Studia Scientiarum Mathematicarum Hungarica 28 (1993), 339–348

MAL'TSEV FUNCTIONS ON SMALL ALGEBRAS

I. CHAJDA and G. CZÉDLI

Abstract

The following problem is considered. Given an n-element set A and a set L of permuting equivalences on A, does there exist a Mal'tsev function $A^3 \to A$ which is compatible with all members of L? The answer is negative in general when $n \geqq 25$, it remains open for $9 \leqq n \leqq 24$, and it is shown to be affirmative for $n \leqq 8$. Moreover, there is even a commutative Mal'tsev function when $n \leqq 8$.

Introduction and result

Given a set A, a function $p\colon A^3 \to A$ is called a Mal'tsev function if $p(x,y,y)=p(y,y,x)=x$ holds for any $x,y\in A$. If an algebra A has a Mal'tsev function $p\colon A^3 \to A$ which is compatible with all congruences of A then A is congruence permutable. However, the converse is not true in general (cf. Gumm [3]). The purpose of the present paper is to furnish the converse statement under the additional condition $|A|\leqq 8$. In order to obtain a somewhat stronger statement we formulate our result not only for algebras. Then it may be of some interest in studying intersections of certain maximal clones on a finite set with less than nine elements. A Mal'tsev function $p\colon A^3 \to A$ is called commutative if $p(x_{1\pi},x_{2\pi},x_{3\pi})=p(x_1,x_2,x_3)$ holds for any $(x_1,x_2,x_3)\in A^3$ and any permutation $\pi\colon \{1,2,3\}\to\{1,2,3\}$.

THEOREM. *Let A be a set with $|A|\leqq 8$ and let L be a sublattice of the lattice of equivalences on A. Then the following three conditions are equivalent:*

(a) *the equivalences belonging to L permute, i.e., for any $\rho,\nu\in L$, $\rho\circ\nu=\nu\circ\rho$;*

(b) *there exists a Mal'tsev function $A^3\to A$ which is compatible with any member of L;*

1980 *Mathematics Subject Classification* (1985 Revision). Primary 08B05.
Key words and phrases. Congruence permutability, Mal'tsev term, finite algebra.

The second author's work was partially supported by the Hungarian Foundation for Scientific Research Grant No. 1813.

(c) *there is a commutative Mal'tsev function* $A^3 \to A$ *which is compatible with (any member of)* L.

Our method yielding the equivalence of (a) and (b) for $|A| \leqq 8$ is possibly applicable for $|A| = 9$ or $|A| = 10$ or even more. However, the length of the proof would grow rather fast with $|A|$ and we do not want to make it astronomically long. Another excuse for stopping at eight is that for $|A| = 9$ (a) and (c) are not equivalent. Really, if A is the square of the three element group and L is its congruence lattice then (a) holds but (c) does not (cf. Gumm [4, Thm. 3.2]).

While the equivalence of (a) and (b) is an open problem for $|A| \in \{9, 10, \dots, 24\}$, they are not equivalent for $|A| \geqq 25$. Moreover, we have the following

OBSERVATION. *For any natural number* $n \geqq 25$ *there is an* n*-element algebra* A *such that* A *has permutable congruences but no Mal'tsev function* $A^3 \to A$ *is compatible with all congruences of* A.

PROOF. Starting from a five-element non-associative loop (cf. Gumm [4, Fig. 2.4]) Gumm constructed a twentyfive-element A with the required property in [3]. Suppose we already have an n-element algebra $A = (A, F)$ as required, then we construct an $(n+1)$-element algebra B in the following way. Put $B = A \cup \{w\}$ where $w \notin A$. For $f\colon A^k \to A$ in F define $f_B\colon B^k \to B$,

$$f_B(b_1, \dots, b_k) = \begin{cases} f(b_1, \dots, b_k) & \text{if } b_1, \dots, b_k \in A \\ w & \text{otherwise.} \end{cases}$$

Further, for any $c \in A$, define $g_c\colon B \to B$ by

$$g_c(x) = \begin{cases} x & \text{if } x \neq w \\ c & \text{if } x = w. \end{cases}$$

Now put $B = (B, \{f_B : f \in F\} \cup \{g_c : c \in A\})$. Then for any nontrivial congruence α of B the block $[w]\alpha$ is a singleton and $\alpha|_A$ is a congruence of A. Thus the congruences of B permute. We can observe that any congruence of A is the restriction of a (unique) congruence of B. In particular, B has a congruence κ with exactly two blocks: A and $\{w\}$. Suppose B permits a compatible Mal'tsev function $p\colon B^3 \to B$. Then, for $x, y, z \in A$, $p(x, y, z)\,\kappa\, p(x, x, x) = x$ whence $p(x, y, z) \in A$. Therefore the restriction of p to A is a compatible Mal'tsev function on A, contradicting the induction hypothesis. Q.e.d.

