MEDIANS ARE BELOW JOINS IN SEMIMODULAR LATTICES OF BREADTH 2

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Abstract. Let L be a lattice of finite length and let d denote the minimum path length metric on the covering graph of L. For any $\xi = (x_1, \dots, x_k) \in L^k$, an element y belonging to L is called a *median* of ξ if the sum $d(y, x_1) + \dots + d(y, x_k)$ is minimal. The lattice L satisfies the c_1*-median property* if, for any $\xi = (x_1, \dots, x_k) \in L^k$ and for any median y of ξ, $y \le x_1 \vee \dots \vee x_k$. Our main theorem asserts that if L is an upper semimodular lattice of finite length and the breadth of L is less than or equal to 2, then L satisfies the c_1-median property. Also, we give a construction that yields semimodular lattices, and we use a particular case of this construction to prove that our theorem is sharp in the sense that 2 cannot be replaced by 3.

1. Introduction

Given a lattice L of finite length and $\xi = (x_1, \dots, x_k) \in L^k$, an element $y \in L$ is called a *median* of ξ if the sum $d(y, x_1) + \dots + d(y, x_k)$ is minimal, where $d(y, x_i)$ stands for the path distance in the Hasse diagram of L. Our goal is to prove that

whenever L is, in addition, upper semimodular and of breadth at most 2, to be defined in (1.10), then $y \le x_1 \vee \dots \vee x_k$ holds for every $k \ge 2$ and for any median y of every $\xi = (x_1, \dots, x_k) \in L^k$; (1.1)

see our main result, Theorem 4.1, for more details.

1.1. **Outline.** The paper is structured as follows. In Subsection 1.2, we survey some earlier results on medians in lattices. Subsection 1.3 recalls some definitions, whereby the paper is readable with minimal knowledge of Lattice Theory. In Section 2, we give a new way of constructing semimodular lattices; see Proposition 2.1, which can be of separate interest. As a particular case of our construction, we present a semimodular lattice $L(n, k)$ with breadth k and size $|L(n, k)| = 2n^k - (n-1)^k$ for any integers $k \ge 3$ and $n \ge 4$ such that $L(n, k)$ fails to satisfy the c_1-median property. Section 3 is devoted to two technical lemmas that will be used later. Finally, Section 4 presents our main result, Theorem 4.1, which asserts somewhat more than (1.1). Using the auxiliary statements proved in Sections 2 and 3, Section 4 concludes with the proof of Theorem 4.1. Note

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that the survey given in Subsection 1.2 is mainly for lattice theorists; this is why some well-known lattice theoretical concepts occurring there are only explained thereafter.

1.2. **Survey.** For any metric space (X, d) and for any k-tuple $\xi = (x_1, \ldots, x_k)$ belonging to X^k, $y \in X$ is called a *median of* ξ if

$$r(y, \xi) = \sum_{i=1}^{k} d(y, x_i) \tag{1.2}$$

is minimal. Medians are frequently used numerical attributes of, say, (discrete) probability distributions, and they are interesting in other areas of mathematics and even outside mathematics; see, for example, Monjardet [15].

The k-tuple ξ above is called a *profile* and $\{\xi\}$ denotes the set of all elements belonging to the profile. Repetition among the x_i's is permitted, so $|\{\xi\}| \leq k$. The notation $M(\xi)$ is used for the set of all medians of ξ and $r(y, \xi)$ is called the *remoteness* of y from ξ. One can view M as a function with domain the set of all possible profiles and range the set of all nonempty subsets of X. In this case, M is called the *median function* or the *median procedure*. The median function has been extensively studied and we refer the reader to Day and McMorris [8] for more information about this function.

If X is a lattice L of finite length and d is the minimum path length metric on the covering graph of L, then it is sometimes possible to describe a median set $M(\xi)$ explicitly. For example, if L is a finite distributive lattice and $\xi = (x_1, \ldots, x_k) \in L^k$, then

$$\begin{aligned} M(\xi) &= [m(\xi), m'(\xi)] = \{z \in L : m(\xi) \leq z \leq m'(\xi)\} \text{ where} \\ m(\xi) &= \bigvee\{ \bigwedge_{i \in I} x_i : I \subseteq \{1, \ldots, k\}, |I| \geq \frac{k}{2} + 1\} \text{ and} \\ m'(\xi) &= \bigwedge\{ \bigvee_{i \in I} x_i : I \subseteq \{1, \ldots, k\}, |I| \geq \frac{k}{2} + 1\}. \end{aligned}$$

This result is due to Barbut [2] and Monjardet [15]. Their result was extended by Bandelt and Barthélemy to median semilattices [1]. In addition, Barthélemy showed that $M(\xi)$ is a sublattice of the *interval* $[m(\xi), m'(\xi)]$ if L is a finite modular lattice [3]. In the case where L is assumed to be a finite upper semimodular lattice, Leclerc [14] proved that $M(\xi) \subseteq [m(\xi), 1_L]$ for every $\xi \in L^k$. Leclerc also showed the converse. Specifically, if a finite lattice L has the property that $M(\xi) \subseteq [m(\xi), 1_L]$ for every $\xi \in L^k$, then L is upper semimodular. Leclerc's work was generalized to finite upper semimodular posets in [17].

