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HOW TO GENERATE THE INVOLUTION LATTICE OF QUASIORDERS?

Dedicated to E. Tamás Schmidt on his 60th birthday

I. CHAJDA and G. CZÉDLI

Abstract

Given a set A, let Quord(A) denote the set of all quasiorders (i.e., reflexive and transitive relations) on A. Equipped with meet (intersection), join (transitive hull of union) and involution ($\rho \mapsto \{\langle x,y\rangle : \langle y,x\rangle \in \rho\}$), Quord($A$) is an involution lattice. When A is infinite, Quord(A) is considered a complete involution lattice. Let $\kappa_0 = \aleph_0$, the smallest infinite cardinal, and define $\kappa_{n+1} = 2^{\kappa_n}$. It is shown that if $|A| \leqq \kappa_n$ for some integer n, then Quord(A) has a three-element generating set.

Given a set A, let Quord(A) denote the set of all quasiorders (i.e., reflexive and transitive relations) on A. Similarly, the set of equivalences on A will be denoted by Equ(A). Both Quord(A) and Equ(A) are algebraic lattices if we define meet and join as intersection and transitive hull of union, respectively. According to the following table, which was partly produced by a computer program, Equ(A) and especially Quord(A) have quite many elements:

$\lvert A\rvert$	1	2	3	4	5	6	7
$\lvert\text{Equ}(A)\rvert$	1	2	5	15	52	203	877
$\lvert\text{Quord}(A)\rvert$	1	4	29	355	6942	?	?

It was proved by Strietz [8] (cf. also Zádori [10]) that the lattice Equ(A), $4 < |A| < \infty$, has a four-element generating set, but cannot be generated by three elements.

By an *involution lattice* we mean a lattice L equipped with an additional unary operation * such that * is an involutory automorphism of the lattice reduct. I.e., $L = \langle L; \vee, \wedge, {}^*\rangle$ is an involution lattice if $\langle L; \vee, \wedge\rangle$ is a lattice and $(x \vee y)^* = x^* \vee y^*$, $(x \wedge y)^* = x^* \wedge y^*$ and $x^{**} = x$ hold for all $x, y \in L$. If the lattice reduct of L is a complete lattice, then L is called a *complete involution lattice*. The most typical example is Quord(A) where α^* for $\alpha \in$ Quord(A) is defined to be $\{\langle x,y\rangle \in A^2 : \langle y,x\rangle \in \alpha\}$. From now on, Quord($A$) will always

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be considered a complete involution lattice. Involution lattices and Quord(A) have recently been studied in [1], [2], [4], [5] and [6]. The relation between involution lattices and Quord(A) is similar to but not quite the same as that between lattices and Equ(A). E.g., while each lattice can be embedded in some Equ(A) by Whitman [9], there are involution lattices that can be embedded in no Quord(A), cf. [4].

Now for any ordinal number ν we define a cardinal number κ_ν via induction. Set $\kappa_0 = \aleph_0$, the cardinality of $\mathbf{N}_0 = \{0,1,2,3,\dots\}$. If κ_ν is defined, then let $\kappa_{\nu+1} = 2^{\kappa_\nu}$. If ν is a limit ordinal, then let κ_ν be the sum of all κ_μ, $\mu < \nu$. For example, κ_ω is the sum of all cardinals κ_n, $n \in \mathbf{N}_0$. The goal of the present paper is to prove the following

THEOREM 1. *Let A be a set with $3 \leqq |A| < \kappa_\omega$. Then* Quord($A$)*, as a complete involution lattice, has a 3-element generating set. In fact,* Quord(A) *can be generated by three partial orders.*

Before proving this theorem, some remarks are worth formulating.

If A is finite then Theorem 1 holds for Quord(A) as an involution lattice in the usual sense (when the operations are the *binary* join and meet, and the unary involution).

The proof of Theorem 1 will (more or less) give the right feeling that there are many countable ordinals $\nu > \omega$ such that Quord(A) is 3-generated for $|A| < \kappa_\nu$. But proving this stronger statement would require a much more complicated proof without proving the result for all sets A; therefore the present paper is restricted to $\nu = \omega$.

If $\{\alpha,\beta,\gamma\}$ generates Quord(A) as a (complete) involution lattice, then $\{\alpha,\beta,\gamma,\alpha^*,\beta^*,\gamma^*\}$ generates it as a complete lattice. Thus Theorem 1 offers a six-element generating set for the lattice reduct of Quord(A).

If $|A| \in \{3,4\}$, then a straightforward computer program shows that Quord(A) cannot be generated by two elements. This encourages us to conjecture that Theorem 1 is sharp in the sense that Quord(A) has no two-element generating set for $|A| \geqq 3$.

Besides the mentioned computer program, there is manual proof of the fact that no $\{\alpha,\beta\} \subseteqq$ Quord(A) generates Quord(A) for $|A| = 3$. We can list all possible $\{\alpha,\beta\}$, apart from symmetries and duality, and we can associate a nontrivial unary operation $f_{\{\alpha,\beta\}}\colon A \to A$ with $\{\alpha,\beta\}$ such that α and β are compatible with $f_{\{\alpha,\beta\}}$. Then all elements of $[\{\alpha,\beta\}]$, the involution sublattice generated by α and β, are compatible with $f_{\{\alpha,\beta\}}$. Hence $[\{\alpha,\beta\}] \neq$ Quord(A), for all members of Quord(A) (or Equ(A)) are simultaneously compatible only with trivial (unary) operations (i.e., projections and constants). The long but easy details of this argument will not be presented here.

