$(1+1+2)$-GENERATED LATTICES OF QUASIORDERS

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ABSTRACT. A lattice is $(1+1+2)$-generated if it has a four-element generating set such that exactly two of the four generators are comparable. We prove that the lattice $\mathrm{Quo}(n)$ of all quasiorders (also known as preorders) of an n-element set is $(1+1+2)$-generated for $n=3$ (trivially), $n=6$ (when $\mathrm{Quo}(6)$ consists of 209 527 elements), $n=11$, and for every natural number $n \geq 13$. In 2017, the second author and J. Kulin proved that $\mathrm{Quo}(n)$ is $(1+1+2)$-generated if either n is odd and at least 13 or n is even and at least 56. Compared to the 2017 result, this paper presents twenty-four new numbers n such that $\mathrm{Quo}(n)$ is $(1+1+2)$-generated. Except for $\mathrm{Quo}(6)$, an extension of Zádori's method is used.

1. Introduction

Postponing the basic but well-known definitions to Subsection 1.2, we are going to prove that for $n \in \{3, 6, 11\}$ and also for any natural number $n \geq 13$, the lattice $\mathrm{Quo}(n)$ of quasiorders of an n-element set has a four-element generating set of order type $1+1+2$. Shortly saying, if $n \in \{3, 6, 11\} \cup \{n \in \mathbb{N}^+ : n \geq 13\}$, then $\mathrm{Quo}(n)$ is $(1+1+2)$-generated.

1.1. **Outline.** Subsection 1.2 of the present section contains the basic concepts used in the paper. Subsection 1.3 gives a short historical survey. Subsection 1.4 is a comment on the joint authorship. Sections 2, the longest section, proves that $\mathrm{Quo}(n)$ is $(1+1+2)$-generated for $n \in \{3, 6\}$. Finally, Section 3 proves the same for $n \in \{11\} \cup \{n \in \mathbb{N}^+ : n \geq 13\}$. At the end of Section 3, Remark 3.4 summarizes which $\mathrm{Quo}(n)$ are known to be $(1+1+2)$-generated and which are not.

1.2. **Basic concepts.** Given a set A, a relation $\rho \subseteq A^2$ is a *quasiorder* (also known as a *preorder*) if ρ is reflexive and transitive. With respect to set inclusion, the set of all quasiorders of A form a lattice $\mathrm{Quo}(A) = \langle \mathrm{Quo}(A), \subseteq \rangle$, the *quasiorder lattice* of A. The meet and the join of two elements in this lattice are the intersection and the transitive closure of the union of the two elements, respectively. Symmetric quasiorders are *equivalences* (also known as *equivalence relations*). The equivalences of A also form a lattice, the *equivalence lattice* $\mathrm{Equ}(A)$ of A, which is a sublattice of $\mathrm{Quo}(A)$. Since we are only interested in these lattices up to isomorphism, we will often write $\mathrm{Equ}(|A|)$ and $\mathrm{Quo}(|A|)$ instead of $\mathrm{Equ}(A)$ and $\mathrm{Quo}(A)$, respectively.

A four-element subset X of a poset (partially ordered set) Y is a $(1+1+2)$-*subset* of Y if exactly two elements of X are comparable. A subset X of a lattice L is a

Date: May 3, 2021.

2010 *Mathematics Subject Classification.* 06B99.

Key words and phrases. Quasiorder lattice, lattice of preorders, minimum-sized generating set, four-generated lattice, $(1+1+2)$-generated lattice, Zádori's method.

This research of the second author was supported by the National Research, Development and Innovation Fund of Hungary under funding scheme K 134851.

n	1	2	3	4	5	6	7
$\vert\mathrm{Equ}(n)\vert$	1	2	5	15	52	203	877
$\vert\mathrm{Quo}(n)\vert$	1	4	29	355	6 942	209 527	9 535 241

TABLE 1. $|\mathrm{Equ}(n)|$ and $|\mathrm{Quo}(n)|$ for $n \in \{1, 2, \dots, 7\}$

$(1+1+2)$-*generating set* of L if X is a $(1+1+2)$-subset of L that generates L. If a lattice L has a $(1+1+2)$-generating set, then we say that L is $(1+1+2)$-*generated*. Lattices having a four-element generating set are said to be *four-generated*.

1.3. **Earlier results that motivate the present paper.** In the seventies, Strietz [13] and [14] proved that $\mathrm{Equ}(n)$ is four-generated for $3 \leq n \in \mathbb{N}^+$ and it is (1+1+2)-generated for $10 \leq n \in \mathbb{N}^+$. In 1983, Zádori [16] gave an entirely new method to find four-element generating sets of $\mathrm{Equ}(n)$ and extended Strietz's result by proving that $\mathrm{Equ}(n)$ is $(1 + 1 + 2)$-generated even for $7 \leq n \in \mathbb{N}^+$.

Except for a $(1+1+2)$-generating set of $\mathrm{Equ}(6)$ given by Czédli and Oluoch [9], Zádori's method was the basis of all the more involved methods that were used to find small generating sets of $\mathrm{Equ}(A)$ and $\mathrm{Quo}(A)$ in the last three and a half decades; see Chajda and Czédli [1], Czédli [2], [3], and [4], Czédli [6], Czédli and Kulin [8], and Takách [15] . Even the methods used by Dolgos [10] and Kulin [11] show lots of similarity with Zádori's method.