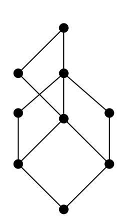

FIGURE 1. A nonplanar semimodular lattice of breadth two

While Leclerc [14] above gives a lower bound of $M(\xi)$, here we are interested in a reasonable upper bound. Namely, following White [21], we will say that a lattice L satisfies the c_1-*median property* if for any positive integer k and any $\xi = (x_1, \dots, x_k) \in L^k$,

$$y \le c_1(\xi) := \bigvee_{i=1}^{k} x_i \tag{1.3}$$

for all $y \in M(\xi)$. This is obviously equivalent to $\bigvee M(\xi) \le c_1(\xi)$. Since $m'(\xi) \le c_1(\xi)$ for all ξ, it follows that every finite *modular* lattice satisfies the c_1-median property. Finite (upper) semimodular lattices are known to be graded. (As usual, "semimodular" will always mean "upper semimodular".) Czédli, Powers, and White [5] proved that

every planar graded lattice satisfies the c_1-median property. (1.4)

Let us emphasize that a planar lattice is finite by definition; see Grätzer and Knapp [11, page 447] or Czédli and Grätzer [4, page 92]. Clearly, (1.4) implies immediately that

planar semimodular lattices satisfy the c_1-median property. (1.5)

It belongs to the folklore and we will prove in Section 3 that

every planar lattice is of breadth at most 2. (1.6)

Hence (1.1) is a generalization of (1.5). Furthermore, this is a proper generalization since there are non-planar finite semimodular lattices of breadth 2; see Figure 1 for an example. Note at this point that the class of all semimodular lattices of finite length and breadth 2 is plentiful since, for example, Rival [18] proved that this class contains lattices with arbitrarily large finite width and length. Note also that a graded lattice need not be semimodular, and so it is easy to see that none of (1.1) and (1.4) implies the other one.

In 2000, Li and Boukaabar [13] gave a semimodular lattice with 101 elements that fails to satisfy the c_1-median property; we will denote this lattice by L_{LiBou}. Hence, (1.1) cannot be extended to all semimodular lattices of finite length. Our Theorem 4.1 will assert even more: as $L(n,3)$ in Section 2 exemplifies, (1.1) cannot be extended to finite length semimodular lattices of breadth 3. Note that Section 2 builds on the essence of L_{LiBou} but, in addition that we will show that $L(n,3)$ is of breadth 3, there is a significant difference between the two approaches. Namely, as opposed to [13], where L_{LiBou} is defined by its involved Hasse diagram, tedious work is needed to show that it is a lattice and it is semimodular, and most of this work is left to the reader, our argument proving the same properties of $L(n,3)$ does not rely on any diagram and it is easy to read.

It was proved in White [21] that

semimodular lattices of height at most 6 satisfy the c_1-median property. (1.7)

Each of the conditions given in (1.1), (1.4), (1.5), and (1.7) determines an interesting class of semimodular lattices of finite length satisfying the c_1-median property. Although interesting additional such classes of semimodular lattices will hopefully be discovered in the future, we do not see much hope for a reasonable characterization of semimodular lattices of finite length that satisfy the c_1-median property.

1.3. **Basic concepts.** All the elementary concepts and notation not defined in this paper can easily be found in Grätzer [9] or in its freely downloadable *Part I. A Brief Introduction to Lattices and Glossary of Notation* at `tinyurl.com/lattices101`, and also in Nation [16], freely available again. Alternatively, the reader can look into Davey and Priestley [7] or Stern [20]. However, for convenience, we recall the following. A lattice L is of *finite length* if there is a nonnegative integer n such that every chain of L consists of at most $n+1$ elements; if so, then the smallest such n is the *length* of the lattice, denoted by $\ell(L)$. A lattice of finite length is *graded* if any two of its maximal chains have the same (finite) number of elements. A lattice L is *upper semimodular*, or simply *semimodular*, if for every $x, y \in L$, the covering $x \wedge y \prec x$ implies $y \prec x \vee y$. The condition *lower semimodular* is defined dually. It is well known that every semimodular lattice of finite length is graded. For $x, y \in L$, the distance between x and y in the undirected covering graph associated with L is denoted by $d(x, y)$. It is straightforward to see that in a *semimodular* lattice L of finite length, for any $x, y, u, v, w \in L$,

$$d(x,y) = d(x, x\vee y) + d(x\vee y, y) = \ell([x, x\vee y]) + \ell([y, x\vee y]) \tag{1.8}$$

$$\text{and } u \le v \le w \text{ implies that } d(u,w) = d(u,v) + d(v,w). \tag{1.9}$$

Finally, recall that

$$\left.\begin{array}{l}\text{the } \textit{breadth} \text{ of a lattice } L, \text{ to be denoted by } \mathrm{br}(L), \text{ is the}\\ \text{least positive integer } n \text{ such that any join } \bigvee_{i=1}^m x_i,\ x_i \in L,\\ m \ge n, \text{ is always a join of } n \text{ of the joinands } x_i.\end{array}\right\} \tag{1.10}$$

2. Semimodular constructs and an example

An element u in a lattice L is *join-irreducible* if for every $x, y \in L$, $u = x \vee y$ implies that $u = x$ or $u = y$. Similarly, if $u \le x \vee y$ implies that $u \le x$ or $u \le y$, then u is *join-prime*. Finally, u is *codistributive* (or *dually distributive*) if for every $x, y \in L$, $u \wedge (x \vee y) = (u\wedge x) \vee (u \wedge y)$; see, for example, Šešelja and Tepavčević [19] and Grätzer[10]. Clearly, a join-prime element is join-distributive. If an element is codistributive and join-irreducible, then it is join-prime; see (the easy proof of) Nation [16, Theorem 8.6(1)]. So there are many examples of join-prime elements in lattices. Note that each of the three free generators of the 28-element free modular lattice is join-prime, join-irreducible, but not codistributive; see Grätzer [10, Figure 20 in page 85]. Observe that, for every positive integer t and any lattices $K_1, \ldots, K_t$ of finite length,

$$\left.\begin{array}{l}\text{a nonzero element } e = (e_1, \ldots, e_t) \in K_1 \times \cdots \times K_t \text{ is join-prime}\\ \text{if and only if there exists a unique } i = i(e) \in \{1, \ldots, t\} \text{ such that}\\ e_i \text{ is a nonzero join-prime element of } K_i \text{ and } e_j \text{ is the bottom}\\ \text{element } 0_j \text{ of } K_j \text{ for all } j \in \{1, \ldots, t\} \setminus \{i\}.\end{array}\right\} \tag{2.1}$$

In order to verify (2.1), assume that e has at least two nonzero coordinates, say, e_1 and e_2. Then $e \le (e_1, 0_2, \ldots 0_t) \vee (0_1, e_2, \ldots, e_t)$ witnesses that e is not join-prime. The rest of the argument proving (2.1) is even more trivial and will not be detailed.