Unfortunately, the above idea, borrowed from Zádori [10], does not seem to work for $|A| \geqq 4$. By Demetrovics and Rónyai [7], for $|A| \geqq 4$ there are $\alpha,\beta \in$ Quord(A) such that they are simultaneously compatible only with trivial $A^n \to A$ operations. At present, there is no good description of these $\{\alpha, \beta\}$. E.g., both α and β can be a three-element chain (cf. [7]), but (as it is not too hard to check) the choice $\alpha = \{\langle 1,2\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 5,4\rangle, \langle 5,6\rangle\} \cup \Delta$ and $\beta = \{\langle 1,3\rangle, \langle 1,5\rangle, \langle 1,6\rangle, \langle 2,6\rangle, \langle 4,6\rangle\} \cup \Delta$ for $A = \{1,2,3,4,5,6\}$ is also possible. (Here and in the sequel Δ stands for the diagonal relation $\{\langle x,x\rangle : x \in A\}$; since this is the smallest element of Quord(A), it will also be denoted by 0.)

PROOF of Theorem 1. For a relation $\mu \subseteq A^2$, let μ^{qo} denote the smallest quasiorder including μ, i.e., the transitive hull of $\mu \cup \Delta$. As usual, $P(X)$ will stand for the set of all subsets of X, and let $P^+(X) = P(X) \setminus \{\emptyset\}$. First we deal with the infinite case.

For each nonnegative integer n we will define an *n-scheme*

$$S_n = \langle A_n; e_n^n, e_n^{n+1}, \dots ; D_n^{(n)}, D_n^{(n+1)}, \dots ; \alpha_n \beta^{(n)}, \gamma_n\rangle$$

via induction on n. (The meaning of its components will be given soon.) This n-scheme will depend only on n. Further, associated with S_n and $U \in P(D_n^{(n)})$, we will define a (unique) *n-box*

$$B_n = B_n(U) = \langle A_n; e_n^n, e_n^{n+1}, \dots ; U; D_n^{(n)}, D_n^{(n+1)}, \dots ; \alpha_n, \beta_n, \gamma_n\rangle.$$

In danger of confusion, the more accurate notation

$$\begin{aligned} B_n(U) = \langle &A_n(U); e_n^n(U), e_n^{n+1}(U), \dots ; U; \\ &D_n^{(n)}(U), D_n^{(n+1)}(U), \dots ; \alpha_n(U), \beta_n(U), \gamma_n(U)\rangle \end{aligned}$$

will be used, even if most of the components do not depend on U. For $m < n$, we will also define the *sub-m-boxes* of $B_n(U)$ or S_n; and for $m = n$, $B_n(U)$ will be considered the only sub-n-box of itself. After the necessary definitions and preliminaries we will show that A_n is a set with power κ_n, and $\{\alpha_n, \beta_n, \gamma_n\}$, no matter which $U \in P(D_n^{(n)})$ is considered, is a generating set of Quord(A_n).

Now we define the 0-scheme S_0, cf. Figure 1. Let $A_0 = \{a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, \dots\}$. We define three partial orders on A_0:

$$\begin{aligned} \alpha_0 &= \{\langle a_i, a_j\rangle : 0 \leqq i \leqq j\} \cup \{\langle b_i, b_j\rangle : 0 \leqq j \leqq i\} \cup \{\langle c_1, c_0\rangle\} \cup \\ &\qquad \{\langle c_i, c_j\rangle : 1 \leqq i \leqq j\} \cup \{\langle d_i, d_j\rangle : 0 \leqq j \leqq i\} \cup \Delta, \\ \gamma_0 &= \{\langle b_i, a_{i+1}\rangle : 0 \leqq i\} \cup \{\langle d_i, c_{i+1}\rangle : 0 \leqq i\} \cup \Delta, \\ \beta^{(0)} &= \Big(\{\langle a_0, b_0\rangle, \langle b_0, c_0\rangle, \langle c_0, d_0\rangle, \langle c_3, d_3\rangle, \langle c_6, b_6\rangle\} \cup \\ &\qquad \{\langle b_i, a_i\rangle : 1 \leqq i\} \cup \{\langle d_i, c_i\rangle : 1 \leqq i\}\Big)^{qo}. \end{aligned}$$

These quasiorders are represented by horizontal, southwest $\to$ northeast, and (solid) vertical directed edges, respectively. For $k > 1$, $e = \langle\langle b_{9k}, c_{9k}\rangle,$

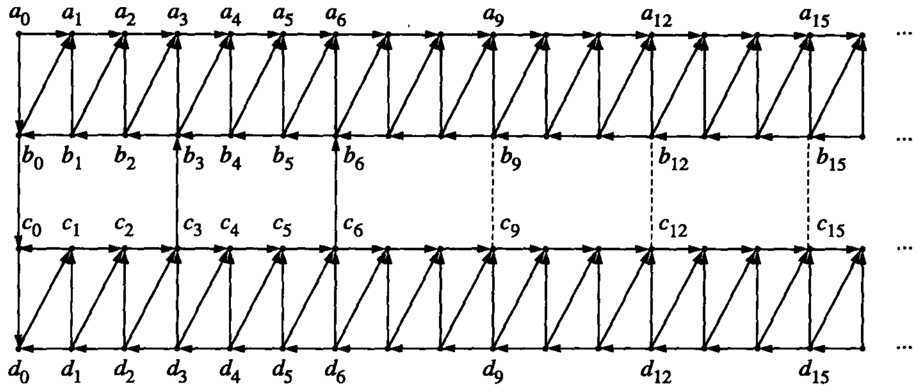

Fig. 1

$\langle b_{9k+3}, c_{9k+3}\rangle, \langle b_{9k+6}, c_{9k+6}\rangle\rangle$ will be called an *edge triplet.* (For $k=1$, this is represented by three dotted vertical lines on Figure 1.) Associated with this e we will use the notation

$$e=\langle\langle b_e, c_e\rangle, \langle b'_e, c'_e\rangle, \langle b''_e, c''_e\rangle\rangle.$$

The binary relations

$$\delta(e)=\{\langle b_e, c_e\rangle, \langle c'_e, b'_e\rangle, \langle c''_e, b''_e\rangle\} \quad \text{and}$$
$$\delta^*(e)=(\delta(e))^*$$

will have special role. (Sometimes we use the notation $\rho^*=\{\langle y,x\rangle\colon \langle x,y\rangle\in\rho\}$ even when ρ is not a quasiorder.) Let $\{D_0^{(-1)}, D_0^{(0)}, D_0^{(1)}, D_0^{(2)}, \dots\}$ be a fixed partition on the set of edge triplets of S_0 such that all the classes $D_0^{(i)}$ are infinite. Let $e_0^0, e_0^1, e_0^2, e_0^3, \dots$ be a fixed enumeration of the elements in $D_0^{(-1)}$. We have defined S_0, and clearly $|A_0|=|D_0^{(0)}|=|D_0^{(1)}|=|D_0^{(2)}|=\dots=\kappa_0$.

