FOUR NOTES ON QUASIORDER LATTICES

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Abstract. The quasiorders, i.e. reflexive, transitive and compatible relations, of a (partial) algebra A form a lattice $\mathrm{Quord}(A)$ with an involution $\rho \mapsto \rho^{-1} = \{\langle x, y\rangle\colon\ \langle y, x\rangle \in \rho\}$. It is shown that every algebraic lattice with involution is isomorphic to $\mathrm{Quord}(A)$ for some partial algebra A. Any finite distributive lattice with involution is isomorphic to $\mathrm{Quord}(A)$ for some finite algebra A such that the quasiorders of A are 3-permutable. Every distributive lattice with involution can be embedded in $\mathrm{Quord}(A)$ for some set A. Any algebraic lattice is isomorphic to $\mathrm{Quord}(A)$ for some algebra A such that $\mathrm{Quord}(A) = \mathrm{Con}(A)$.

Introduction

A triplet $L = \langle L; \leq, {}^{-1}\rangle$ is called an *involution lattice* or a *lattice with involution* if ${}^{-1}\colon L \to L$ is a lattice automorphism such that $(x^{-1})^{-1} = x$ holds for all $x \in L$. The fixed points of the involution form a sublattice $\{x \in L\colon\ x^{-1} = x\}$, whose elements will be called the *fixed elements* (of the involution). If the context is involution lattices then embeddings, isomorphisms and homomorphisms are always supposed to preserve the involution operation ${}^{-1}$. Every lattice can be turned into an involution lattice by considering the identical map as involution. To present a natural but less trivial example, let us consider a partial algebra $A = \langle A; F\rangle$. A binary relation $\rho \subseteq A^2$ is called a *quasiorder* of A if ρ is reflexive, transitive, and compatible, i.e. for any $f \in F$ and any $\langle a_1, \dots, a_n\rangle, \langle b_1, \dots, b_n\rangle$ in the domain of f if $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in \rho$ then $\langle f(a_1, \dots, a_n), f(b_1, \dots, b_n)\rangle \in \rho$. Defining $\rho^{-1} = \{\langle x, y\rangle\colon\ \langle y, x\rangle \in \rho\}$ as usual, the set $\mathrm{Quord}(A)$ of quasiorders of A becomes an involution lattice $\mathrm{Quord}(A) = \langle \mathrm{Quord}(A); \subseteq, {}^{-1}\rangle$. The fixed elements of this lattice are just the congruences of A. Like congruences of algebras, quasiorders arise naturally in case of *ordered algebras* as homomorphism kernels, cf. [4] and Bloom [1]. Our aim is to deal with the following two problems of [3].

Problem A. Which algebraic lattices are isomorphic to $\mathrm{Quord}(A)$ for some algebra A?

Problem B. Characterize pairs $\langle L_1, L_2\rangle$ of (algebraic) lattices such that $L_1 \subseteq L_2$ and there exist an algebra A and a lattice isomorphism $\varphi\colon L_2 \to \mathrm{Quord}(A)$ with $\varphi(L_1) = \mathrm{Con}(A)$.

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It is pointed out in [3] that L_1 cannot be an arbitrary complete sublattice of L_2. In connection with Problem A it is worth mentioning that the analogous characterization of $\mathrm{Con}(A)$ is solved by a celebrated theorem of Grätzer and Schmidt [6].

While our first theorem solves Problem A, we are still far from solving Problem B. A recent result [12] shows that not every algebraic lattice with involution is isomorphic to $\mathrm{Quord}(A)$ for some algebra A. Moreover, certain algebraic lattices with involution cannot be embedded in $\mathrm{Quord}(A)$ for any set A. This is a bit surprising in the view of Theorems 2,3 and 4 of the present paper.

Results and proofs

Theorem 1. *For any algebraic lattice L there is an algebra A such that $L \cong$ Quord(A) and, in addition, Quord(A) coincides with Con(A).*

