Tolerances as images of congruences in varieties defined by linear identities

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Abstract. An identity $s = t$ is *linear* if each variable occurs at most once in each of the terms s and t. Let T be a tolerance relation of an algebra $\mathcal{A}$ in a variety defined by a set of linear identities. We prove that there exist an algebra $\mathcal{B}$ in the same variety and a congruence $\boldsymbol{\theta}$ of $\mathcal{B}$ such that a homomorphism from $\mathcal{B}$ onto $\mathcal{A}$ maps $\boldsymbol{\theta}$ onto T.

An identity $s = t$ is *linear* if each variable occurs at most once in each of the terms s and t, see, for example, M. N. Bleicher, H. Schneider and R. L. Wilson [1, Theorem 4.19], W. Taylor [8], I. Bošnjak and R. Madarász [2], A. Pilitowska [7], and their references. In the particular case where every variable occurs exactly twice, once in s and once in t, we speak of a *balanced linear* identity, see M. V. Lawson [6]. For example, the variety of semigroups and that of commutative semigroups are defined by balanced linear identities. Binary reflexive, symmetric, and compatible relations are called *tolerances*; see I. Chajda [3]. If $\varphi\colon \mathcal{B} \to \mathcal{A}$ is a surjective homomorphism and $\boldsymbol{\theta}$ is a congruence of the algebra $\mathcal{B}$, then $\varphi(\boldsymbol{\theta}) = \{(\varphi(x), \varphi(y)) : (x, y) \in \boldsymbol{\theta}\}$ is a tolerance of $\mathcal{A}$. Each tolerance of $\mathcal{A}$ is obtained this way; this follows from our result below (applied for the variety defined by the empty set of linear identities). Sometimes, like in I. Chajda, G. Czédli, and R. Halaš [4] or G. Czédli and G. Grätzer [5], we can choose an appropriate $\mathcal{B}$ from a given variety. We have the following additional result of this kind.

Theorem. *Assume that $\boldsymbol{\mathcal{V}}$ is a variety defined by a set of linear identities, that $\mathcal{A} = (A, F) \in \boldsymbol{\mathcal{V}}$, and that T is a tolerance of $\mathcal{A}$. Then there exist an algebra $\mathcal{B} \in \boldsymbol{\mathcal{V}}$, a congruence $\boldsymbol{\theta}$ of $\mathcal{B}$, and a surjective homomorphism $\varphi\colon \mathcal{B} \to \mathcal{A}$ such that $T = \varphi(\boldsymbol{\theta})$.*

Proof. We generalize the idea of G. Czédli and G. Grätzer [5].

If $\mathcal{D}$ is an arbitrary algebra (not necessarily in $\boldsymbol{\mathcal{V}}$), then the *complex algebra* $\mathcal{C}$ of $\mathcal{D}$, in other words the *algebra of complexes of* $\mathcal{D}$, has the underlying set $\{X \subseteq D : X \neq \varnothing\}$, and for each basic operation f of $\mathcal{D}$, the corresponding operation of $\mathcal{C}$ is defined by

$$f(X_1, \dots, X_n) = \{f(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\},$$

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where n is the arity of f. If s is a linear term, which means that each variable occurs in s at most once, then it can be shown that

$$s(X_1, \dots, X_n) = \{s(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}$$

holds for arbitrary $X_i \in C$ (but this does not hold for arbitrary terms in general). This implies, as proved in [1] and [8], that if a variety is defined by linear identities, then it contains the complex algebra of each of its members.

Next, let E denote the set $\{X \subseteq A : X^2 \subseteq T \text{ and } X \neq \varnothing\}$. Since it is clearly a subalgebra of the complex algebra of $\mathcal{A}$, the paragraph above implies that $\mathcal{E} = (E, F)$ belongs to $\boldsymbol{\mathcal{V}}$. Let $B = \{(x, Y) \in A \times E : x \in Y\}$. Then $\mathcal{B} = (B, F)$ also belongs to $\boldsymbol{\mathcal{V}}$ since it is a subalgebra of $\mathcal{A} \times \mathcal{E}$. Define $\boldsymbol{\theta} = \big\{\big((x_1, Y_1), (x_2, Y_2)\big) \in B^2 : Y_1 = Y_2\big\}$. As the kernel of the second projection from $\mathcal{B}$ to $\mathcal{E}$, it is a congruence of $\mathcal{B}$. The first projection $\varphi\colon \mathcal{B} \to \mathcal{A}$, $(x, Y) \mapsto x$, is a surjective homomorphism since, for every $x \in A$, $x = \varphi\big((x, \{x\})\big)$.

Clearly, if $\big((x_1, Y_1), (x_2, Y_2)\big) \in \boldsymbol{\theta}$, then $\{x_1, x_2\} \subseteq Y_1 = Y_2 \in E$ implies that $\big(\varphi(x_1, Y_1), \varphi(x_2, Y_2)\big) = (x_1, x_2) \in T$. Hence $\varphi(\boldsymbol{\theta}) \subseteq T$. Conversely, let $(x_1, x_2) \in T$. Then, with $Y = \{x_1, x_2\}$, we have that $(x_1, Y), (x_2, Y) \in B$, $\big((x_1, Y), (x_2, Y)\big) \in \boldsymbol{\theta}$, and $x_i = \varphi((x_i, Y))$. This implies that $(x_1, x_2) \in \varphi(\boldsymbol{\theta})$, and we conclude that $T \subseteq \varphi(\boldsymbol{\theta})$. $\square$

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