PROOF OF THE THEOREM. The implication (b) $\Rightarrow$ (a) follows from the classical argument of Mal'tsev [5]. Namely, if $u, v \in A$, $\alpha, \beta \in L$ and $(u, v) \in \alpha \circ \beta$ then there is an element $w \in A$ with $u\alpha w\beta v$. If p is a compatible Mal'tsev function then

$$u = p(u, v, v)\,\beta\, p(u, w, v)\,\alpha\, p(u, u, v) = v$$

whence $(u, v) \in \beta \circ \alpha$. The implication (c) $\Rightarrow$ (b) being trivial we have to show only that (a) implies (c). This will need several preliminaries.

We will often consider diamonds (five-element non-distributive modular sublattices) in L; their elements will be denoted by $\omega, \alpha, \beta, \gamma, \iota$ such that $\omega -\!\!< \alpha -\!\!< \iota$, $\omega -\!\!< \beta -\!\!< \iota$, $\omega -\!\!< \gamma -\!\!< \iota$. The bottom and the top of L is denoted by 0 and 1, respectively.

Let $n \leqq 8$ and assume that (a) $\Rightarrow$ (c) for sets consisting of less than n elements. We fix an n-element set A and a permutable sublattice L of the equivalence lattice of A. We have to show the existence of a commutative Mal'tsev function which is compatible with L. A particular case is settled by the following

LEMMA 1. *If there exists a $\mu \in L \setminus \{0\}$ such that $\mu \leqq \omega$ holds for every diamond $\{\omega, \alpha, \beta, \gamma, \iota\}$ in L then we are done. (I.e., then there is a commutative Mal'tsev function which is compatible with L.)*

PROOF. The proof of this lemma borrows a lot of ideas from Pixley [6, p. 183]. By the induction hypothesis, there is a commutative Mal'tsev function $P_\mu : (A/\mu)^3 \to A/\mu$ preserving all ν/μ where $\mu \leqq \nu \in L$. For each $\lambda \in L$ we intend to define a commutative Mal'tsev function $p_\lambda : (A/\lambda)^3 \to A/\lambda$ preserving all ν/λ ($\lambda \leqq \nu \in L$) such that for any $\lambda_1 \leqq \lambda_2 \in L$

$$p_{\lambda_1}([x]\lambda_1, [y]\lambda_1, [z]\lambda_1) \subseteqq p_{\lambda_2}([x]\lambda_2, [y]\lambda_2, [z]\lambda_2) \tag{1}$$

for any $x, y, z \in A$. Then we will be ready as $p_0 : A^3 \to A$ is what we are looking for.

Let us fix a linear order on A. First we define p_λ for $\lambda \geqq \mu$ as follows:

$$p_\lambda([x]\lambda, [y]\lambda, [z]\lambda) = \{t \in A : ([t]\mu, p_\mu([x]\mu, [y]\mu, [z]\mu)) \in \lambda/\mu\}.$$

Roughly speaking, this is $[p_\mu([x]\mu, [y]\mu, [z]\mu)]\lambda/\mu$ apart from the canonical correspondence between A/λ and $(A/\mu)/(\lambda/\mu)$. Then for $\lambda = \mu$ p_λ is just the previously defined p_μ. A routine calculation shows that p_λ is a commutative Mal'tsev function preserving all ν/λ ($\nu \geqq \lambda$) and (1) holds for $\mu \leqq \lambda_1 \leqq \lambda_2$.

Now we define p_λ for $\lambda \ngeqq \mu$ via a downward induction on the height of λ. (Note that L is a modular lattice, for its members permute.) Assume that $\lambda \ngeqq \mu$ and $p_{\lambda'}$ is already defined for each $\lambda' > \lambda$ such that the required properties, including (1), are satisfied for these λ'. Let $\nu_1, \dots, \nu_k$ be the upper covers of λ and define p_λ as follows.

Let $p_\lambda([x]\lambda, [y]\lambda, [z]\lambda) = [a]\lambda$ where if two of the blocks $[x]\lambda, [y]\lambda$ and $[z]\lambda$ coincide then a is the first element in the remaining block. Otherwise let a be the first element in the intersection

$$\bigcap_{i=1}^{k} p_{\nu_i}([x]\nu_i, [y]\nu_i, [z]\nu_i). \tag{2}$$

(This will be shown nonempty later.)