Four-generated quasiorder lattices were first given in Czédli [6]. Not much later, Czédli and Kulin [8] proved even more: for an odd natural number $n \geq 13$ and also for an even number $n \geq 56$, $\mathrm{Quo}(n)$ is $(1 + 1 + 2)$-generated. (Generating sets of infinite *complete* quasiorder lattices have also been considered in [8] and in some of the previously mentioned papers, but these details are not relevant here.) Compared to Czédli and Kulin [8], the construction for a large n in this paper is simpler (even for all $n \geq 13$ odd an $n \geq 56$ even), and we give twenty-four new values of n such that $\mathrm{Quo}(n)$ is $(1 + 1 + 2)$-generated; see Remark 3.4.

While the argument showing that $\mathrm{Quo}(3)$ is (1+1+2)-generated is almost trivial, see Corollary 2.5, the case of $\mathrm{Quo}(6)$ is different. The analogous problem for $\mathrm{Equ}(6)$ was raised by Zádori [16], and it took thirty-seven years to prove that $\mathrm{Equ}(6)$ is $(1 + 1 + 2)$-generated; see Czédli and Oluoch [9]. These thirty-seven years and Table 1 explain that the lion's share of the paper is Section 2, where we prove that $\mathrm{Quo}(6)$ is $(1 + 1 + 2)$-generated.

1.4. **Joint authorship.** Sections 1 and 2 are joint work of the two authors. The contribution of the first author to Section 2 is about sixty percent. Section 3 is due to the second author.

2. A $(1+1+2)$-GENERATING SET OF $\mathrm{Quo}(6)$

For a set A and $x, y \in A$, we let $\Delta = \Delta_A := \{\langle x, x\rangle : x \in A\} \in \mathrm{Quo}(A)$,

$$q(x, y) := \{\langle x, y\rangle\} \cup \Delta \quad\text{and}\quad e(x, y) = e(y, x) := \{\langle x, y\rangle, \langle y, x\rangle\} \cup \Delta. \tag{2.1}$$

The atoms of $\mathrm{Quo}(A)$ and those of $\mathrm{Equ}(A)$ are exactly the $q(x, y)$ and the $e(x, y)$ with $x \neq y \in A$. These two lattices are *atomistic*, that is, for every $\rho \in \mathrm{Quo}(A)$

and $\theta \in \mathrm{Equ}(A)$,

$$\rho = \bigvee\{q(x,y) : \langle x,y\rangle \in \rho\} \quad \text{and} \quad \theta = \bigvee\{e(x,y) : \langle x,y\rangle \in \theta\}. \tag{2.2}$$

Next, let $A = \{a,b,c,d,f,g\}$. We define the following quasiorders of A:

$$\begin{aligned} \alpha &:= e(d,f) \vee e(f,g), & \beta &:= \alpha \vee e(b,c) \vee q(b,a) \\ \gamma &:= e(a,b) \vee e(a,d) \vee e(c,f), & \delta &:= e(b,c) \vee e(c,g) \vee e(a,f). \end{aligned} \tag{2.3}$$

Remark 2.1. We know from Czédli and Oluoch [9] that $\{\alpha, \beta \vee q(a,b), \gamma, \delta\}$ is a $(1+1+2)$-generating set of $\mathrm{Equ}(6)$; see Proposition 2.1 and Figure 1 in [9] with $\langle u_1,u_2,u_3,u_4,u_5,u_6\rangle := \langle b,a,c,d,f,g\rangle$.

While only six equations were necessary in [9] to prove this remark above, we need twenty-five equations, (2.8)–(2.32), to prove the following theorem.

Theorem 2.2. *The quasiorder lattice* $\mathrm{Quo}(6)$ *is* $(1+1+2)$*-generated. The set* $\{\alpha,\beta,\gamma,\delta\}$*, see* (2.3)*, is a* $(1+1+2)$*-generating set of* $\mathrm{Quo}(6) = \mathrm{Quo}(\{a,b,c,d,f,g\})$.

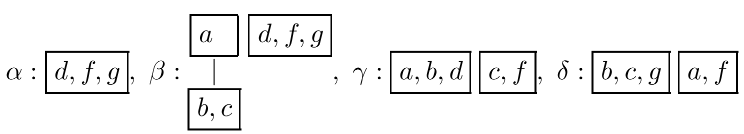
Proof. For $\rho \in \mathrm{Quo}(A)$, let $\Theta(\rho) := \rho \cap \rho^{-1} = \{\langle x,y\rangle : \langle x,y\rangle \in \rho \text{ and } \langle y,x\rangle \in \rho\} \in \mathrm{Equ}(A)$. On the quotient set $A/\Theta(\rho)$, we define a relation $\rho/\Theta(\rho)$ as follows: for $\Theta(\rho)$-blocks $x/\Theta(\rho)$ and $y/\Theta(\rho)$ in $A/\Theta(\rho)$, we let

$$\langle x/\Theta(\rho), y/\Theta(\rho)\rangle \in \rho/\Theta(\rho) \overset{\text{def}}{\iff} \langle x,y\rangle \in \rho. \tag{2.4}$$