Proposition 2.1. *Let K be a lattice of finite length.*

(i) *If e is a nonzero join-prime element of K, $f \in K$, and $e \le f$, then the subposet $L := K \setminus [e, f]$ of L is a lattice.*

(ii) *If t is a positive integer, $K_1, \dots, K_t$ are semimodular lattices of finite length, $K = K_1 \times \dots \times K_t$ is their direct product, $e = (e_1, \dots, e_t) \in K$ is a nonzero join-prime element, $i = i(e)$ denotes the subscript defined in* (2.1)*, and $f = (f_1, \dots, f_t)$ is an element of K such that f_i is the top element 1_i of K_i, then the subposet $L := K \setminus [e, f]$ of K is a semimodular lattice, and it is a join-subsemilattice of K.*

Note that (2.1) and the assumptions of part (ii) above imply that $e \le f$, whereby the interval $[e, f]$ in (ii) makes sense. Note also that the case $t = 1$ is also interesting, but this case would be easier to prove than the general case $t \in \{1, 2, 3, \dots\}$.

Proof. First, we are going to prove (i). Since $0_K < e$, the subposet L has a least element, $0 := 0_K$. Observe that L is of finite length since so is K. Thus, to prove that L is a lattice, it suffices to prove that L is join-closed. So it suffices to show that L is a join-subsemilattice of K. Suppose, for a contradiction, that $x, y \in L$ but $x \vee y \notin L$. Then $e \le x \vee y \le f$. Since e is join-prime, we obtain that $e \le x$ or $e \le y$, and we can assume that $e \le x$ by symmetry. This with $x \le x \vee y \le f$ lead to $x \in [e, f]$, contradicting $x \in L$. Thus, L is join-closed and part (i) holds.

Next, we turn our attention to (ii). We can assume that $i = 1$. Then, by (2.1),

$$e_1 > 0_1,\ e_2 = 0_2,\ \dots,\ e_t = 0_t. \tag{2.2}$$

We obtain from part (i) that L is a lattice. We are going to show that

$$\left.\begin{array}{l}\text{whenever } \{x, y\} \subseteq L \text{ and } y \text{ covers } x \text{ in } L,\\ \text{then } y \text{ covers } x \text{ in } K.\end{array}\right\} \tag{2.3}$$

First of all, observe that for any $a, b \in K$, we trivially have that

$$\left.\begin{array}{l}a \prec_K b \text{ if and only if } a_j \prec b_j \text{ for exactly one subscript } j \text{ and}\\ a_s = b_s \text{ for every other subscript } s\text{; note that this holds even}\\ \text{if } K_1, \dots, K_t \text{ are not assumed to be semimodular.}\end{array}\right\} \tag{2.4}$$

For the sake of contradiction, suppose that $x \prec_L y$ but $x \not\prec_K y$. Then there is at least one element in $[e, f] \cap [x, y]$. Hence, for $a := e \vee x$ and $b := f \wedge y$, we have that $a \le b$. Note that $x \le a \le b \le f$, so $x \notin [e, f]$ yields that $e \not\le x$. Similarly, $e \le a \le b \le y$ and $y \notin [e, f]$ give that $y \not\le f$. Since $a \in [e, f]$ but $x \notin [e, f]$, we have that $x < a$. If we had an $x' \in K$ such that $x < x' < a$, then $x < x' < a \le b < y$ and $x \prec_L y$ would imply that $x' \notin L$, whereby $e \le x'$ would lead to the contradiction $a = e \vee x \le x' < a$. Thus, $x \prec_K a$ in K. Similarly, $b \prec_K y$. Let us summarize:

$$\left.\begin{array}{l}x \prec_K x \vee e = a \le b = y \wedge f \prec_K y,\\ e \not\le x,\ \ y \not\le f,\ \ e \le y,\ \ x \le f.\end{array}\right\} \tag{2.5}$$

Since $e \not\le x$, (2.2) gives that $e_1 \not\le x_1$. We know from (2.5) that $x \le f$, and so we obtain that $x_2 \le f_2, \dots, x_t \le f_t$. Hence, if we had that $x_2 = y_2, \dots, x_t = y_t$, then we would get that $y \le f$ since $f_1 = 1_1$, but $y \le f$ would contradict (2.5). Thus, there is a subscript $j \in \{2, \dots, t\}$ such that $x_j < y_j$. By symmetry, we can assume that $j = 2$, that is, $x_2 < y_2$. Take the element $z := (x_1, y_2, x_3, \dots, x_t)$ in K. Since $e_1 \not\le x_1 = z_1$, we have that $e \not\le z$, whereby $z \in L$. Using $x_2 < y_2 = z_2$, we obtain that $x < z$. Since $x < y$, we have that $z \le y$. Using that $e_1 \not\le x_1 = z_1$ but (2.5) gives that $e_1 \le y_1$, it follows that $z \ne y$. So $z < y$. Since $x < z$, $z < y$, and $z \in L$ contradict $x \prec_L y$, we conclude (2.3).

Next, recall from Czédli and Walendziak [6] that

$$\text{the direct product of finitely many semimodular lattices is semimodular.} \tag{2.6}$$

This yields that K is semimodular. This fact, (2.3), and Exercise 3.1 in [4] imply the semimodularity of L. This proves part (ii) and completes the proof of Proposition 2.1. □

Lemma 2.2. *For any integer $t \geq 2$ and non-singleton lattices $L_1, \dots, L_t$ of finite breadth,*

$$\mathrm{br}(L_1 \times \cdots \times L_t) = \mathrm{br}(L_1) + \cdots + \mathrm{br}(L_t).$$

Having no reference at hand, we present a straightforward proof of this easy lemma.