Now let $U\in P(D_0^{(0)})$, and define

$$\beta_0=\beta_0(U)=\Big(\beta^{(0)}\cup\bigcup_{e\in U}\delta(e)\cup\bigcup_{e\in D_0^{(0)}\setminus U}\delta^*(e)\Big)^{\mathrm{qo}}.$$

Thus we obtain the 0-box

$$B_0=B_0(U)=\langle A_0; e_0^0, e_0^1, \dots; U; D_0^{(0)}, D_0^{(1)}, \dots; \alpha_0, \beta_0, \gamma_0\rangle.$$

Now let us assume that S_n, $B_n(U)$ for $U\in P(D_n^{(n)})$ and their sub-m-boxes for $m<n$ are already defined. We may assume that $A_n(U)\cap A_n(V)=\emptyset$ for distinct $U,V\in P(D_n^{(n)})$. Let

$$A_{n+1}=\bigcup_{U\in P(D_n^{(n)})} A_n(U),$$

$$\alpha_{n+1} = \bigcup_{U \in P(D_n^{(n)})} \alpha_n(U),$$

$$\gamma_{n+1} = \bigcup_{U \in P(D_n^{(n)})} \gamma_n(U), \quad \text{and}$$

$$D_{n+1}^{(i)} = \bigcup_{U \in P(D_n^{(n)})} D_n^{(i)}(U), \quad \text{for } i \geqq n+1.$$

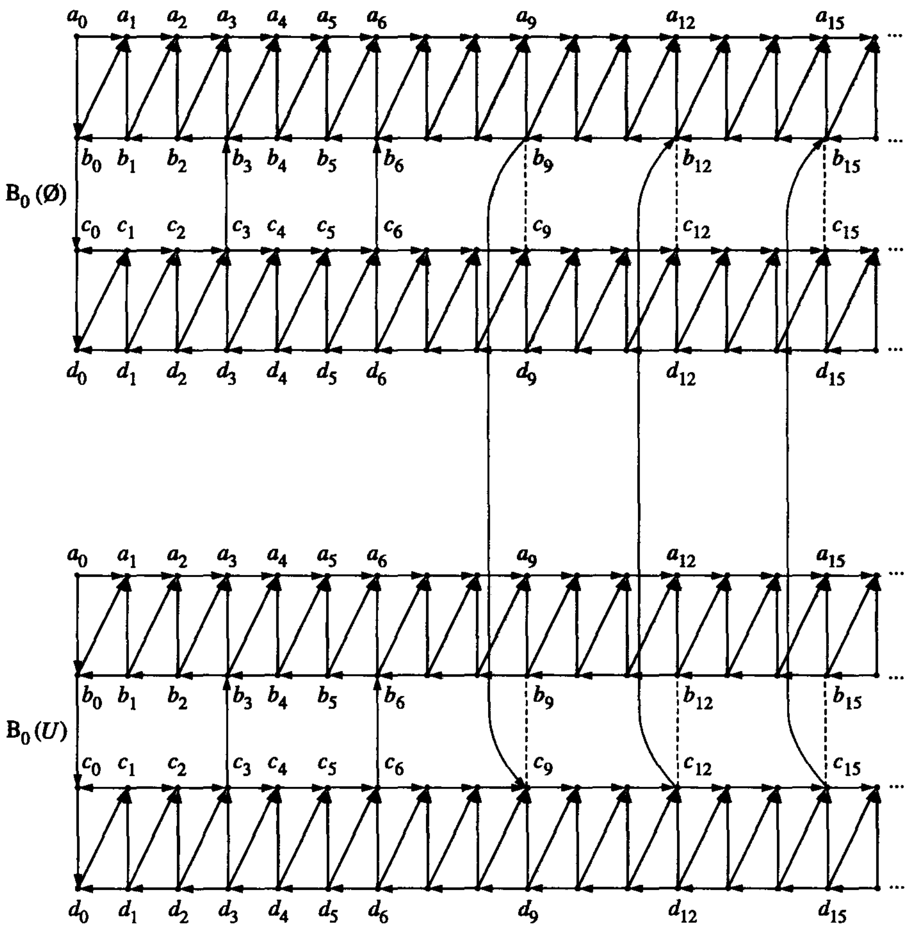

Fig. 2

(Of course, all these unions are unions of pairwise disjoint sets. For $n=0$, the situation is outlined on Figure 2, where the three dotted lines stand for the edge triplet e_0^0, and only one of $B_0(U)$, $U \neq \emptyset$, is indicated.) Define

$e^i_{n+1} = e^i_n(\emptyset)$ for $i \geqq n+1$ and

$$\varepsilon(n,\emptyset,U) = \left\{ \langle b_{e^n_n(\emptyset)}, c_{e^n_n(U)} \rangle, \langle c'_{e^n_n(U)}, b'_{e^n_n(\emptyset)} \rangle, \langle c''_{e^n_n(U)}, b''_{e^n_n(\emptyset)} \rangle \right\}$$

for $U \in P^+(D^{(n)}_n)$. Set

$$\beta^{(n+1)} = \Big(\bigcup_{U \in P(D^{(n)}_n)} \beta_n(U) \cup \bigcup_{U \in P^+(D^{(n)}_n)} \varepsilon(n,\emptyset,U) \Big)^{\mathrm{qo}}.$$

This way we have defined

$$S_{n+1} = \left\langle A_{n+1}; e^{n+1}_{n+1}, e^{n+2}_{n+1}, \dots ; D^{(n+1)}_{n+1}, D^{(n+2)}_{n+1}, \dots ; \alpha_{n+1}, \beta^{(n+1)}, \gamma_{n+1} \right\rangle.$$

Clearly, $|A_{n+1}| = |D^{(n+1)}_{n+1}| = |D^{(n+2)}_{n+1}| = \ldots = \kappa_{n+1}$. The sub-$n$-boxes of S_{n+1} are just the $B_n(U)$, $U \in P(D^{(n)}_n)$. For $m < n$, the sub-m-boxes of S_{n+1} are the sub-m-boxes of its sub-n-boxes. Now let $U \in P(D^{(n+1)}_{n+1})$, and define

$$\beta_{n+1} = \Big(\beta^{(n+1)} \cup \bigcup_{e \in U} \delta(e) \cup \bigcup_{e \in D^{(n+1)}_{n+1} \setminus U} \delta^*(e) \Big)^{\mathrm{qo}}.$$

Thus we obtain the $(n+1)$-box

$$B_{n+1} = B_{n+1}(U) = \left\langle A_{n+1}; e^{n+1}_{n+1}, e^{n+2}_{n+1}, \dots ; U; D^{(n+1)}_{n+1}, D^{(n+2)}_{n+1}, \dots ; \alpha_{n+1}, \beta_{n+1}, \gamma_{n+1} \right\rangle.$$

For $m \leqq n$, the sub-m-boxes of $B_{n+1}(U)$ are the same as that of S_{n+1}.

In order to show that $\alpha_n, \beta_n = \beta_n(U)$ and γ_n generate $\mathrm{Quord}(A_n)$ (no matter which $U \in P(D^{(n)}_n)$ is considered), we introduce certain binary terms $f^n_{p,q} = f^n_{p,q}(x,y,z)$ ($n \in \mathbf{N}_0, p, q \in A_n$). While the $f^0_{p,q}$ will be involution lattice terms in the usual sense, for $n > 0$ the $f^n_{p,q}$ will contain the infinitary join and/or meet operations as well. Instead of developing the exact definition of "terms" (like in [3, Chapter 2]) prior to their usage, we only note that all complete involution sublattices are closed with respect to the "term functions" they induce, and we will not make a distinction between two terms if they induce the same term function on each complete involution lattice. Set $f^n_{p,q} = x \wedge y \wedge z \wedge x^* \wedge y^* \wedge z^*$, and notice that $f^n_{p,p}(\alpha_n, \beta_n, \gamma_n) = 0$ in $\mathrm{Quord}(A_n)$. (This follows from $\alpha_n \wedge \alpha^*_n = \beta_n \wedge \beta^*_n = \gamma_n \wedge \gamma^*_n = 0$.) When we define the $f^n_{p,q}$ in the sequel, we implicitly always assume on $\langle n,p,q \rangle$ that neither $f^n_{p,q}$ nor $f^n_{q,p}$ has previously been defined. Further, for $p \neq q$, $f^n_{q,p} = (f^n_{p,q})^*$. (Remember, we do not make a distinction between $f^n_{p,q}$ and $(f^n_{p,q})^{**}$.) When defining our terms, we keep in mind that the final purpose is to show

$$f^n_{p,q}(\alpha_n, \beta_n, \gamma_n) = \{\langle p,q \rangle\}^{\mathrm{qo}}. \tag{1}$$

Then α_n, β_n and γ_n will evidently generate Quord(A), for any element μ of Quord(A) is the join of all $\{\langle p,q\rangle\}^{\mathrm{qo}}$ below μ. However, (1) is not appropriate to be an induction hypothesis; something stronger is necessary. For $p,q\in A_n$ and $n\leqq m$, let H be the set of all sub-n-boxes of $B_m=B_m(U)$. These sub-n-boxes are pairwise disjoint, of course. For $h\in H$, let p_h and q_h denote (the elements corresponding to) p and q in the h-th copy of A_n (i.e., in the base set of the h-th sub-n-box). Define

$$\langle p,q\rangle^{(n,m)}=\Big(\bigcup_{h\in H}\{\langle p_h,q_h\rangle\}\Big)^{\mathrm{qo}}\in \mathrm{Quord}(A_m).$$

Note that $\{\langle p,q\rangle\}^{\mathrm{qo}}=\langle p,q\rangle^{(n,n)}$ in Quord(A_n) and $\langle p,q\rangle^{(n,m)}=\Delta\cup\bigcup_{h\in H}\{\langle p_h,q_h\rangle\}$. We will define terms $f^n_{p,q}$ such that

$$f^n_{p,q}(\alpha_m,\beta_m,\gamma_m)=\langle p,q\rangle^{(n,m)}\quad\text{in Quord}(A_m) \tag{2}$$

holds for all $0\leqq n\leqq m$ and $p,q\in A_n$. Note that (2) implies (1), and therefore it implies Theorem 1 for $|A|=|A_n|=\kappa_n$.

The verification of (2) will be based on the geometric arrangement of elements in A_m. These elements are in κ_m rows and $\kappa_0=\aleph_0$ columns. The subset $\{a_0,a_1,a_2,\dots\}$, $\{b_0,b_1,b_2,\dots\}$, $\{c_0,c_1,c_2,\dots\}$, and $\{d_0,d_1,d_2,\dots\}$ of sub-0-boxes of S_m are called *rows* (a-row, b-row, c-row and d-row), while a_j, b_j, c_j and d_j of sub-0-boxes belong to the j-the *column*. For $u\in A_m$ we introduce the notation $\mathrm{col}(u)=j$ to express the fact that u is in the j-th column. For an edge-triplet e, let col(e) denote $\{\mathrm{col}(b_e),\mathrm{col}(b'_e),\mathrm{col}(b''_e)\}$. It is worth mentioning that for $\tau\in\{\beta_m,\beta^*_m\}$, $\rho\in\{\gamma_m,\gamma^*_m\}$ and $p,q\in A_m$

$$\langle p,q\rangle\in\tau\vee\rho\Longrightarrow|\mathrm{col}(p)-\mathrm{col}(q)|<3. \tag{3}$$

This explains why the "column distance" of edges in an edge triplet is chosen to be three in the construction. Some other, more or less self-explaining, terminology induced by the "geometry" of A_m will also be used. For example, α_m is *row preserving* and β_m is *column preserving*. If $\tau\in\mathrm{Quord}(A_m)$ and $X\subseteqq A_m$ has the property that $u\in X$ and $\langle u,v\rangle\in\tau$ imply $v\in X$, then X is said to be *closed with respect to* τ. E.g., columns are closed with respect to β_m and rows are closed with respect to α^*_m. If $\langle u,v\rangle\in\tau$ and $u\neq v$ imply $\mathrm{col}(u)\neq\mathrm{col}(v)$ resp. $\mathrm{col}(u)<\mathrm{col}(v)$, then τ is said to be *column changing* resp. *column increasing*. If, for some $i\neq j$, $\langle u,v\rangle\in\tau$ and $u\neq v$ imply $\mathrm{col}(u)=i$ and $\mathrm{col}(v)=j$, then we say that τ *changes the column* from i to j. Associated with a 0-box or 0-scheme we may speak of its *halves*; the a-row and b-row form the *upper half* while c-row and d-row constitute the *lower half*.

Now define

$$f^0_{a_0,b_0}=y\wedge(x\vee z^*)\quad\text{and}\quad f^0_{c_0,d_0}=y\wedge(x^*\vee z^*).$$

In order to show (2) for $f^0_{a_0,b_0}$, suppose $u,v\in A_m$ are distinct elements and $\langle u,v\rangle\in f^0_{a_0,b_0}(\alpha_m,\beta_m,\gamma_m)=\beta_m\wedge(\alpha_m\vee\gamma^*_m)$. From $\langle u,v\rangle\in\beta_m$ we conclude $\mathrm{col}(u)=\mathrm{col}(v)$, whence u and v are in distinct rows. Since $\langle u,v\rangle\in\alpha_m\vee\gamma^*_m$, u and v belong to the same sub-0-box B_0, and even to the same half of B_0. Since the γ-arrows "go up" (cf. Figure 1), either u is in the a-row and v is in the b-row or u is in the c-row and v is in the d-row of B_0. Therefore $\mathrm{col}(u)=0$, for otherwise the β-arrow would go up between the rows of u and v. Thus $\langle u,v\rangle\in\{\langle a_0,b_0\rangle,\langle c_0,d_0\rangle\}$. But $\langle c_0,d_0\rangle\notin\alpha_m\vee\gamma^*_m$, for c_0 is a maximal element with respect to α_m and it is isolated with respect to γ_m. So $\langle u,v\rangle\in\langle a_0,b_0\rangle\in\langle a_0,b_0\rangle^{(0,m)}$. The inclusion $\langle a_0,b_0\rangle^{(0,m)}\subseteq f^0_{a_0,b_0}(\alpha_m,\beta_m,\gamma_m)$ is evident, hence we have shown that (2) holds for $f^0_{a_0,b_0}$. The treatment for $f^0_{c_0,d_0}$ is very similar.

Simple considerations like the above for $f^0_{a_0,b_0}$ will not be detailed usually. Moreover, when we define a term $f^n_{p,q}$ in the sequel without further reasoning, this definition should be understood also as a statement claiming (2) for the term in question; the proof of this implicit assertion is left to the reader.

Now we assume that $f^0_{a_i,b_i}$ and $f^0_{c_i,d_i}$ satisfying (2) are already defined. Let

$$\begin{aligned}
f^0_{a_i,a_{i+1}}&=x\wedge(f^0_{a_i,b_i}\vee z),\\
f^0_{c_i,c_{i+1}}&=\begin{cases} x^*\wedge(f^0_{c_0,d_0}\vee z), & \text{if } i=0\\ x\wedge(f^0_{c_i,d_i}\vee z), & \text{if } i>0,\end{cases}\\
f^0_{b_i,a_{i+1}}&=z\wedge(f^0_{b_i,a_i}\vee f^0_{a_i,a_{i+1}}),\\
f^0_{d_i,c_{i+1}}&=z\wedge(f^0_{d_i,c_i}\vee f^0_{c_i,c_{i+1}}),\\
f^0_{b_i,b_{i+1}}&=x^*\wedge(f^0_{b_i,a_{i+1}}\vee y^*),\\
f^0_{d_i,d_{i+1}}&=x^*\wedge(f^0_{d_i,c_{i+1}}\vee y^*),\\
f^0_{a_{i+1},b_{i+1}}&=y^*\wedge(f^0_{a_{i+1},b_i}\vee f^0_{b_i,b_{i+1}}),\\
f^0_{c_{i+1},d_{i+1}}&=y^*\wedge(f^0_{c_{i+1},d_i}\vee f^0_{d_i,d_{i+1}}).
\end{aligned}$$