We know from the folklore that $A/\Theta(\rho) = \langle A/\Theta(\rho), \rho/\Theta(\rho)\rangle$ is a poset. For several choices of ρ, we will frequently draw the Hasse diagram of this poset in order to give a visual description of ρ. In such a diagram, the $\Theta(\rho)$-blocks are indicated by rectangles. However, we adopt the following convention:

$$\begin{array}{l} \text{if } (\forall y \in A)\big(\{\langle x,y\rangle,\langle y,x\rangle\} \cap \rho \neq \emptyset \implies x = y\big), \\ \text{then the singleton } \Theta(\rho)\text{-block } \{x\} \text{ is omitted from} \\ \text{the Hasse diagram of } A/\Theta(\rho). \end{array} \tag{2.5}$$

A diagram reduced in the sense of (2.5) still determines ρ by (2.4). For example, the quasiorders defined in (2.3) are visualized by diagrams as follows.



(2.6)

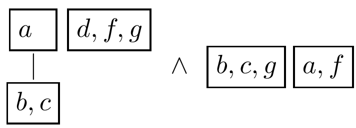
The following observation is quite easy to prove.

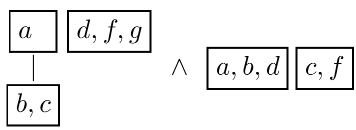
Observation 2.3 (Disjoint Paths Principle)**.** For $k,s \in \mathbb{N}^+$ and a set B, let x,y, $u_0 = x, u_1, \dots, u_{k-1}, u_k = y$, $v_0 = x, v_1, \dots, v_{s-1}, v_s = y$ be elements of B such that $\{u_1,\dots,u_{k-1}\} \cap \{v_1,\dots,v_{s-1}\} = \emptyset$, $|\{u_1,\dots,u_{k-1}\}| = k-1$, and $|\{v_1,\dots,v_{s-1}\}| = s-1$. For $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,s\}$, let $p_i \in \{e,q\}$ and $r_j \in \{e,q\}$; see (2.1). Assume that there is an $i' \in \{1,\dots,k\}$ such that $p_{i'} = q$ or there is a $j' \in \{1,\dots,s\}$ such that $r_{j'} = q$. Then

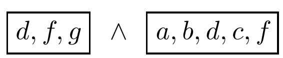
$$q(x,y) = \Big(\bigvee_{i=1}^{k} p(u_{i-1},u_i)\Big) \wedge \Big(\bigvee_{j=1}^{s} r(v_{j-1},v_j)\Big). \tag{2.7}$$

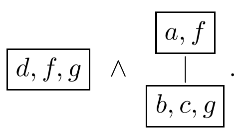
Similar observations have previously been formulated in Czédli [2], [6, Lemma 2.1], Czédli and Kulin [4, Lemma 2.5], and Kulin [11, Lemma 2.2], but Observation 2.3 is slightly stronger than its precursors. To prove it, let ρ denote the quasiorder given on the right of the equality sign in (2.7). Since $q(x, y) \le \rho \le \big(\bigvee_{i=1}^{k} e(u_{i-1}, u_i)\big) \wedge \big(\bigvee_{j=1}^{s} e(v_{j-1}, v_j)\big) = e(x, y)$ and $\langle y, x\rangle \notin \rho$ by the existence of i' or j' if $x \neq y$, we obtain (2.7) and the validity of Observation 2.3. Note that, for brevity, we will often reference (2.7) rather than Observation 2.3.

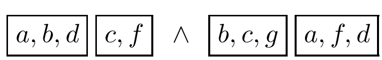
Next, resuming the proof of Theorem 2.2, let S denote the sublattice generated by $\{\alpha, \beta, \gamma, \delta\}$ in $\mathrm{Quo}(6) = \mathrm{Quo}(\{a, b, c, d, f, g\})$. To see that (2.8)–(2.32) below give quasiorders belonging to S, we are going to reference the relevant earlier members of S except possibly (2.6). The equalities in (2.8)–(2.32) will follow either from (2.7) or by using the diagrams of the meetands.

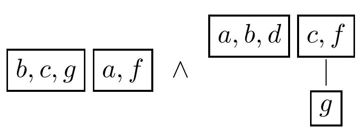
$e(b, c) = \beta \wedge \delta$ by (2.6);  (2.8)

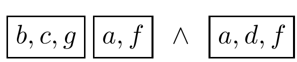
$q(b, a) = \beta \wedge \gamma$ by (2.6);  (2.9)

$e(d, f) = \alpha \wedge (\gamma \vee e(b, c))$ by (2.6) and (2.8);  (2.10)

$q(g, f) = \alpha \wedge (\delta \vee q(b, a))$ by (2.6) and (2.9);  (2.11)

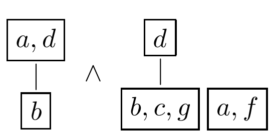
$e(a, d) = \gamma \wedge (e(d, f) \vee \delta)$ by (2.6) and (2.10);  (2.12)