Proof. We can assume that $t = 2$, because then the lemma follows by induction. For $i \in \{1, 2\}$, denote $\mathrm{br}(L_i)$ by n_i, and pick an n_i-element subset $\{a(i)_1, \dots, a(i)_{n_i}\}$ of L_i such that no element of this subset is the smallest element of L_i (which need not exist), and $b(i) := a(i)_1 \vee \cdots \vee a(i)_{n_i} \in L_i$ is an *irredundant join*, that is, none of the joinands can be omitted without making the equality false. Pick $c(i) \in L_i$ such that $c(i) < b(i)$ and $c(i) \leq a(i)_j$ for all $j \in \{1, \dots, n_i\}$; this is possible either because $n_i > 1$ and we can let $c(i) = a(i)_1 \wedge \cdots \wedge a(i)_{n_i}$, or because $n_i = 1$ and we can pick an element smaller than $a(i)_1$. Since the join $(b(1), b(2))$ of the elements $(a(1)_1, c(2))$, $(a(1)_2, c(2))$, $\dots$, $(a(1)_{n_1}, c(2))$, $(c(1), a(2)_1)$, $(c(1), a(2)_2)$, $\dots$, $(c(1), a(2)_{n_2})$ is clearly an irredundant join, $\mathrm{br}(L_1 \times L_2) \geq n_1 + n_2 = \mathrm{br}(L_1) + \mathrm{br}(L_2)$. To prove the converse inequality, assume that $(w_1, w_2) = \bigvee S$ in $L_1 \times L_2$ with $|S| \geq n_1 + n_2$. For each $i \in \{1, 2\}$, we can pick an n_i-element subset T_i of S such that $w_i = \bigvee_{v \in T_i} v_i$. Letting T be an $(n_1 + n_2)$-element subset of S such that $T_1 \cup T_2 \subseteq T$, we have that $(w_1, w_2) \leq \bigvee T \leq \bigvee S = (w_1, w_2)$. Thus, $\mathrm{br}(L_1 \times L_2) \leq n_1 + n_2 = \mathrm{br}(L_1) + \mathrm{br}(L_2)$. □

For integers $n \geq 4$ and $k \geq 3$, we define a lattice $L(n, k)$ as follows. Let $C_n = \{0, 1, 2, \dots, n-1\}$ be the n-element chain with the usual ordering from $\mathbb{Z}$. Let $K = K(n, k)$ be the $(k+1)$-fold direct product

$$K = K(n, k) = C_n \times C_n \times \cdots \times C_n \times C_2.$$

After defining $e = (e_1, \dots, e_{k+1})$ and $f = (f_1, \dots, f_{k+1})$ by

$$e := (0, \dots, 0, 1, 0) \text{ and } f := (n-2, \dots, n-2, n-1, 0),$$

we define $L = L(n, k)$ as $K \setminus [e, f]$. At present, $L(n, k)$ is only a poset.

Proposition 2.3. *For integers $n \geq 4$ and $k \geq 3$, $L(n, k)$ is a $(2n^k - (n-1)^k)$-element semimodular lattice of breadth k, and this lattice fails to satisfy the c_1-median property.*

Proof. In a chain, every element is join-prime. Thus, it follows from Proposition 2.1 that $L = L(n, k)$ is a semimodular lattice. Clearly, $|L| = |K| - |[e, f]| = 2n^k - (n-1)^k$.

The 2^k-element boolean lattice is isomorphic to, say, $\{2, 3\} \times \cdots \times \{2, 3\} \times \{1\}$, which is a join-subsemilattice of L. Hence, we obtain from Lemma 2.2 (or we conclude easily even without this lemma) that $\mathrm{br}(L) \geq k$. In order to prove the converse inequality, let $\mathcal{W} = \{w(1), w(2), \dots, w(m)\}$ with $m \geq k+1$ be a collection of elements from L. (In order to avoid avoid four-level formulas with microscopic subscripts of superscripts later, we prefer $w(i)$ to the notation $w^{(i)}$.) The j-th component of $w(i)$ will be denoted by $w(i)_j$. Denote $\bigvee \mathcal{W}$ by y. It suffices to find an at most k-element subset $\mathcal{W}^*$ of $\mathcal{W}$ such that $\bigvee \mathcal{W}^* = y$. For each $i = 1, \dots, k+1$, we can find at least one $w(j_i) \in \mathcal{W}$ such that $y_i = w(j_i)_i$. Let $\mathcal{W}' := \{w(j_1), \dots, w(j_{k+1})\}$. Clearly, $\bigvee \mathcal{W}' = y$ and $|\mathcal{W}'| \leq k+1$.

Suppose that $y_i = 0$ for some $i \in \{1, \dots, k+1\}$. Then $\bigvee(\mathcal{W}' \setminus \{w(j_i)\})$ still equals y, so $\mathcal{W}' \setminus \{w(j_i)\}$ serves as $\mathcal{W}^*$. Now assume that every coordinate of y is nonzero; in particular, $y_{k+1} = 1$. We can also assume that $w(j_k)_{k+1} = 0$ since otherwise the equality $w(j_k)_{k+1} = 1$ would make $w(j_{k+1})$ superfluous, that is, we could let $\mathcal{W}^* := \mathcal{W}' \setminus \{w(j_{k+1})\}$. Since $w(j_k)_k = y_k \neq 0$ gives that $e \leq w(j_k)$ but $w(j_k) \notin [e, f]$, it follows that $w(j_k) \not\leq f$. This fact and $w(j_k)_{k+1} = 0$ give that $w(j_k)_i = n-1$ for some $i \in \{1, \dots, k-1\}$. So $n - 1 = w(j_k)_i \leq y_i = w(j_i)_i$, where the inequality turns into an equality since $n-1$ is the largest element of C_n. Thus, we can let $\mathcal{W}^* := \mathcal{W}' \setminus \{w(j_i)\}$. We have proved that $\mathrm{br}(L) = k$.