Proof. We will use the yeast graph construction given by Pudlák and Tůma [9] which gives an algebra with $\mathrm{Con}(A) \cong L$, we will show $\mathrm{Con}(A) = \mathrm{Quord}(A)$ only. The graph construction in Chapter 1 of [9] is much more general than needed here, so we describe only as much of it as necessary. Let $J = \langle J; \vee, ^{-1}\rangle$ be a semilattice with involution. The elements of J will be denoted by lowercase Greek letters. Let V be a nonempty set, let $P_2(V)$ denote the set of two-element subsets of V and let $E \subseteq J \times P_2(V)$. An element $\langle \alpha, \{a, b\}\rangle$ of E will mostly be denoted by $\langle a, \alpha, b\rangle$; of course $\langle a, \alpha, b\rangle = \langle b, \alpha, a\rangle$ and $a \neq b$. A pair $G = \langle V, E\rangle$ is called a J-graph or simply graph if, for any $a, b \in V$ and $\alpha, \beta \in J$, $\langle a, \alpha, b\rangle, \langle a, \beta, b\rangle \in E$ implies $\alpha = \beta$. The elements of V are called vertices while the elements of E are called edges. Here α resp. a, b are called the colour resp. endpoints of the edge $\langle a, \alpha, b\rangle$. The endpoints of an edge uniquely determine its colour. Our graphs will often have two distinguished vertices referred to as left and right endpoints. Given two graphs, $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, a map $f\colon V_1 \to V_2$ is called a homomorphism if for every $\langle a, \alpha, b\rangle \in E_1$ either $f(a) = f(b)$ or $\langle f(a), \alpha, f(b)\rangle \in E_2$. Isomorphisms, endomorphisms and automorphisms are the usual particular cases of this notion.

With any positive integer k and $\langle \alpha_1, \alpha_2, \dots, \alpha_k\rangle \in J^k$ we associate a graph $R(\alpha_1, \dots, \alpha_k)$, called arc, such that the vertex set of $R(\alpha_1, \dots, \alpha_k)$ is $\{a_0, a_1, \dots, a_{2k}\}$ and the edge set is $\{\langle a_0, \alpha_1, a_1\rangle, \langle a_1, \alpha_2, a_2\rangle, \dots, \langle a_{k-1}, \alpha_k, a_k\rangle, \langle a_k, \alpha_1, a_{k+1}\rangle, \langle a_{k+1}, \alpha_2, a_{k+2}\rangle, \dots, \langle a_{2k-1}, \alpha_k, a_{2k}\rangle\}$. The vertices a_0 resp. a_{2k} are the left resp. right endpoints of $R(\alpha_1, \dots, \alpha_k)$. Given an $\alpha \in J$, we define a graph $C(\alpha)$, called α-cell, as follows. We start with $C_0(\alpha) = \langle \{b_0, b_1\}, \{\langle b_0, \alpha, b_1\rangle\}$. I.e., $C_0(\alpha)$ consists of two vertices, which are its endpoints, and a single α-coloured edge connecting them. For each $k \geq 1$ and for each $\langle \alpha_1, \alpha_2, \dots, \alpha_k\rangle \in J^k$ such that $\alpha \leq \alpha_1 \vee \alpha_2 \vee \dots \vee \alpha_k$ let us take (an isomorphic copy of) the arc $R(\alpha_1, \alpha_2, \dots, \alpha_k)$. The arcs we consider must be disjoint from each other and from $C_0(\alpha)$ as well. Now identifying the left endpoints of these arcs with b_0 and their right endpoints with b_1 we obtain $C(\alpha)$. The vertices b_0 and b_1 are the left and right endpoints of $C(\alpha)$, respectively, and the edge $\langle b_0, \alpha, b_1\rangle$ is called the base edge of $C(\alpha)$. Let us cite from [9] that $C(\alpha)$ admits an automorphism interchanging its endpoints. Indeed, we obtain a desired automorphism by mapping the vertices of $R(\alpha_1, \alpha_2, \dots, \alpha_k)$ to the vertices of $R(\alpha_k, \alpha_{k-1}, \dots, \alpha_1)$ in the reverse order.

Now, for all $k \geq 0$ and $\alpha \in J$ we define a graph $G_n(\alpha) = \langle V_n(\alpha), E_n(\alpha)\rangle$ via induction on n as follows. Let $G_0(\alpha)$ be the α-cell $C(\alpha)$ and let $E_{-1}(\alpha) = \emptyset$. We obtain $G_{n+1}(\alpha)$ from $G_n(\alpha)$ as follows. For each edge $\langle a, \beta, b\rangle \in E_n(\alpha) \setminus E_{n-1}(\alpha)$

we take (an isomorphic copy of) the β- cell $C(\beta)$. These cells, even those associated with distinct edges of the same colour, must be disjoint form each other and from $G_n(\alpha)$. Now, for each $\langle a,\beta,b\rangle \in E_n(\alpha)\setminus E_{n-1}(\alpha)$ at the same time, let us identify a resp. b with the left resp. right endpoint of (the copy of) $C(\beta)$ associated with this edge. (In other words, to each edge in $E_n(\alpha)\setminus E_{n-1}(\alpha)$ we glue the base edge of a cell with the same colour, and we use disjoint cells for distinct edges.) The graph we have obtained is $G_{n+1}(\alpha)$.

Now $V_0(\alpha)\subseteq V_1(\alpha)\subseteq V_2(\alpha)\subseteq\dots$ and $E_0(\alpha)\subseteq E_1(\alpha)\subseteq E_2(\alpha)\subseteq\dots$, so we can define $V(\alpha)=\bigcup_{n=0}^{\infty}V_n(\alpha)$, $E(\alpha)=\bigcup_{n=0}^{\infty}E_n(\alpha)$, and let $G(\alpha)=G_\infty(\alpha)$ denote the graph $\langle V(\alpha),E(\alpha)\rangle$. The base edge and the endpoints of $G(\alpha)$ are that of $G_0(\alpha)=C(\alpha)$, respectively. Since $G_0(\alpha)=C(\alpha)$ has an automorphism interchanging its endpoints, a trivial induction shows that so does $G(\alpha)=G_\infty(\alpha)$ as well.

Now we are ready to define the last of our graphs, denoted by $G(J)$. For each $\alpha\in J$ let us take (a copy of) $G(\alpha)$ such that $G(\alpha)$ and $G(\beta)$ be disjoint when $\alpha\neq\beta$. Identifying the left endpoints of these $G(\alpha)$ to a single vertex we obtain $G(J)=\langle V(J),E(J)\rangle$.

Let us consider the algebra $A=\langle V(J),F\rangle$ where F is the set of endomorphisms of the graph $G(J)$. Further, let J be the set of nonzero compact elements of L. It is well-known, cf. Grätzer and Schmidt [6] or Grätzer [8, p. 22], that the ideal lattice $\mathcal{I}(J)$ of J is isomorphic to L. (Here the empty set is also considered an ideal.) Consequently, the first chapter of [9] yields that L is isomorphic to $\mathrm{Con}(A)$. (Indeed, the “quadricle” $\langle J,\le,D,\mathcal{L}\rangle$ in [9] corresponds to $\langle J,=,D,\mathcal{I}(J)\rangle$ in our case where $D=\{\langle\alpha,\{\alpha_1,\dots,\alpha_k\}\colon\ \alpha\in J,\ \ \{\alpha_1,\dots,\alpha_k\}\subseteq J,\ \ \alpha\le\alpha_1\vee\dots\vee\alpha_k\}$.) So we have to show that every quasiorder of A is symmetric, i.e. a congruence.

Suppose ρ is a quasiorder of A, $a\neq b\in A$ and $\langle a,b\rangle\in\rho$. It is shown in [9], cf. RC 5 and the proof of Lemma 1.9, that there is a “path” from a to b, i.e. a sequence

$$\langle c_0,\alpha_1,c_1\rangle,\langle c_1,\alpha_2,c_2\rangle,\dots,\langle c_{k-1},\alpha_k,c_k\rangle\in E(J)$$

of edges such that $c_0=a$, $c_k=b$, and for $i=1,2,\dots,k$ there is an $f_i\in F$ with $\{f_i(a),f_i(b)\}=\{c_{i-1},c_i\}$. We want to show the existence of a $g_i\in F$ such that $g_i(a)=c_i$ and $g_i(b)=c_{i-1}$. For a fixed i let u resp. v denote the left resp. right endpoints of $G(\alpha_i)$, and let h be an endomorphism of $G(\alpha_i)$ interchanging them. Clearly, the map

$$f^{(1)}\colon V(J)\to V(J),\qquad x\mapsto\begin{cases} h(x), & \text{if } x\in V(\alpha_i)\\ v, & \text{if } x\notin V(\alpha_i)\end{cases}$$

belongs to F and interchanges u and v. By [9], cf. RC 4 of Theorem 1.6, there are $f^{(2)},f^{(3)}\in F$ such that $\{f^{(2)}(u),f^{(2)}(v)\}=\{c_{i-1},c_i\}$ and $\{f^{(3)}(c_{i-1}),f^{(3)}(c_i)\}=\{u,v\}$. Since F is closed with respect to composition, $f^{(2)}f^{(1)}f^{(3)}f_i$ and $f^{(2)}f^{(3)}f_i$ belong to F, and one of them is an appropriate g_i.