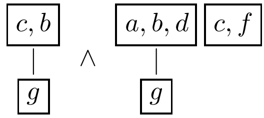
$q(g, c) = \delta \wedge (q(g, f) \vee \gamma)$ by (2.6) and (2.11);  (2.13)

$e(a, f) = \delta \wedge (e(a, d) \vee e(d, f))$ by (2.6), (2.12), and (2.10);  (2.14)

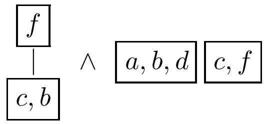
$q(g, a) = (q(g, f) \vee e(f, a)) \wedge (q(g, c) \vee e(c, b) \vee q(b, a))$ by (2.7), (2.11), (2.14), (2.13), (2.8), and (2.9). (2.15)

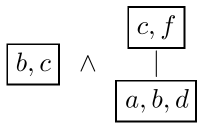
$q(g, d) = (q(g, f) \vee e(f, d)) \wedge (q(g, a) \vee e(a, d))$ by (2.7), (2.11), (2.10), (2.15), and (2.12). (2.16)

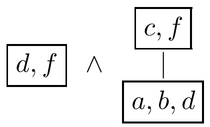
$q(b, d) = (q(b, a) \vee e(a, d)) \wedge (\delta \vee q(g, d))$ by (2.9), (2.12), and (2.16);  (2.17)

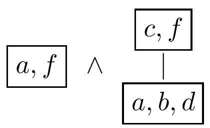
$q(g,b) = (q(g,c) \vee e(c,b)) \wedge (q(g,d) \vee \gamma)$ by (2.13), (2.8), and (2.16);  (2.18)

$q(b,f) = (q(b,a) \vee e(a,f)) \wedge (q(b,d) \vee e(d,f))$ by (2.7), (2.9), (2.14), (2.17), and (2.10). (2.19)

$q(c,f) = (e(c,b) \vee q(b,f)) \wedge \gamma$ by (2.8) and (2.19);  (2.20)

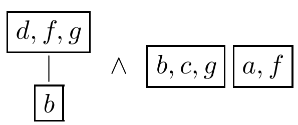
$q(b,c) = e(b,c) \wedge (q(b,f) \vee \gamma)$ by (2.8) and (2.19);  (2.21)

$q(d,f) = e(d,f) \wedge (q(b,f) \vee \gamma)$ by (2.10) and (2.19);  (2.22)

$q(a,f) = e(a,f) \wedge (q(b,f) \vee \gamma)$ by (2.14) and (2.19);  (2.23)

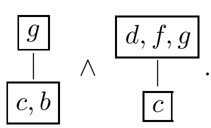
$q(d,a) = e(d,a) \wedge (q(d,f) \vee e(f,a))$ by (2.7), (2.12), (2.22), and (2.14). (2.24)

$q(f,a) = e(f,a) \wedge (e(f,d) \vee q(d,a))$ by (2.7), (2.14), (2.10), and (2.24). (2.25)

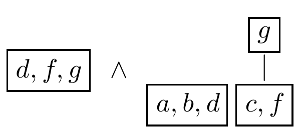
$q(b,g) = (q(b,f) \vee \alpha) \wedge \delta$ by (2.19);  (2.26)

$q(a,d) = e(a,d) \wedge (q(a,f) \vee e(f,d))$ by (2.7), (2.12), (2.23), and (2.10). (2.27)

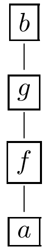
$q(f,d) = e(f,d) \wedge (q(f,a) \vee q(a,d))$ by (2.7), (2.10), (2.25), and (2.27). (2.28)

$q(c,g) = (e(c,b) \vee q(b,g)) \wedge (q(c,f) \vee \alpha)$ by (2.8), (2.26), and (2.20);  (2.29)

$q(c,b) = (q(c,g) \vee q(g,b)) \wedge e(c,b)$ by (2.7), (2.29), (2.18), and (2.8). (2.30)

$q(f,g) = \alpha \wedge (\gamma \vee q(c,g))$ by (2.29);  (2.31)

$q(a, b) = (q(a, f) \vee q(f, g) \vee q(g, b)) \wedge \gamma$ by (2.23), (2.31), and (2.18);



$\wedge$ $\boxed{a, b, d}\boxed{c, f}$. (2.32)

Since the twelve atoms of $\mathrm{Quo}(6) = \mathrm{Quo}(A)$ that are indicated in Figure 1 belong to S, (2.7) yields that $q(x, y) \in S$ for all $x, y \in A$. Hence, $S = \mathrm{Quo}(A)$ by (2.2). Since $\{\alpha, \beta, \gamma, \delta\}$ is a $(1 + 1 + 2)$-subset, the proof of Theorem 2.2 is complete. □

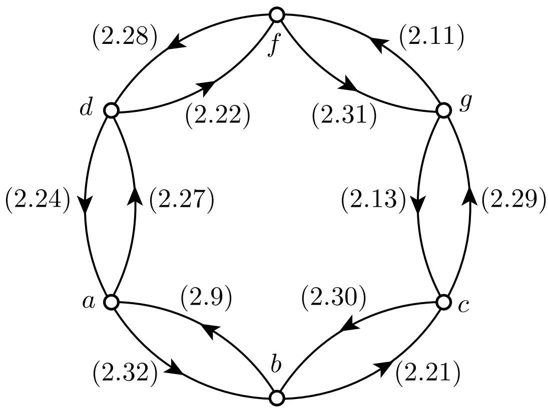


FIGURE 1. Twelve atoms of $\mathrm{Quo}(A)$

The following lemma is implicit in Kulin [11, proof of Thm. 2.1(i)]. For $a \neq b$, $\mathrm{Equ}(\{a, b\}) \cup \{q(a, b)\}$ does not generate $\mathrm{Quo}(\{a, b\})$; so $|A| \geq 3$ will be essential.

Lemma 2.4 (Kulin [11])**.** *If A is a set consisting of at least three elements and ρ belongs to $\mathrm{Quo}(A) \setminus \mathrm{Equ}(A)$, then $\mathrm{Equ}(A) \cup \{\rho\}$ generates the lattice $\mathrm{Quo}(A)$.*

Based on Observation 2.3, we give a slightly new proof for the particular case when A is finite.

Proof of Lemma 2.4. We can assume that A consists of the vertices $a_0, a_1, \dots, a_{n-1}$, listed counterclockwise, of a regular n-gon such that $\langle a_0, a_1\rangle \in \rho$ but $\langle a_1, a_0\rangle \notin \rho$. If $i, j \in \{0, \dots, n-1\}$ and $j \equiv i+1 \pmod n$, then $e(a_i, a_j)$, $q(a_i, a_j)$, and $q(a_j, a_i)$ are called an *undirected edge*, a *counterclockwise edge*, and a *clockwise edge* of the n-gon, respectively. Let S be denote sublattice of $\mathrm{Quo}(A)$ generated by $\mathrm{Equ}(A) \cup \{\rho\}$. Then all the undirected edges of the n-gon are in S. (2.7) yields that if all the counterclockwise edges and all the clockwise edges of the n-gon are in S, then all the atoms of $\mathrm{Quo}(A)$ are in S and so $S = \mathrm{Quo}(A)$ by (2.2). Also, (2.7) implies that if the counterclockwise version of an (undirected) edge belongs to S, then the clockwise versions of all other edges are in S. Similarly with "clockwise" and "counterclockwise" interchanged. Consequently, if at least one directed edge is in S, then all directed edges are in S and $S = \mathrm{Quo}(A)$. Thus, $q(a_0, a_1) = e(a_0, a_1) \wedge \rho \in S$ completes the proof. □

Corollary 2.5. Quo(3) *is* $(1+1+2)$*-generated.*

Proof. Since $\mathrm{Equ}(3) = \mathrm{Equ}(\{a,b,c\})$ is generated by the set $\{e(a,b), e(b,c), e(c,a)\}$ of its atoms, $\{q(a,b), e(a,b), e(b,c), e(c,a)\}$ is a $(1+1+2)$-generating set of the lattice $\mathrm{Quo}(3) = \mathrm{Quo}(\{a,b,c\})$ by Lemma 2.4. □

3. $(1+1+2)$-generating sets of Quo(n) for $n = 11$ and $n \geq 13$

Definition 3.1 (Zádori configuration)**.** For $2 \leq k \in \mathbb{N}^+$, let $a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_{k-1}$ be pairwise distinct elements of a finite set B. Using (2.1), let

$$\alpha = \bigvee_{i=1}^{k} e(a_{i-1}, a_i) \vee \bigvee_{i=1}^{k-1} e(b_{i-1}, b_i), \quad \beta = \bigvee_{i=0}^{k-1} e(a_i, b_i)$$
$$\gamma = \bigvee_{i=1}^{k} e(a_i, b_{i-1}), \quad \epsilon_0 = e(a_0, b_0), \quad \text{and} \quad \eta = e(a_k, b_{k-1}); \tag{3.1}$$

they are members of $\mathrm{Equ}(B)$. The system of these $2k+1$ elements and five equivalences of B is called a *Zádori configuration* of (odd) size $2k+1$ in B. The set

$$A := \{a_0, \ldots, a_k, b_0, \ldots, b_{k-1}\} \tag{3.2}$$

is the *support* of this configuration.

A Zádori configuration is easy to visualize; following Zádori's original drawing, we do this with the help of a graph in the following way. We say that a path in a graph is horizontal, is of slope 1, and is of slope -1 if all of the edges constituting the path are such. For vertices x and y in the graph,

$$\begin{aligned} \langle x, y\rangle \in \alpha &\overset{\text{def}}{\iff} \text{there is a horizontal path from } x \text{ to } y;\\ \langle x, y\rangle \in \beta &\overset{\text{def}}{\iff} \text{there is a path of slope } -1 \text{ from } x \text{ to } y;\\ \langle x, y\rangle \in \gamma &\overset{\text{def}}{\iff} \text{there is a path of slope 1 from } x \text{ to } y; \end{aligned} \tag{3.3}$$

note that a path of length 0 is simultaneously of slope 1 and of slope -1, and it is also horizontal. Also, note that (3.3) complies with (3.1).