Next, to prove that L does not satisfy the c_1-median property, let

$$\left.\begin{array}{l} x(0) = (\quad 0, \quad 0, \quad 0, \; \dots, \; 0, \quad 0, \quad 0 \;), \\ x(1) = (\; n-1, \quad 0, \quad 0, \; \dots, \; 0, \; n-1, \; 0 \;), \\ x(2) = (\quad 0, \; n-1, \; 0, \; \dots, \; 0, \; n-1, \; 0 \;), \end{array}\right\} \tag{2.7}$$

and define $\xi := (x(0), x(1), x(2)) \in L^3$. Clearly, $c_1(\xi) = (n-1, n-1, 0, \dots, 0, n-1, 0)$; see (1.3). By (1.2) and (1.8), the remoteness of an arbitrary $y = (y_1, y_2, \dots, y_k, y_{k+1}) \in L$ with respect to ξ is

$$\begin{aligned} r(y, \xi) = \sum_{i=1}^{2} [(n-1) - y_i + 2y_i] + \sum_{i=3}^{k-1} 3y_i + 2(n-1) - y_k \\ + 3y_{k+1} = 4(n-1) + y_1 + y_2 - y_k + 3y_{k+1} + \sum_{i=3}^{k-1} 3y_i. \end{aligned} \tag{2.8}$$

Consider $z = (0, 0, 0, \dots, 0, n-1, 1) \in L$. By (2.8) or trivially,

$$r(z, \xi) = 2(n-1) + n - 1 + 3 = 3n. \tag{2.9}$$

We are going to show that, for every $y \in K = K(n, k)$,

$$r(y, \xi) < r(z, \xi) \text{ implies } y \notin L. \tag{2.10}$$

Suppose that $r(y, \xi) < r(z, \xi)$. Thus, using $y_k \leq n-1$, (2.8), and (2.9), we obtain after rearranging and simplifying that

$$n + y_1 + y_2 + 3y_{k+1} + \sum_{i=3}^{k-1} 3y_i < y_k + 4 \leq n - 3. \tag{2.11}$$

This implies that $y_1 + y_2 + 3 \cdot \left(y_{k+1} + \sum_{i=3}^{k-1} y_i\right) < 3$, whereby

$$\left.\begin{array}{l} y_i = 0 \text{ for } i \in \{3, 4 \dots, k-1, k+1\} \text{ and} \\ y_i \leq 2 \leq n-2 \text{ for } i = 1, 2. \end{array}\right\} \tag{2.12}$$

The first inequality in (2.11) together with $n \geq 4$ yield that that $1 \leq y_k$. This fact and (2.12) imply that $y \in [e, f]$, that is, $y \notin L$. Consequently, (2.10) holds, and so $z \in M(\xi)$. Since $z \not\leq c_1(\xi)$, it follows that L does not satisfy the c_1-median property. □

For lattices $(L'; \leq')$ with top $1'$ and $(L''; \leq'')$ with bottom $0''$, their *glued sum* is defined to be $\big((L' \setminus \{1'\}) \cup \{1' = 0''\} \cup (L'' \setminus \{0''\}); \leq\big)$ where $x' \leq y''$ for any $(x', y'') \in L' \times L''$ and the restriction of $\leq$ to L' and that to L'' are $\leq'$ and $\leq''$, respectively. Saying in a pragmatical way for the finite case: we put the diagram of L'' atop that of L' and we identify $1'$ with $0''$. For example, the glued sum of the 2-element chain and the 3-element chain is the 4-element chain. The following remark is a trivial consequence of the case $(n, k) = (4, 3)$ of Proposition 2.3; note that the proof of this particular case would not be significantly shorter than that of Lemma 2.3.

Remark 2.4. For $k > 3$, we can easily construct a finite semimodular lattice $G(k)$ of breadth k such that $G(k)$ does not satisfy the c_1-median property and its size is less than $|L(4, k)| = 2 \cdot 4^k - 3^k$. Namely, let $G(k)$ be the glued sum of $L(4, 3)$ and the 2^k-element boolean lattice; its size is $|G(k)| = 2 \cdot 4^3 - 3^3 + 2^k - 1 = 2^k + 100$.

3. Two technical lemmas

Before formulating two technical lemmas, we prove (1.6), simply because we could not find any reference to this almost trivial statement.

Proof of (1.6). For the sake of contradiction, suppose that L is a planar lattice but not of breadth at most 2. Then we can take a join $x_1 \vee \cdots \vee x_n =: y$ in L such that $n \geq 3$ but $y \neq x_i \vee x_j$ for any $i, j \in \{1, \dots, n\}$. Since $\{x_1, \dots, x_n\}$ is clearly not a chain, we can assume that x_1 and x_2 are incomparable (in notation, $x_1 \parallel x_2$) and $x_1 \vee x_2$ is a maximal element of $\{x_i \vee x_j : \{i, j\} \subseteq \{1, \dots, n\}\}$. There is a $t \in \{3, \dots, n\}$ such that $x_t \nleq x_1 \vee x_2$ since otherwise we would have that $y = x_1 \vee x_2$. We claim that $H := \{x_1 \vee x_2, x_1 \vee x_t, x_2 \vee x_t\}$ is a three-element antichain. Since $x_t \nleq x_1 \vee x_2$, we have that $x_i \vee x_t \nleq x_1 \vee x_2$ for $i \in \{1, 2\}$. In particular, $x_i \vee x_t \neq x_1 \vee x_2$. So if we had $x_1 \vee x_2 \leq x_i \vee x_t$, then $x_1 \vee x_2 < x_i \vee x_t$ would contradict the maximality of $x_1 \vee x_2$. If we had that $x_1 \vee x_t \nparallel x_2 \vee x_t$, say, $x_1 \vee x_t \leq x_2 \vee x_t$, then $x_1 \vee x_2 \leq (x_1 \vee x_t) \vee (x_2 \vee x_t) = x_2 \vee x_t$ would lead to an already excluded case. So H is a three-element antichain. We know from, say, Grätzer [10, Lemma 73] that H generates a sublattice isomorphic to the eight-element boolean lattice. This contradicts the planarity of L by Kelly and Rival [12]. □

The next two lemmas will be needed later in the paper.

Lemma 3.1 (White [21])**.** *Let L be a semimodular lattice of finite length. If $\xi = (x_1, x_2) \in L^2$, then for all $x \in M(\xi)$, $x \leq x_1 \vee x_2$.*

Let L be a lattice and $\xi = (x_1, \dots, x_k) \in L^k$. Recall that $\{\xi\}$ denotes the set $\{x_1, \dots, x_k\}$. Suppose $z \in L$ with $z \nleq c_1(\xi)$. We note that for each $x_i \in \{\xi\}$ it is the case that $x_i \parallel z$ or $x_i < z$. Let

$$\left.\begin{aligned} \xi_{\mathrm{P}} &= \{i : x_i \in \{\xi\} \text{ and } x_i \parallel z\} \text{ and} \\ \xi_{\mathrm{B}} &= \{i : x_i \in \{\xi\} \text{ and } x_i < z\}; \end{aligned}\right\} \tag{3.1}$$

the subscripts come from "parallel" and "below", respectively. Note that $|\xi_{\mathrm{P}}| + |\xi_{\mathrm{B}}| = k$.