Now if, e.g., $[x]\lambda = [y]\lambda$ then $[x]\nu_i = [y]\nu_i$ yields that $[z]\lambda$ is a subset of (2). Therefore a always belongs to the intersection (2). Thus p_λ is a commutative

Mal'tsev function. The property (1) extends to λ easily. Indeed, if $\lambda < \lambda_2$ then $\lambda -\!< \nu_i \leqq \lambda_2$ for some i and $p_\lambda([x]\lambda, [y]\lambda, [z]\lambda) = [a]\lambda \subseteqq [a]\nu_i = p_{\nu_i}([x]\nu_i, [y]\nu_i, [z]\nu_i) \subseteqq p_{\lambda_2}([x]\lambda_2, [y]\lambda_2, [z]\lambda_2)$. Using a routine calculation or referring to Pixley's proof [6, p. 183] we can see that p_λ is compatible with all ν/λ, $\nu \geqq \lambda$.

Now we set off to prove that (2) is not empty. We claim that

$$\prod_{i=1}^{j-1}(\nu_j + \nu_i) = \nu_j \quad \text{for } 2 < j < k. \tag{3}$$

(Here and in the sequel $+$ and $\cdot$ stand for the lattice operations join and meet, respectively.) Since the role of the ν_l ($1 \leqq l \leqq k$) is symmetric, it suffices to deal with $j = 3$. Then (3) turns into $(\nu_3 + \nu_1)(\nu_3 + \nu_2) = \nu_3$. It belongs to the folklore of lattice theory that if $(x_3 + x_1)(x_3 + x_2) > x_3$ for distinct atoms x_1, x_2, x_3 in a modular lattice M then $\{x_1, x_2, x_3\}$ generates a diamond with bottom 0_M and top $x_3 + x_1$. Indeed, by the properties of the height function (cf., e.g., Grätzer [2]), $x_3 + x_1$ and $x_3 + x_2$ are of height two and so is their meet by the assumption. Thus $x_3 + x_1 = x_3 + x_2$. Since $x_1 + x_2$ is of height two either and $x_1 + x_2 \leqq (x_3 + x_1) + (x_3 + x_2) = x_3 + x_1$, $x_1 + x_2 = x_3 + x_1$. Since L is modular (cf., e.g., Grätzer [2, Thm. IV.4.10 and the remark after its proof]), we can apply the above observation for the interval $[\lambda, 1]$. Therefore $(\nu_3 + \nu_1)(\nu_3 + \nu_2) = \nu_3$ as otherwise λ would be the bottom of a diamond in spite of $\lambda \ngeqq \mu$.

The next step is to show

(4) If $a_i \in A$ and for all $i, j \leqq k$ $(a_i, a_j) \in \nu_i + \nu_j$ then there exists an element $b \in A$ such that $(a_i, b) \in \nu_i$ for all $i \leqq k$.

Indeed, this says nothing for $k = 1$ and follows from $\nu_1 + \nu_2 = \nu_1 \circ \nu_2$ for $k = 2$. If we have found an element b already such that $(a_i, b) \in \nu_i$ for $i = 1, 2, \ldots, j$ ($2 \leqq j < k$) then $(b, a_{j+1}) \in \nu_i \circ (\nu_i + \nu_{j+1}) = \nu_i + \nu_{j+1}$ for all $i \leqq j$ and (3) yields $(b, a_{j+1}) \in \prod_{i \leqq j}(\nu_{j+1} + \nu_i) = \nu_{j+1}$. Therefore $(a_i, b) \in \nu_i$ holds for all $i \leqq k$.

Now, returning to (2), pick an element a_i in $p_{\nu_i}([x]\nu_i, [y]\nu_i, [z]\nu_i)$, $i = 1, 2, \ldots, k$. By the induction hypothesis made on λ, for $i, j \leqq k$ we have

$$a_i \in p_{\nu_i}([x]\nu_i, [y]\nu_i, [z]\nu_i) \subseteqq$$
$$\subseteqq p_{\nu_i + \nu_j}([x](\nu_i + \nu_j), [y](\nu_i + \nu_j), [z](\nu_i + \nu_j)),$$

and a_j belongs there, too. Hence $(a_i, a_j) \in \nu_i + \nu_j$. Now (4) supplies us with an element b such that $b\nu_i a_i$ for all i. I.e., $b \in [a_i]\nu_i = p_{\nu_i}([x]\nu_i, [y]\nu_i, [z]\nu_i)$. This b belongs to the intersection (2). Q.e.d.

Let us call an element $\mu \in L$ semicentral if $\mu \circ \nu = \mu \cup \nu$ (set theoretic union) holds for every $\nu \in L$. (Note that $\mu \circ \nu = \mu + \nu$ by permutability.)

LEMMA 2. *If there exists a semicentral* $\mu \in L \backslash \{0, 1\}$ *then we are done.*

PROOF. Let $B_1, B_2, \dots, B_t$ be the μ-blocks. Since μ is not in $\{0, 1\}$, we have $t < n$ and $|B_i| < n$ for all i. