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# A Note on Representation of Lattices by Tolerances

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While the lattice $T(A)$ of tolerance relations of a universal algebra $A$ has been known to be algebraic for a long time (cf., e.g., [1]), the converse statement has been unsettled. The aim of the present note is to point out that a suitable modification of the construction in Grätzer and Lampe [2] leads to the following.

THEOREM. *Let L be an algebraic lattice. Then there exists an algebra A such that*

(a) $L \cong T(A)$;

(b) *every subalgebra of* $A^2$ *is a tolerance on* $A$.

Here a tolerance means a reflexive, symmetric, and compatible relation. The set $T(A)$ of all tolerances of $A$ constitutes a lattice under the set-theoretic inclusion.

Grätzer and Lampe [2] produce an algebra $A$ such that the subalgebra lattice $S(A^2)$ of $A^2$ is isomorphic to $L$. This $A$ is defined as a limit of a series of partial algebras $B'_0 = B$, $B'_1$, $B'_2$, $B'_3$, ... such that $S(B_i'^2) \cong L$ for all $i$. Every $B'_i$, $i > 0$, is derived from $B'_{i-1}$ in a canonical way. In what follows we modify the construction of $B$ so that every subalgebra of $B^2$ is a tolerance on $B$. Then, as it will be straightforward by checking through [2], every subalgebra of $A^2$ will be a tolerance on $A$, proving our theorem.

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Let $L$ be an algebraic lattice with more than one element, and let $C$ be the set of compact elements of $L$. Let $B$ be an arbitrary set with $C \subset B$, and fix an element $b_0 \in B \setminus C$ and an element $c_0 \in C \setminus \{0\}$. We define a mapping $\psi$ from $C$ to the set of all subsets of $B^2$ as follows:

$$0\psi = \omega_B = \{\langle b, b \rangle : b \in B\};$$

$$c\psi = \omega_B \cup \{\langle b_0, c \rangle, \langle c, b_0 \rangle\} \qquad \text{for} \quad c \in C \setminus \{0, c_0\};$$

$$c_0\psi = \Big( B^2 \setminus \bigcup_{c \in C \setminus \{c_0\}} c\psi \Big) \cup \omega_B.$$

Note that, for $c \in C$, $c\psi$ is a reflexive and symmetric relation on $B$, and $c\psi \cap d\psi = \omega_B$ holds when $c, d \in C$ are distinct. We define two kinds of partial operations on $B$:

(A) Each $b \in B$ is a value of a nullary operation;

(B) For any $c_1$, $c_2$, $c_3 \in C \setminus \{0\}$ with $c_1 \leqslant c_2 \vee c_3$, and for every $\langle a_i, b_i \rangle \in c_i\psi \setminus \omega_B$, $i = 1, 2, 3$, we define a binary partial operation $f$ by $f(a_2, a_3) = a_1$, $f(b_2, b_3) = b_1$ and $f$ is not defined elsewhere.

Now let $F$ be the set of all partial operations defined in (A) and (B) and consider the partial algebra $\mathbf{B} = (B, F)$. Following the proof of Lemma 1 in [2], it is straightforward to check that, besides $S(\mathbf{B}^2) \cong L$ (and other properties stated in [2, Lemma 1]), all subalgebras of $\mathbf{B}^2$ are tolerances on $\mathbf{B}$. Hence our theorem follows as indicated before.

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