Lemma 3.2. *Let L be a semimodular lattice of finite length. Let $\xi = (x_1, \dots, x_k) \in L^k$ and $z \in L$ such that $z \nleq c_1(\xi)$. If $|\xi_{\mathrm{P}}| \leq |\xi_{\mathrm{B}}|$, then $z \notin M(\xi)$.*

Proof. If $|\xi_{\mathrm{P}}| = 0$, then $z > c_1(\xi)$. By Lemma 2.2 in [5], $z \notin M(\xi)$. From now on we will assume that $|\xi_{\mathrm{P}}| \geq 1$ and so $z \parallel c_1(\xi)$. If $|\xi_{\mathrm{P}}| = |\xi_{\mathrm{B}}| = 1$, then $z \notin M(\xi)$ follows from Lemma 3.1. Assume that $|\xi_{\mathrm{B}}| \geq 2$ and let $y := \bigvee\{x_i \in \{\xi\} : x_i < z\} = \bigvee\{x_i : i \in \xi_{\mathrm{B}}\}$. Since $y \leq c_1(\xi)$, $y \leq z$, and $z \parallel c_1(\xi)$, it is the case that $y < z$. We observe that for each $x_i \in \{\xi\}$ with $x_i \parallel z$ (that is, for each $i \in \xi_{\mathrm{P}}$) the triangle inequality gives that

$$d(y, x_i) \leq d(y, z) + d(z, x_i), \tag{3.2}$$

and for each $x_i \in \{\xi\}$ with $x_i < z$ (that is, for each $i \in \xi_{\rm B}$), (1.9) implies that

$$d(y, x_i) = d(z, x_i) - d(y, z). \tag{3.3}$$

We may assume without loss of generality that $1 \in \xi_{\rm P}$ and so $x_1 \parallel z$. Note that $y \vee x_1 \leq z \vee x_1$. Since $y \vee x_1 \leq c_1(\xi)$ and $z \vee x_1 \not\leq c_1(\xi)$, it follows that $y \vee x_1 < z \vee x_1$. Thus

$$d(y, y \vee x_1) < d(y, z \vee x_1) \text{ and } d(y \vee x_1, x_1) < d(z \vee x_1, x_1). \tag{3.4}$$

We may assume that $2 \in \xi_{\rm B}$ and so $x_2 < z$. Using (1.8), (3.4), and the triangle inequality at $\leq'$, we get

$$\begin{aligned}
d(y, x_1) + d(y, x_2) &\overset{(1.8)}{=} d(y, y \vee x_1) + d(y \vee x_1, x_1) + d(y, x_2) \\
&\overset{(3.4)}{<} d(y, z \vee x_1) + d(z \vee x_1, x_1) + d(y, x_2) \\
&\leq' d(y, z) + d(z, z \vee x_1) + d(z \vee x_1, x_1) + d(y, x_2) \\
&\overset{(1.8)}{=} d(z, x_1) + d(z, y) + d(y, x_2) \\
&\overset{(1.9)}{=} d(z, x_1) + d(z, x_2), \qquad \text{whereby}
\end{aligned}$$

$$d(y, x_1) + d(y, x_2) < d(z, x_1) + d(z, x_2). \tag{3.5}$$

Finally, let $\xi'_{\rm P} = \xi_{\rm P} \setminus \{1\}$ and let $\xi'_{\rm B} = \xi_{\rm B} \setminus \{2\}$. Using the inequality $|\xi'_{\rm P}| \leq |\xi'_{\rm B}|$ at $\leq'$, we get the following calculation.

$$\begin{aligned}
r(y, \xi) &= \sum_{i \in \xi_{\rm P}} d(y, x_i) + \sum_{i \in \xi_{\rm B}} d(y, x_i) \\
&= \sum_{i \in \xi'_{\rm P}} d(y, x_i) + d(y, x_1) + \sum_{i \in \xi'_{\rm B}} d(y, x_i) + d(y, x_2) \\
&\overset{(3.2,3.3)}{\leq} \sum_{i \in \xi'_{\rm P}} d(z, x_i) + |\xi'_{\rm P}| \cdot d(y, z) + d(y, x_1) + \\
&\qquad \sum_{i \in \xi'_{\rm B}} d(z, x_i) - |\xi'_{\rm B}| \cdot d(z, y) + d(y, x_2) \\
&\leq' \sum_{i \in \xi'_{\rm P}} d(z, x_i) + d(y, x_1) + \sum_{i \in \xi'_{\rm B}} d(z, x_i) + d(y, x_2) \\
&\overset{(3.5)}{<} \sum_{i \in \xi'_{\rm P}} d(z, x_i) + d(z, x_1) + \sum_{i \in \xi'_{\rm B}} d(z, x_i) + d(z, x_2) = r(z, \xi).
\end{aligned}$$

Hence $r(y, \xi) < r(z, \xi)$, and so $z \notin M(\xi)$, as required. □

Note that in the proof of Proposition 2.3, where ξ is given in (2.7) modulo notational changes and $z = (0, \ldots, 0, n-1, 1)$, we have $|\xi_{\rm P}| = 2 > 1 = |\xi_{\rm B}|$. Therefore the restriction $|\xi_{\rm P}| \leq |\xi_{\rm B}|$ given in Lemma 3.2 cannot be dropped.

4. Main result

In harmony with the general convention that the empty join is the least element, note that the breadth of the singleton lattice is 0.

Theorem 4.1.

