, n$ *consists of* $m+1$ *members, holds in* $\mathbf{V}$ *for some* $m \geq 1$.

The proof is involved in the previous section. This corollary is not surprising in itself, since the existence of the Mal'cev condition occurring in it follows from the Taylor-Neumann characterization [16, 18] (cf. Jónsson [14, Theorem 2.16]). Even a concrete form of an appropriate Mal'cev condition is known for any Horn sentence (cf. [3]). However Corollary 5.2 gives simpler Mal'cev conditions than those in [3]. This can be demonstrated by the following.

COROLLARY 5.3. *For any Horn sentence $L \Rightarrow p \leq q$ satisfying (W) there is an infinite sequence of lattice identities $(\kappa_0, \kappa_1, \kappa_2, \ldots)$ such that for any $n \geq 2$ and for any n-permutable variety* **V** *$L \Rightarrow p \leq q$ holds in the congruence lattices of* **V** *if and only if there exists an integer $m < \omega$ such that κ_m holds in the congruence lattices of* **V**.

Proof. Let p_m denote the lattice term for which $G(p, L, n, \ldots, n)$ (n occurring $m+1$ times) coincides with $G_n(p_m)$. (If we modify the definition of $G(p, L, \mathbf{s})$ such that, for all m, we add an appropriate α-β-coloured new path of length $s_{m+1}(=n)$ to *any* γ-coloured edge of G_m ($\gamma = \alpha \vee \beta$) then all the previous statements remain valid but the existence of p_m and its independence from n can be seen more easily. Really, the substitution of $\gamma \wedge (\alpha \vee \beta)$ to γ corresponds to the above mentioned new paths.) Now we can define κ_m to be the identity $p_m \leq q(L_m)$, and apply Corollary 5.2 (both to $L \Rightarrow p \leq q$ and to κ_m).

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