Since the g_i preserve ρ, we obtain $\langle c_i,c_{i-1}\rangle=\langle g_i(a),g_i(b)\rangle\in\rho$, and $\langle b,a\rangle=\langle c_k,c_0\rangle\in\rho$ follows by transitivity. $\square$

The quasiorders of an algebra A are called 3-permutable if $\alpha\circ\beta\circ\alpha=\beta\circ\alpha\circ\beta$ holds for any $\alpha,\beta\in\mathrm{Quord}(A)$.

Theorem 2. *For any finite distributive involution lattice L there exists a finite algebra A such that L and Quord(A) are isomorphic as involution lattices and, in addition, the quasiorders of A are 3-permutable.*

We remark that if the quasiorders of all algebras in a given variety V are 3-permutable then $\mathrm{Con}(A) = \mathrm{Quord}(A)$ for all $A \in V$, cf. [2].

Proof. Let J be the set of join-irreducible elements of L, 0 is included. For each $a \in J \setminus \{0\}$ we define a unary operation

$$f_a\colon J \to J, \qquad x \mapsto \begin{cases} 0, & \text{if } x = a, \\ a^{-1}, & \text{if } x \neq a. \end{cases}$$

Let us call a map $g\colon J \to J$ a contraction of J if $g(x) \leq x$ holds for all $x \in J$. Let F consist of all contractions of J and all f_a, $a \in J \setminus \{0\}$. Consider the algebra $A = \langle J; F\rangle$; we intend to show that L and Quord(A) are isomorphic.

A subset Y of J is called hereditary if for any $x \in J$ and $y \in Y$ if $x \leq y$ then $x \in Y$. Let $\mathcal{H}(J)$ denote the set of nonempty hereditary subsets of J. It is well-known, cf. Grätzer [7, Theorem II.1.9 on page 61], that the map $a \mapsto \{x \in J\colon x \leq a\}$ is a lattice isomorphism from L to the lattice $\mathcal{H}(J) = \langle \mathcal{H}(J); \cup, \cap\rangle$. Clearly, $\mathcal{H}(J)$ becomes an involution lattice by defining $Y^{-1} = \{y^{-1}\colon y \in Y\}$ and the above-mentioned map preserves this involution. So it suffices to prove that the map $\psi\colon \mathcal{H}(J) \to \mathrm{Quord}(A)$, $Y \mapsto (Y \times Y^{-1}) \cup \{\langle x, x\rangle\colon x \in J\}$ is an isomorphism. Clearly, $\psi(Y)$ is reflexive, transitive and preserved by all contractions of J. To show that f_a preserves $\psi(Y)$ suppose that $\langle u, v\rangle \in \psi(Y)$ and, without loss of generality, $f_a(u) \neq f_a(v)$. Then either $f_a(u) = 0$, $u = a$ and $\langle f_a(u), f_a(v)\rangle = \langle 0, a^{-1}\rangle \in \psi(Y)$ since $a = u \in Y$, or $f_a(v) = 0$, $v = a$ and $\langle f_a(u), f_a(v)\rangle = \langle a^{-1}, 0\rangle \in \psi(Y)$ since $a^{-1} = v^{-1} \in (Y^{-1})^{-1} = Y$. Thus $\psi(Y)$ is a quasiorder of A. Clearly, ψ is meet-preserving, whence it is monotone. Assume that $\langle u, v\rangle \in \psi(X \cup Y)$ and $u \neq v$. Then $u \in X \cup Y$, $v \in (X \cup Y)^{-1} = X^{-1} \cup Y^{-1}$. There are four cases depending on the location of u and v but each of these cases can be treated similarly, so we detail the case $u \in Y$, $v \in X^{-1}$ only. Then $\langle u, 0\rangle \in \psi(Y)$ and $\langle 0, v\rangle \in \psi(X)$, so by reflexivity we obtain $\langle u, v\rangle \in \psi(X) \circ \psi(Y) \circ \psi(X) \subseteq \psi(X) \vee \psi(Y)$ and $\langle u, v\rangle \in \psi(Y) \circ \psi(X) \circ \psi(Y) \subseteq \psi(X) \vee \psi(Y)$. Besides proving that ψ is join-preserving, this also shows that $\psi(X)$ and $\psi(Y)$ 3-permute. Clearly, $\psi(X^{-1}) = (\psi(X))^{-1}$, therefore ψ is a homomorphism. If $x \in Y \setminus X$ then $\langle x, 0\rangle \in \psi(Y) \setminus \psi(X)$, whence ψ is injective.