(i) *Let L be a semimodular lattice of finite length. If L is of breadth at most 2, then L satisfies the c_1-median property.*

(ii) *For each integer $k \geq 3$, there exists a finite semimodular lattice of breadth k that fails to satisfy the c_1-median property.*

(iii) *Let t be a positive integer. For $i = 1, \ldots, t$, let L_i be a lattice of finite length satisfying the c_1-median property. Then the direct product $L := L_1 \times \cdots \times L_t$ is a lattice of finite length and it also satisfies the c_1-median property. If all the L_i are of finite breadth, then $\mathrm{br}(L) = \mathrm{br}(L_1) + \cdots + \mathrm{br}(L_t)$. Furthermore, if all the L_i are semimodular, then so is L.*

Proof. In order to prove part (i), let L be a semimodular lattice of finite length with breadth 2. Let $\xi = (x_1, \ldots, x_k) \in L^k$ and $z \in L$ with $z \not\leq c_1(\xi)$; we need to show that $z \notin M(\xi)$. If $k = 2$, then $z \notin M(\xi)$ follows from Lemma 3.1. From now on we will assume that $k \geq 3$. With the notation of (3.1), $|\xi_{\mathrm{P}}| \leq |\xi_{\mathrm{B}}|$ implies $z \notin M(\xi)$ by Lemma 3.2. Now suppose that $|\xi_{\mathrm{P}}| > |\xi_{\mathrm{B}}|$. Consider the set $T = \{z \vee x_i : i \in \xi_{\mathrm{P}}\}$. Let $z \vee x_i, z \vee x_j \in T$. Breadth 2 implies that $(z \vee x_i) \vee (z \vee x_j) = z \vee x_i \vee x_j \in \{x_i \vee x_j, z \vee x_i, z \vee x_j\}$. Note that $z \vee x_i \vee x_j = x_i \vee x_j$ would imply that $z < x_i \vee x_j \leq c_1(\xi)$, a contradiction. So $(z \vee x_i) \vee (z \vee x_j) \in \{z \vee x_i, z \vee x_j\}$. Thus T is a chain; let $z \vee x_j$ be its least element.

We claim that for each $x_i \in \{\xi\}$ with $x_i \parallel z$ (that is, for each $i \in \xi_{\mathrm{P}}$),

$$d(z \vee x_j, x_i) \leq d(z, x_i) - d(z, z \vee x_j). \tag{4.1}$$

To see this consider that for each $i \in \xi_{\mathrm{P}}$ we have that

$$\begin{aligned} d(z, x_i) &\overset{(1.8)}{=} d(z, z \vee x_i) + d(z \vee x_i, x_i) \\ &\overset{(1.9)}{=} d(z, z \vee x_j) + d(z \vee x_j, z \vee x_i) + d(z \vee x_i, x_i). \end{aligned}$$

Hence $d(z, x_i) - d(z, z \vee x_j) = d(z \vee x_j, z \vee x_i) + d(z \vee x_i, x_i)$, which implies (4.1) by the triangle inequality. Further, for each $x_i \in \{\xi\}$ with $x_i < z$ (that is, for $i \in \xi_{\mathrm{B}}$),

$$d(z \vee x_j, x_i) \overset{(1.9)}{=} d(z, x_i) + d(z, z \vee x_j) \tag{4.2}$$

since $x_i < z < z \vee x_j$. Armed with (4.1) and (4.2), we have that

$$\begin{aligned} r(z \vee x_j, \xi) &= \sum_{i \in \xi_{\mathrm{P}}} d(z \vee x_j, x_i) + \sum_{i \in \xi_{\mathrm{B}}} d(z \vee x_j, x_i) \\ &\leq \sum_{i \in \xi_{\mathrm{P}}} d(z, x_i) - |\xi_{\mathrm{P}}| \cdot d(z, z \vee x_j) + \\ &\quad \sum_{i \in \xi_{\mathrm{B}}} d(z, x_i) + |\xi_{\mathrm{B}}| \cdot d(z, z \vee x_j) \\ &= r(z, \xi) - d(z, z \vee x_j) \cdot (|\xi_{\mathrm{P}}| - |\xi_{\mathrm{B}}|) \\ &< r(z, \xi) \qquad (\text{since } d(z, z \vee x_j) > 0 \text{ and } |\xi_{\mathrm{P}}| > |\xi_{\mathrm{B}}|). \end{aligned}$$

Hence $r(z \vee x_j, \xi) < r(z, \xi)$, and so $z \notin M(\xi)$. This proves part (i).

Part (ii) of the theorem follows from Proposition 2.3 or from Remark 2.4.

Next, to prove part (iii), assume that $L := L_1 \times \cdots \times L_t$ such that L_i is a lattice of finite length satisfying the c_1-median property for $i = 1, \dots, t$. Clearly, we can assume that $t = 2$ since then the case $t > 2$ follows by a trivial induction. So, $L = L_1 \times L_2$. We can assume that none of L_1 and L_2 is a singleton. We claim that for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in L,

$$d(x, y) = d(x_1, y_1) + d(x_2, y_2). \tag{4.3}$$

To prove this, let $n := d(x_1, y_1)$ and $m := d(x_2, y_2)$. The *neighboring relation* "$\prec$" $\cup$ "$\succ$", which means connectivity by an edge in the Hasse diagram, will be denoted by $\multimap\!\!\circ$. By the definition of our distance function d, there are sequences $x_1 = a_0, a_1, \dots, a_n = y_1$ in L_1 and $x_2 = b_0, b_1, \dots, b_m = y_2$ in L_2 such that $a_i \circ\!\!-\!\!\circ_{L_1} a_{i+1}$ for all $i < n$ and $b_j \circ\!\!-\!\!\circ_{L_2} b_{j+1}$ for all $j < m$. Since the pair of any two consecutive members of the sequence $x = (x_1, x_2) = (a_0, b_0), (a_1, b_0), \dots (a_n, b_0), (a_n, b_1), \dots, (a_n, b_m) = (y_1, y_2) = y$ belongs to $\circ\!\!-\!\!\circ$, we obtain that $d(x, y) \le m + n = d(x_1, y_1) + d(x_2, y_2)$. Conversely, let $x = (x_1, x_2) = (u_0, v_0), (u_1, v_1), \dots, (u_s, v_s) = (y_1, y_2) = y$ be a sequence in L such that the pairs of its consecutive members belong to $\circ\!\!-\!\!\circ$. Let

$$A := \{i : 0 \le i < s,\ u_i \circ\!\!-\!\!\circ_{L_1} u_{i+1},\ v_i = v_{i+1}\} \text{ and}$$
$$B := \{i : 0 \le i < s,\ v_i \circ\!\!-\!\!\circ_{L_2} v_{i+1},\ u_i = u_{i+1}\}.$$