For example, the argument proving (2) for $f^0_{b_i,b_{i+1}}$ runs as follows. Suppose $u,v\in A_m$ are distinct elements and $\langle u,v\rangle\in f^0_{b_i,b_{i+1}}(\alpha_m,\beta_m,\gamma_m)=\alpha^*_m\wedge(\langle b_i,a_{i+1}\rangle^{(0,m)}\vee\beta^*_m)$. Since α^*_m is row preserving, u and v are in the same row (and in the same sub-0-box) but in distinct columns. There are distinct elements $w_0=u, w_1,\dots,w_t=v$ in A_m such that $\langle w_{j-1},w_j\rangle\in\langle b_i,a_{i+1}\rangle^{(0,m)}\cup\beta^*_m$ for all j. Since β^*_m is column preserving and $\langle b_i,a_{i+1}\rangle^{(0,m)}$ changes the column from i to $i+1$, there is a k such that $\langle w_{k-1},w_k\rangle\in\langle b_i,a_{i+1}\rangle^{(0,m)}$ and $\langle w_{j-1},w_j\rangle\in\beta^*_m$ for all $j\neq k$. Hence $\langle u,w_{k-1}\rangle$, $\langle w_k,v\rangle\in\beta^*_m$, $\mathrm{col}(u)=\mathrm{col}(w_{k-1})=i$ and $\mathrm{col}(v)=\mathrm{col}(w_k)=i+1$. Suppose i is not a multiple of 3 (the other case, when 3 does not divide $i+1$, is similar). The intersection of the i-th column with an arbitrary sub-0-box (and the upper half of this sub-0-box) is closed with respect to β_m^* and $\langle b_i, a_{i+1}\rangle^{(0,m)}$. Hence u, w_{k-1}, w_k and v belong to the upper half of the same sub-0-box B_0 of S_m, and $w_{k-1} = b_i$, $w_k = a_{i+1}$ in B_0. Therefore (2) for $f^0_{b_i,b_{i+1}}$ follows easily from Figure 1.

Now let $p \neq q$ belong to the same half (upper or lower) of S_0, and consider the smallest circle in the undirected variant of the graph on Figure 1 which contains p, q and consists of

(4) vertical and horizontal

edges only, and goes within the same half of S_0 that contains p and q. Let $\{p = r_0, r_1, r_2, \ldots, r_i = q, r_{i+1}, \ldots, r_{k-1}, r_k = r_0 = p\}$ be this circle (which is uniquely determined, the elements are listed anti-clockwise); the elements $r_0, r_1, \ldots, r_{k-1}$ are pairwise distinct. Define

$$f^0_{p,q} = (f^0_{r_0,r_1} \vee f^0_{r_1,r_2} \vee \ldots \vee f^0_{r_{i-1},r_i}) \wedge (f^0_{r_k,r_{k-1}} \vee f^0_{r_{k-1},r_{k-2}} \vee \ldots \vee f^0_{r_{i+1},r_i}).$$

Now we can set

$$f^0_{b_0,c_0} = y \wedge (f^0_{b_0,b_3} \vee y^* \vee f^0_{c_3,c_0}) \wedge (f^0_{b_0,b_6} \vee y^* \vee f^0_{c_6,c_0}) \quad \text{and}$$
$$f^0_{b_3,c_3} = y^* \wedge (f^0_{b_3,b_0} \vee f^0_{b_0,c_0} \vee f^0_{c_0,c_3}).$$

Now suppose that p is in the upper half and q is in the lower half of S_0, and define

$$f^0_{p,q} = (f^0_{p,b_0} \vee f^0_{b_0,c_0} \vee f^0_{c_0,q}) \wedge (f^0_{p,b_3} \vee f^0_{b_3,c_3} \vee f^0_{c_3,q}).$$

We have defined all the f^0 terms, and these terms satisfy (2).

Now let us assume that appropriate ternary terms $f^n_{p,q}$ $(p, q \in A_n)$ are already defined (and they satisfy (2)); we start defining the f^{n+1} terms.

First assume that $p, q \in A_{n+1}$ belong to the same sub-n-box $B_n(U)$ of S_{n+1} such that $\mathrm{col}(p) \neq \mathrm{col}(q)$ and neither $\mathrm{col}(p)$ nor $\mathrm{col}(q)$ is divisible by 3. Here $U \in P(D_n^{(n)})$. Let

$$f^{n+1}_{p,q} = f^n_{p,q} \wedge \bigwedge_{e \in U} (f^n_{p,b_e} \vee y \vee f^n_{c_e,q}) \wedge \bigwedge_{e \in D_n^{(n)} \setminus U} (f^n_{p,b_e} \vee y^* \vee f^n_{c_e,q}).$$

To show that this term satisfies (2), let $m \geqq n+1$, and consider an m-box B_m. By definitions and the validity of (2) for f^n-terms we obtain

$$(5) \qquad \langle p, q\rangle^{(n+1,m)} \subseteqq f^{n+1}_{p,q}(\alpha_m, \beta_m, \gamma_m) \subseteqq f^n_{p,q}(\alpha_m, \beta_m, \gamma_m) = \langle p, q\rangle^{(n,m)}.$$

To show that the first inclusion in (5) is in fact an equality, suppose there is a pair $\langle u,v\rangle \in f^{n+1}_{p,q}(\alpha_m,\beta_m,\gamma_m)\setminus\langle p,q\rangle^{(n+1,m)}$. It follows from (5) that $u=p(V)$ and $v=q(V)$ in some sub-n-box $B_n(V)$ of B_m. Here $B_n(V)$ belongs to a unique sub-$(n+1)$-box of B_m, $V\in P(D^{(n)}_n)$, and $p(V)$, $q(V)$ are the elements of $B_n(V)$ that correspond to $p,q\in B_n(U)$. From $\langle u,v\rangle\notin\langle p,q\rangle^{(n+1,m)}$ we conclude that $U\neq V$. Let e be an edge triplet in $(U\setminus V)\cup(V\setminus U)$. Since $\mathrm{col}(b_e)$ is divisible by 3, the elements u, v and b_e belong to distinct columns.