To prove surjectivity, assume that $\rho \in \mathrm{Quord}(A)$ and let $X = \{x \in J\colon \langle x, 0\rangle \in \rho\}$ and $Y = \{y \in J\colon \langle 0, y\rangle \in \rho\}$. Thanks to the fact that ρ is preserved by the contractions we conclude that $X, Y \in \mathcal{H}(J)$. If $x \in X \setminus \{0\}$ then $\langle 0, x^{-1}\rangle = \langle f_x(x), f_x(0)\rangle \in \rho$, whence $x = (x^{-1})^{-1} \in Y^{-1}$. Similarly, if $y \in Y \setminus \{0\}$ then $\langle y^{-1}, 0\rangle = \langle f_y(0), f_y(y)\rangle \in \rho$, whence $y^{-1} \in X$ gives $y \in X^{-1}$. From $X \subseteq Y^{-1}$ and $Y \subseteq X^{-1}$ we obtain $Y = X^{-1}$.

Now, to show that $\rho = \psi(X)$, suppose $a \neq b$ and $\langle a, b\rangle \in \rho$. Then $\langle b^{-1}, 0\rangle = \langle f_b(a), f_b(b)\rangle \in \rho$ gives $b^{-1} \in X$, i.e. $b \in X^{-1}$, while $\langle 0, a^{-1}\rangle = \langle f_a(a), f_a(b)\rangle \in \rho$ gives $a^{-1} \in Y$, i.e. $a \in Y^{-1} = X$, yielding $\langle a, b\rangle \in X \times X^{-1} \subseteq \psi(X)$. Conversely, suppose that $a \neq b$ and $\langle a, b\rangle \in \psi(X)$. Then, by definitions and $Y = X^{-1}$, $\langle a, 0\rangle \in \rho$ and $\langle 0, b\rangle \in \rho$, yielding $\langle a, b\rangle \in \rho$ by transitivity. ☐

Whitman [11] has shown that every lattice can be embedded in a partition lattice. The preceding theorem trivially gives a corollary stating that each finite distributive involution lattice L can be embedded in Quord(A) for an appropriate set A. We have even proved that L has a type 2 representation in Jónsson's sense, cf. [5], which means that L is isomorphic to a sublattice S of Quord(A) such that the members of S are 3-permutable. However, the assumption of finiteness can be easily removed, for we have

Theorem 3. *For each distributive involution lattice L there is a set A such that L has a type 2 representation in Quord(A).*

Proof. Knowing the canonical bijection between prime filters (i.e. dual prime ideals) and nonzero join-irreducible elements of a finite distributive lattice, cf. Grätzer [7, p. 63], it is easy to adapt the previous proof to the present theorem. Let $A = \{P\colon P$ is a prime filter of L or $P = L\}$. We claim that the map $\psi\colon L \to$ Quord(A), $x \mapsto \{\langle P, Q\rangle\colon x \in P$ and $x^{-1} \in Q$, or $P = Q\}$ is an embedding. By Stone's prime ideal theorem, cf. [10] or [7, p. 63], ψ is injective. Using the basic properties of prime filters and some ideas of the previous proof, Theorem 3 follows easily. □

Theorem 4. *For any algebraic involution lattice L there is a partial algebra A such that L is isomorphic to Quord(A).*

Proof. Let S be the set of compact elements of L. Then S is a join-sub-semilattice of L and clearly S is closed with respect to the involution of L. The set $\mathcal{I}(S)$ of ideals, i.e. hereditary nonempty $\vee$-closed subsets, of S form an algebraic lattice with involution where $Y^{-1} = \{a^{-1}\colon a \in Y\}$. It is known that $\varphi\colon L \to \mathcal{I}(S)$, $x \mapsto \{a \in S\colon a \le x\}$ is a lattice isomorphism, cf. Grätzer and Schmidt [6] or [8, p. 22]. Evidently, φ preserves the involution, too. The rest of our proof borrows a lot of ideas from the congruence lattice counterpart of our theorem, cf. Grätzer and Schmidt [6] or [8, pp. 96-97]. We define the following partial operations on S, each of them has a two-element domain as indicated:

(1) for $a, b \in S \setminus \{0\}$ $\quad f_{ab}\colon \langle a, b\rangle \mapsto a \vee b$, $\langle 0, 0\rangle \mapsto 0$;
(2) for $a > b \in S$ $\quad g_{ab}\colon a \mapsto b$, $0 \mapsto 0$;
(3) for $a \ne b \in S$ $\quad h_{ab}\colon a \mapsto a$, $b \mapsto 0$;
(4) for $a \in S \setminus \{0\}$ $\quad p_a\colon a \mapsto 0$, $0 \mapsto a^{-1}$.