For example, a Zádori configuration of size 11 is given in Figure 2; disregard the dashed curved edges for a while. Some of the horizontal edges are labeled by α but, to avoid crowdedness, not all. The same convention applies for edges of slope -1 and β, and edges of slope 1 and γ.

Given a Zádori configuration in B with support set A, see (3.1)–(3.2), we define

$$\mathrm{Equ}(B\rceil_A) := \{\theta \in \mathrm{Equ}(B) : \text{if } \langle x,y\rangle \in \theta \text{ and } \{x,y\} \not\subseteq A, \text{ then } x = y\}. \tag{3.4}$$

In Zádori [16], this configuration and the following lemma assumed that $B = A$. However, this assumption is not a real restriction since we have an isomorphism

$$\mathrm{Equ}(B\rceil_A) \to \mathrm{Equ}(A) \text{ defined by } \theta \mapsto \theta \cap (A \times A). \tag{3.5}$$

Hence, the following lemma follows from its original version proved in Zádori [16].

Lemma 3.2 (Zádori [16])**.** *Assume that a Zádori configuration of size* $2k+1$ *with support* A *is given in* B*; see* (3.1) *and* (3.2)*. Then* $\{\alpha, \beta, \gamma, \epsilon_0, \eta_k\}$ *generates* $\mathrm{Equ}(B\rceil_A)$.

Note that this lemma is explicitly stated in Czédli [7] and Czédli and Kulin [8]. The aim of the present section is to prove the following theorem.

Theorem 3.3. *Let $n \in \mathbb{N}^+$ be a natural number. If $n = 11$ or $n \geq 13$, then the quasiorder lattice* $\mathrm{Quo}(n)$ *is* $(1+1+2)$*-generated.*

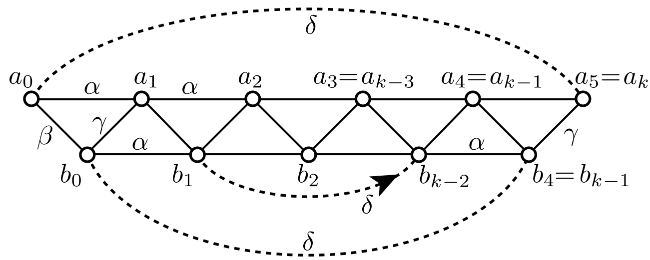


FIGURE 2. $\{\alpha, \beta, \gamma, \delta\}$ is a $(1+1+2)$-generating set of $\mathrm{Quo}(11)$

Proof. First, to prove the statement for an odd number n. Assume that $n = 2k+1 \geq 11$. Take a Zádori configuration of size $2k+1$ as described in Definition 3.1. Similarly to (3.4), we let

$$\mathrm{Quo}(B\rceil_A) := \{\rho \in \mathrm{Quo}(B) : \text{ if } \langle x, y\rangle \in \rho \text{ and } \{x, y\} \not\subseteq A, \text{ then } x = y\}. \tag{3.6}$$

Since the map

$$\mathrm{Quo}(B\rceil_A) \to \mathrm{Quo}(A) \text{ defined by } \rho \mapsto \rho \cap (A \times A) \tag{3.7}$$

is an isomorphism and $|A| = 2k + 1 = n$, it suffices to show that $\mathrm{Quo}(B\rceil_A)$ is $(1 + 1 + 2)$-generated. Define the following members of $\mathrm{Quo}(B\rceil_A)$:

$$\begin{aligned} &\delta_* := e(a_0, a_k) \vee e(b_0, b_{k-1}), \quad \delta := \delta_* \vee q(b_1, b_{k-2}), \text{ and} \\ &\delta^+ := \delta_* \vee e(b_1, b_{k-2}) = \delta \vee q(b_{k-2}, b_1). \end{aligned} \tag{3.8}$$

The quasiorders δ_* and δ^+ are equivalences but δ is not. For $n = 11$, δ is visualized by dashed curved edges in Figure 2. In addition to (3.3) and complying with (3.8), our convention for δ in Figure 2 (and later in Figure 3) is that

$$\begin{aligned} \langle x, y\rangle \in \delta \overset{\text{def}}{\iff} &\text{ there is a directed path} \\ &\text{of curved dashed edges from } x \text{ to } y; \end{aligned} \tag{3.9}$$

the edges without arrow are directed in both ways. Again, paths of length zero are permitted. By the peculiarities of δ, the path in (3.9) has to be of length 1 or 0.