Suppose first that $e\in U\setminus V$; we claim that $\langle u,v\rangle=\langle p(V),q(V)\rangle$ does not belong to $f^n_{p,b_e}(\alpha_m,\beta_m,\gamma_m)\vee\beta_m\vee f^n_{c_e,q}(\alpha_m,\beta_m,\gamma_m)=\langle p,b_e\rangle^{(n,m)}\vee\beta_m\vee\langle c_e,q\rangle^{(n,m)}$. Indeed, let us assume the opposite. Then there is a shortest sequence (of distinct elements) $w_0=p(V)$, $w_1,w_2,\dots,w_t=q(V)$ such that $\langle w_{i-1},w_i\rangle\in\langle p,b_e\rangle^{(n,m)}\cup\beta_m\cup\langle c_e,q\rangle^{(n,m)}$ for all i. Since β_m is column preserving, $\langle p,b_e\rangle^{(n,m)}$ changes the column (only) from $\mathrm{col}(p)$ to $\mathrm{col}(b_e)=\mathrm{col}(c_e)$ and $\langle c_e,q\rangle^{(n,m)}$ changes the column from $\mathrm{col}(c_e)$ to $\mathrm{col}(q)$, all the w_i belong to the $\mathrm{col}(p)$-th, $\mathrm{col}(q)$-th and $\mathrm{col}(c_e)$-th columns. By the construction, no e^l_k has an element in these three columns, whence the intersection of these columns with $B_n(V)$ (or even with any sub-0-box) is closed with respect to β_m. Consequently, all the w_i belong to the same sub-n-box, i.e., to $B_n(V)$. Our present information on the columns $\mathrm{col}(w_i)$ imply that, within $B_n(V)$,

$$p(V)=w_0\ \langle p,b_e\rangle^{(n,m)}\ w_1=b_e(V)\ \beta_m\ w_2=c_e(V)\ \langle c_e,q\rangle^{(n,m)}\ w_3=q(V)$$

is the only possibility. But this is a contradiction, for $\langle b_e(V),c_e(V)\rangle$ is not in β_m (in fact, it is in β^*_m) by the construction. For $e\in V\setminus U$, $\langle u,v\rangle\notin f^n_{p,b_e}(\alpha_m,\beta_m,\gamma_m)\vee\beta^*_m\vee f^n_{c_e,q}(\alpha_m,\beta_m,\gamma_m)$ follows similarly. Thus (2) holds for $f^{n+1}_{p,q}$.

Now let us assume that $p,q\in A_{n+1}$, $p\neq q$, still belong to the same sub-n-box $B_n(U)$ of S_{n+1} but the previous additional assumption does not hold (i.e., $\mathrm{col}(p)=\mathrm{col}(q)$ or $3\mid\mathrm{col}(p)$ or $3\mid\mathrm{col}(q)$). Choose elements $p',p'',q',q''\in A_{n+1}$ such that p,p',p'' are in the same row, q,q',q'' are in the same (possibly another) row, none of $\mathrm{col}(p')$, $\mathrm{col}(p'')$, $\mathrm{col}(q')$, $\mathrm{col}(q'')$ is divisible by 3, $|\{\mathrm{col}(p'),\mathrm{col}(p''),\mathrm{col}(q'),\mathrm{col}(q'')\}|=4$ and $\{\mathrm{col}(p'),\mathrm{col}(p''),\mathrm{col}(q'),\mathrm{col}(q'')\}\cap\{\mathrm{col}(p),\mathrm{col}(q)\}=\emptyset$. Note that this choice can be made unique by fixing an appropriate $\mathbf{N}^2_0\to\mathbf{N}^4_0$ map and requiring $\langle\mathrm{col}(p),\mathrm{col}(q)\rangle\mapsto\langle\mathrm{col}(p'),\mathrm{col}(p''),\mathrm{col}(q'),\mathrm{col}(q'')\rangle$, but the explicit knowledge of this map is unimportant for us. Now we can define

$$f^{n+1}_{p,q}=(f^n_{p,p'}\vee f^{n+1}_{p',q'}\vee f^n_{q',q})\wedge(f^n_{p,p''}\vee f^{n+1}_{p'',q''}\vee f^n_{q'',q}).$$

This way we have defined all $f^{n+1}_{p,q}$ when p and q belong to the same sub-n-box of S_{n+1}. Now let us consider the sub-n-boxes $B_n(\emptyset)$ and $B_n(U)$ of S_{n+1}, $U\in P^+(D^{(n)}_n)$. The e^n_n edge triplets in these sub-n-boxes will be denoted by

$$e^n_n(\emptyset)=\langle\langle b_\emptyset,c_\emptyset\rangle,\langle b'_\emptyset,c'_\emptyset\rangle,\langle b''_\emptyset,c''_\emptyset\rangle\rangle\quad\text{and}\quad e^n_n(U)=\langle\langle b_U,c_U\rangle,\langle b'_U,c'_U\rangle,\langle b''_U,c''_U\rangle\rangle,$$

respectively. Let

$$f^{n+1}_{b_\emptyset,c_U}=y\wedge\left(f^{n+1}_{b_\emptyset,b'_\emptyset}\vee y^*\vee f^{n+1}_{c'_U,c_U}\right)\wedge\left(f^{n+1}_{b_\emptyset,b''_\emptyset}\vee y^*\vee f^{n+1}_{c''_U,c_U}\right).$$

To show that this term satisfies (2), suppose that B_{n+1} is a sub-$(n+1)$-box of some m-box B_m and, for distinct $u,v\in A_m$, $\langle u,v\rangle\in f^{n+1}_{b_\emptyset,c_U}(\alpha_m,\beta_m,\gamma_m)=\beta_m\wedge(\langle b_\emptyset,b'_\emptyset\rangle^{(n+1,m)}\vee\beta^*_m\vee\langle c'_U,c_U\rangle^{(n+1,m)})\wedge(\langle b_\emptyset,b''_\emptyset\rangle^{(n+1,m)}\vee\beta^*_m\vee$ $\vee\langle c''_U,c_U\rangle^{(n+1,m)})$. Since $\langle u,v\rangle\in\beta_m$, $\mathrm{col}(u)=\mathrm{col}(v)$ and $\langle u,v\rangle\notin\beta^*_m$. Hence in any sequence $w_0=u$, $w_1,\dots,w_t=v$ in A_m such that $\langle w_{i-1},w_i\rangle\in\langle b_\emptyset,b'_\emptyset\rangle^{(n+1,m)}\cup\beta^*_m\cup\langle c'_U,c_U\rangle^{(n+1,m)}$ for all i not all the $\langle w_{i-1},w_i\rangle$ belong to β^*_m. Therefore $\{\mathrm{col}(w_0),\mathrm{col}(w_1),\dots,\mathrm{col}(w_t)\}=\{\mathrm{col}(b_\emptyset),\mathrm{col}(b'_\emptyset)\}$ (which is the same as $\{\mathrm{col}(c_\emptyset),\mathrm{col}(c'_\emptyset)\}$). In particular, $\mathrm{col}(u)=\mathrm{col}(v)\in\{\mathrm{col}(b_\emptyset),\mathrm{col}(b'_\emptyset)\}$. Since $\mathrm{col}(u)\in\{\mathrm{col}(b_\emptyset),\mathrm{col}(b''_\emptyset)\}$ comes similarly, we obtain $\mathrm{col}(u)=\mathrm{col}(v)=\mathrm{col}(b_\emptyset)$. We can assume on the sequence that $\{\langle w_{i-1},w_i\rangle,\langle w_i,w_{i+1}\rangle\}\subseteqq\beta^*_m$ holds for no i. Now the only possibility concerning the elements w_i is the following:

$$u\ \beta^*_m\ w_1=b_\emptyset\ \langle b_\emptyset,b'_\emptyset\rangle^{(n+1,m)}\ w_2=b'_\emptyset\ \beta^*_m\ w_3=c'_U\ \langle c'_U,c_U\rangle^{(n+1,m)}\ w_4=c_U\ \beta^*_m\ v.$$

Since $\mathrm{col}(e^n_n)\cap\mathrm{col}(e^k_k)=\emptyset$ for $k>n$, the intersection of a sub-$(n+1)$-box with the $\mathrm{col}(b_\emptyset)$-th or the $\mathrm{col}(b'_\emptyset)$-th column is closed with respect to β^*_m. Hence all the w_i, including u and v, belong to the same sub-$(n+1)$-box. Working within this sub-$(n+1)$-box, from

$$b_\emptyset\ \beta_m\ u\ \beta_m\ v\ \beta_m\ c_U,$$

$u\neq v$ and $b_\emptyset\prec c_U$ (with respect to the partial order β_m) we conclude $u=b_\emptyset$ and $v=c_U$. Thus $\langle u,v\rangle\in\langle b_\emptyset,c_U\rangle^{(n+1,m)}$, proving (2) for $f^{n+1}_{b_\emptyset,c_U}$.

We define

$$f^{n+1}_{b'_\emptyset,c'_U}=y^*\wedge\left(f^{n+1}_{b'_\emptyset,b_\emptyset}\vee f^{n+1}_{b_\emptyset,c_U}\vee f^{n+1}_{c_U,c'_U}\right).$$

Suppose now that $p\in B_n(\emptyset)$ and $q\in B_n(U)$ for $U\in P^+(D^{(n)}_n)$. (The previous cases, $\langle p,q\rangle=\langle b_\emptyset,c_U\rangle$ or $\langle p,q\rangle=\langle b'_\emptyset,c'_U\rangle$, are excluded, of course.) We can define

$$f^{n+1}_{p,q}=\left(f^{n+1}_{p,b_\emptyset}\vee f^{n+1}_{b_\emptyset,c_U}\vee f^{n+1}_{c_U,q}\right)\wedge\left(f^{n+1}_{p,b'_\emptyset}\vee f^{n+1}_{b'_\emptyset,c'_U}\vee f^{n+1}_{c'_U,q}\right).$$

Finally, let $p\in B_n(U_1)$ and $q\in B_n(U_2)$ for distinct $U_1,U_2\in P^+(D^{(n)}_n)$, and let $b_\emptyset,c_\emptyset\in B_n(\emptyset)$ be as before. We define

$$f^{n+1}_{p,q}=\left(f^{n+1}_{p,b_\emptyset}\vee f^{n+1}_{b_\emptyset,q}\right)\wedge\left(f^{n+1}_{p,c_\emptyset}\vee f^{n+1}_{c_\emptyset,q}\right).$$

We have defined $f_{p,q}^{n+1}$ satisfying (2) for all $p, q \in A_{n+1}$. The induction is complete. So Theorem 1 is proved for all Quord(A_n), i.e., for Quord(A) with $|A| = \kappa_n$. Now if A is infinite and $|A| < \kappa_\omega$, then $\kappa_n \leqq |A| < \kappa_{n+1}$ for some n. We may suppose $\kappa_n < |A| < \kappa_{n+1}$ (note that this case would not occur if we assumed the generalized continuum hypothesis). Then simply modifying the construction of A_{n+1} so that we replace $P(D_n^{(n)})$ by a subset $\hat{P}(D_n^{(n)})$ of it such that $|\hat{P}(D_n^{(n)})| = |A|$ and $\emptyset \in \hat{P}(D_n^{(n)})$ we easily obtain the result for Quord(A).

Now let us deal with the finite case. If A consists of three elements a, b and c, then Quord(A) is clearly generated by $\langle a, b\rangle^{\mathrm{qo}}$, $\langle b, c\rangle^{\mathrm{qo}}$ and $\langle c, a\rangle^{\mathrm{qo}}$. If $|A| = 2k \geqq 4$, then we can restrict α_0, β_0 and γ_0 to $A = \{a_0, a_1, \dots, a_{k-1}, b_0, b_1, \dots, b_{k-1}\}$; the terms $f_{p,q}^0$ for $p, q \in A$ still satisfy (1). The odd case, $A = \{a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_{k-1}\}$ $(k \geqq 2)$, is essentially the same, but instead of (4) we have to say

(4') vertical, horizontal and $\{b_{k-1}, b_k\}$.

The proof of Theorem 1 is complete.

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