It follows from (2.4) that $\{1, 2, \dots, s\}$ is the disjoint union of A and B. In particular, $|A| + |B| = s$. Observe that $\{u_i : i \in A\}$ is a sequence of $\circ\!\!-\!\!\circ_{L_1}$-neighboring elements from x_1 to y_1; for example, if $s = 7$ and $A = \{2, 4, 5\}$, then this sequence is $x_1 = u_0 = u_1 = u_2 \circ\!\!-\!\!\circ u_3 = u_4 \circ\!\!-\!\!\circ u_5 \circ\!\!-\!\!\circ u_6 = u_7 = y_1$. Hence, $n = d(x_1, y_1) \le |A|$. Similarly, $m = d(x_2, y_2) \le |B|$. Thus $s = |A| + |B| \ge d(x_1, y_1) + d(x_2, y_2)$, and we conclude that $d(x, y) \ge d(x_1, y_1) + d(x_2, y_2)$, proving (4.3).

Next, for an arbitrary profile $\xi = (x(1), \dots, x(k)) \in L^k$ and $i \in \{1, 2\}$, we let $\xi_i := (x(1)_i, \dots, x(k)_i) \in L_i^k$. For every $y \in L$, (4.3) gives that

$$r(y, \xi) = r(y_1, \xi_1) + r(y_2, \xi_2). \tag{4.4}$$

Now assume that $y \in M(\xi)$, that is, $r(y, \xi)$ is minimal for *this* ξ. Let $i \in \{1, 2\}$. If $r(y_1, \xi_1)$ was not minimal for ξ_1, then we could pick an element $y_1' \in L_1$ with $r(y_1', \xi_1) < r(y_1, \xi_1)$, we could take $\widehat{y} := (y_1', y_2)$ in L, and we would have $r(\widehat{y}, \xi) < r(y, \xi)$ by (4.4), contradicting the minimality of $r(y, \xi)$. Hence, $r(y_1, \xi_1)$ is minimal and $y_1 \in M(\xi_1)$. Since the indices 1 and 2 play a symmetric role, we obtain in the same way that $y_2 \in M(\xi_2)$. Since L_i satisfies the c_1-median property for $i \in \{1, 2\}$, we obtain that $y_i \le c_1(\xi_i) = x(1)_i \vee \cdots \vee x(k)_i$. Consequently, $y \le x(1) \vee \cdots \vee (k)$, which proves that L satisfies the c_1-median property. The assertion on $\mathrm{br}(L)$ is Lemma 2.2. Finally, (2.6) completes the proof of Theorem 4.1. $\square$

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References

[1] Bandelt, H.-J., Barthélemy, J.P.: Medians in median graphs. Discrete Applied Math. **8**, 131–142 (1984)

[2] Barbut, M.: Médiane, distributivité, éloignements. Centre de Mathématique, Sociale, Paris, 1961, Math. Sci. Hum. **70**, 5–31 (1980)
[3] Barthélemy, J.P.: Trois propriétés des médianes dans une treillis modulaire. Math. Sci. Hum. **75**, 83-91 (1981)
[4] Czédli, G., Grätzer, G.: Planar Semimodular Lattices: Structure and Diagrams. In: Grätzer, G., Wehrung, F. (eds.) Lattice Theory: Special Topics and Applications, pp. 91–130, Birkhäuser, Cham (2014)
[5] Czédli, G., Powers, R.C., White, J.M.: Planar graded lattices and the c_1-median property. Order **33**, 365–369 (2016)
[6] Czédli, G., Walendziak, A.: Subdirect representation and semimodularity of weak congruence lattices. Algebra Universalis **44**, 371–373 (2000)
[7] Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, 2nd edition. Cambridge University Press, United Kingdom (2002)
[8] Day, W.H.E., McMorris, F.R.: Axiomatic consensus theory in group choice and biomathematics. SIAM Frontiers of Applied Mathematics, vol. 29, SIAM, Philadelphia, PA (2003)
[9] Grätzer, G.: The Congruences of a Finite Lattice, A Proof-by-Picture Approach, 2nd edn. Birkhäuser, Basel (2016)
[10] Grätzer, G.: Lattice Theory: Foundation. Birkhäuser/Springer. Basel, 2011
[11] Grätzer, G., Knapp, E.: Notes on planar semimodular lattices I. Construction. Acta Sci. Math. (Szeged) **73**, 445–462 (2007)
[12] Kelly, D., Rival, I.: Planar lattices. Canad. J. Math. 27, 636–665 (1975)
[13] Li, J., Boukaabar, K.: Singular points and an upper bound of medians in upper semimodular lattices. Order **17**, 287-299 (2000)
[14] Leclerc, B.: Medians and majorities in semimodular lattices. SIAM J. Disc. Math. **3**, 266–276 (1990)
[15] Monjardet, B.: Théorie et applications de la médiane dans les treillis distributifs finis. Annals Discrete Math. **9**, 87–91 (1980)
[16] Nation, J. B.: Notes on Lattice Theory. `www.math.hawaii.edu/~jb/books.html`
[17] Powers, R.C.: Medians and majorities in semimodular posets. Discrete Applied Math. **127**, 325–336 (2003)
[18] Rival, I.: Combinatorial inequalities for semimodular lattices of breadth two. Algebra Universalis **6**, 303–311 (1976)
[19] Šešelja, B., Tepavčevič, A.: Special elements of the lattice and lattice identities. Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **20**, 21–29 (1990)
[20] Stern, M.: Semimodular Lattices: Theory and Applications. Encyclopedia of Mathematics and its Applications **73**, Cambridge University Press (2009)
[21] White, J.M.: Upper semimodular lattices and the c_1-median property. Ph.D. Thesis, University of Louisville, Louisville (2007)

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