Note that the partial operations (1), (2) and (3) also occur in [8, pp. 96-97]. Let A be the partial algebra $\langle S; F\rangle$ where F is the collection of partial operations (1)–(4). Let $\alpha\colon \mathcal{I}(S) \to$ Quord(A), $Y \to (Y \times Y^{-1}) \cup \{\langle a, a\rangle\colon a \in S\}$ and $\beta\colon$ Quord$(A) \to \mathcal{I}(S)$, $\rho \mapsto \{s \in S\colon \langle s, 0\rangle \in \rho\}$.

It is straightforward to check that $\alpha(Y) \in$ Quord(A) for $Y \in \mathcal{I}(S)$. Using the partial operations (1) and (2) it follows easily that $\beta(\rho) \in \mathcal{I}(S)$ for $\rho \in$ Quord(A). If $s \in \beta(\rho^{-1})$ then $\langle s, 0\rangle \in \rho^{-1} \Longrightarrow \langle 0, s\rangle \in \rho \Longrightarrow \langle s^{-1}, 0\rangle = \langle p_s(0), p_s(s)\rangle \in \rho \Longrightarrow s^{-1} \in \beta(\rho) \Longrightarrow s = (s^{-1})^{-1} \in (\beta(\rho))^{-1}$. Conversely, if $s \in (\beta(\rho))^{-1}$ then $s^{-1} \in \beta(\rho) \Longrightarrow \langle s^{-1}, 0\rangle \in \rho \Longrightarrow \langle 0, s^{-1}\rangle \in \rho^{-1} \Longrightarrow \langle s, 0\rangle = \langle p_{s^{-1}}(0), p_{s^{-1}}(s^{-1}\rangle \in \rho^{-1} \Longrightarrow s \in \beta(\rho^{-1})$. Therefore $\beta(\rho^{-1}) = (\beta(\rho))^{-1}$, i.e. β preserves the involution. Clearly, so does α, too. Since both α and β are monotone, it suffices to show that they are inverses of each other. It is straightforward that $\beta(\alpha(Y)) = Y$ for $Y \in \mathcal{I}(S)$. Now let $\rho \in$ Quord(A), $a, b \in S$ and $a \ne b$. Suppose first that $\langle a, b\rangle \in \rho$. Then $\langle a, 0\rangle = \langle h_{ab}(a), h_{ab}(b)\rangle \in \rho$ gives $a \in \beta(\rho)$ while $\langle b, 0\rangle = \langle h_{ba}(b), h_{ba}(a)\rangle \in \rho^{-1}$ gives $b \in \beta(\rho^{-1}) = (\beta(\rho))^{-1}$, and we infer $\langle a, b\rangle \in \alpha(\beta(\rho))$. Conversely, suppose that $\langle a, b\rangle \in \alpha(\beta(\rho))$. Now $a \in \beta(\rho)$ yields $\langle a, 0\rangle \in \rho$, $b \in (\beta(\rho))^{-1} = \beta(\rho^{-1})$ gives $\langle b, 0\rangle \in \rho^{-1}$ implying $\langle 0, b\rangle \in \rho$, and $\langle a, b\rangle \in \rho$ follows by transitivity. Therefore

$\alpha(\beta(\rho)) = \rho$, and α is an isomorphism. Consequently, $\alpha \circ \varphi\colon L \to \mathrm{Quord}(A)$ is an isomorphism as well. $\square$

Contrary to Theorem 2, Theorem 4 does not lead to any corollary concerning embeddability of involution lattices in Quord(A) for sets A, for the joins are different.

Added at final revision. Recently A. G. Pinus has informed us that he also had proved Theorem 1 independently. His paper "On the lattice of quasiorders on universal algebras" (in Russian) is submitted to Algebra i Logic.

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