Letting S denote the sublattice generated by $\{\alpha, \beta, \gamma, \delta\}$ in $\mathrm{Quo}(B\rceil_A)$, we are going to show that $S = \mathrm{Quo}(B\rceil_A)$. Observe that

$$\begin{aligned} &\text{the blocks of } (\delta^+ \vee \gamma) \text{ are } \{a_0, a_1, a_k, b_0, b_{k-1}\}, \\ &\{a_2, a_{k-1}, b_1, b_{k-2}\}, \text{ and the two-element sets} \\ &\{a_i, b_{i-1}\} \text{ such that } 3 \leq i \leq k-2. \end{aligned} \tag{3.10}$$

Hence, we obtain that $\beta \wedge (\delta^+ \vee \gamma) = e(a_0, b_0)$. Using that the lattice operations are isotone and $\langle a_0, b_0\rangle$ belongs to the equivalence $\beta \wedge (\delta_* \vee \gamma)$, it follows from

$$e(a_0, b_0) \leq \beta \wedge (\delta_* \vee \gamma) \leq \beta \wedge (\delta \vee \gamma) \leq \beta \wedge (\delta^+ \vee \gamma) = e(a_0, b_0) \tag{3.11}$$

that $\epsilon_0 := e(a_0, b_0) = \beta \wedge (\delta \vee \gamma) \in S$.

If we disregard δ (but keep δ_* and δ^+), then β and γ play a symmetric role. This corresponds to the symmetry of Figure 2 across a vertical axis, if the arrow is disregarded. Hence, $\gamma \wedge (\delta^+ \vee \beta) = e(a_k, b_{k-1})$ and $e(a_k, b_{k-1}) \le \gamma \wedge (\delta_* \vee \delta)$. Thus,

$$e(a_k, b_{k-1}) \le \gamma \wedge (\delta_* \vee \beta) \le \gamma \wedge (\delta \vee \beta) \le \gamma \wedge (\delta^+ \vee \beta) = e(a_k, b_{k-1}). \tag{3.12}$$

This implies that

$$\eta_k := e(a_k, b_{k-1}) = \gamma \wedge (\delta \vee \beta) \in S.$$

Therefore, $\mathrm{Equ}(B\rceil_A) \subseteq S$ by Lemma 3.2. Since the restriction of the isomorphism given in (3.7) to $\mathrm{Equ}(B\rceil_A)$ is the isomorphism given in (3.5), it follows from Lemma 2.4, $\mathrm{Equ}(B\rceil_A) \subseteq S$, and $\delta \in \mathrm{Quo}(B\rceil_A) \setminus \mathrm{Equ}(B\rceil_A)$ that $S = \mathrm{Quo}(B\rceil_A)$. Furthermore, $\delta < \alpha$ and $\{\alpha, \beta, \gamma, \delta\}$ is a $(1+1+2)$-subset of $\mathrm{Quo}(B\rceil_A)$. Thus,

$$\{\alpha, \beta, \gamma, \delta\} \text{ is a } (1+1+2)\text{-generating set of } \mathrm{Quo}(B\rceil_A). \tag{3.13}$$

Finally, using that $\mathrm{Quo}(B\rceil_A) \cong \mathrm{Quo}(A)$ by (3.7), or letting $B := A$ when $\mathrm{Quo}(B\rceil_A) = \mathrm{Quo}(A)$, the theorem for n odd follows from (3.13).

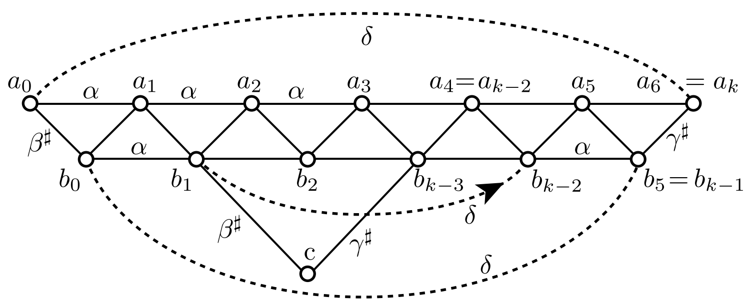


FIGURE 3. $\{\alpha, \beta^\sharp, \gamma^\sharp, \delta\}$ is a $(1+1+2)$-generating set of Quo(14)

Next, we assume that $n \ge 14$ is an even number. We let $k := (n-2)/2 \ge 6$. With this k, we use the same Zádori configuration as in the first part of the proof (where n was odd), but now we specify that $B = A \cup \{c\}$ where c is a new element outside A. So $|B| = |A| + 1 = 2k + 2 = n$. We still need α, β, γ, δ_*, δ, $\delta^+ \in \mathrm{Quo}(B\rceil_A)$ defined in (3.1) and (3.8). Furthermore, we define the following two members of $\mathrm{Quo}(B)$:

$$\beta^\sharp := \beta \vee e(b_1, c) \quad \text{and} \quad \gamma^\sharp := \gamma \vee e(b_{k-3}, c). \tag{3.14}$$

The assumption $k \ge 6$ guarantees that (3.14) makes sense. As opposed to the previously defined quasiorders of $\mathrm{Quo}(B\rceil_A)$, now $\beta^\sharp$ and $\gamma^\sharp$ are not in $\mathrm{Quo}(B\rceil_A)$. Note that the "distance" $(k-2)-1$ between the members of the two-element δ^+-block $\{b_1, b_{k-2}\}$ as well as that between the "suspension points" b_1 and b_{k-3} of c are at least 2 and the δ-block b_{k-3}/δ is a singleton; this is why we had to assume that $n \ge 14$, that is, $k \ge 6$. For the smallest value, $n = 14$, the situation is visualized by Figure 3, where the conventions formulated in (3.3) and (3.9) are valid for $\langle \alpha, \beta^\sharp, \gamma^\sharp, \delta \rangle$ instead of $\langle \alpha, \beta, \gamma, \delta \rangle$.

Let S be the sublattice generated by the (1+1+2)-subset $\{\alpha, \beta^\sharp, \gamma^\sharp, \delta\}$ of $\mathrm{Quo}(B)$. We are going to show that $S = \mathrm{Quo}(B)$. Similarly to (3.10), we observe that

the blocks of $(\delta^+ \vee \gamma^\sharp)$ are $\{a_0, a_1, a_k, b_0, b_{k-1}\}$, $\{a_2, a_{k-1}, b_1, b_{k-2}\}$, $\{a_{k-2}, b_{k-3}, c\}$, and the two-element sets $\{a_i, b_{i-1}\}$ such that $3 \le i \le k-3$. (3.15)

Hence, similarly to the three sentences containing (3.10) and (3.11), we have that $\beta^\sharp \wedge (\delta^+ \vee \gamma^\sharp) = e(a_0, b_0)$. Thus,

$$e(a_0, b_0) \le \beta^\sharp \wedge (\delta_* \vee \gamma^\sharp) \le \beta^\sharp \wedge (\delta \vee \gamma^\sharp) \le \beta^\sharp \wedge (\delta^+ \vee \gamma^\sharp) = e(a_0, b_0), \tag{3.16}$$

implying that $\epsilon_0 := e(a_0, b_0) = \beta^\sharp \wedge (\delta \vee \gamma^\sharp) \in S$. Observe that $\beta = (\epsilon_0 \vee \alpha) \wedge \beta^\sharp \in S$ and $\gamma = (\epsilon_0 \vee \alpha) \wedge \gamma^\sharp \in S$. Thus, $\{\alpha, \beta, \gamma, \delta\} \subseteq S$, and it follows from (3.13) that

$$\mathrm{Quo}(B\rceil_A) \subseteq S. \tag{3.17}$$

In particular, $e(b_1, b_{k-3}) \in S$. Hence, using that the only $(e(b_1, b_{k-3}) \vee \gamma)$-block that is not a γ-block is $\{a_2, a_{k-2}, b_1, b_{k-3}, c\}$, we obtain that $e(b_1, c) = \beta \wedge (e(b_1, b_{k-3}) \vee \gamma) \in S$. Similarly, using that $e(b_1, b_{k-3}) \in S$ by (3.17) and the only $(e(b_1, b_{k-3}) \vee \beta)$-block that is not a β-block is $\{a_1, a_{k-3}, b_1, b_{k-3}, c\}$, we obtain that $e(b_{k-3}, c) = \gamma \wedge (e(b_1, b_{k-3}) \vee \beta) \in S$. If $x \in A \setminus \{b_1, b_{k-3}\}$, then

$$e(x, c) = \big(e(x, b_1) \vee e(b_1, c)\big) \wedge \big(e(x, b_{k-3}) \vee e(b_{k-3}, c)\big) \in S \tag{3.18}$$

by (3.17), $e(b_1, c) \in S$, and $e(b_{k-3}, c) \in S$. We obtain from $e(b_1, c) \in S$, $e(b_{k-3}, c) \in S$, and (3.18) that $e(x, c) \in S$ for all $x \in A$. This fact and (3.17) yield that S contains all atoms of $\mathrm{Equ}(B)$. Hence, (2.2) gives that $\mathrm{Equ}(B) \subseteq S$. Finally, $\delta \in S \setminus \mathrm{Equ}(B)$ and Lemma 2.4 (applied to B instead of A) imply that $S = \mathrm{Quo}(B)$. So $\mathrm{Quo}(B)$ is generated by its $(1+1+2)$-subset $\{\alpha, \beta^\sharp, \gamma^\sharp, \delta\}$ and $|B| = 2k+2 = n$, completing the proof of Theorem 3.3. ☐

Remark 3.4. Now, at the end of this writing, the following is known on the existence of $(1+1+2)$-generating sets of $\mathrm{Quo}(n)$ for $n \in \mathbb{N}^+$. As new results, this paper proves that $\mathrm{Quo}(n)$ is $(1+1+2)$-generated for

$$n \in \{3, 6, 11\} \cup \{14, 16, 18, 20, 22, \dots, 50, 52, 54\}; \tag{3.19}$$

that is, for twenty-four new values of n; see Theorems 2.2 and 3.3 and Corollary 2.5. We know that, trivially, $\mathrm{Quo}(1)$ and $\mathrm{Quo}(2)$ are not (1+1+2)-generated. In addition to (3.19), $\mathrm{Quo}(n)$ is $(1+1+2)$-generated for all $n \ge 13$; see Theorem 3.3. For

$$n \in \{4, 5, 7, 8, 9, 10, 12\}, \tag{3.20}$$

we do not know whether $\mathrm{Quo}(n)$ is $(1+1+2)$-generated.

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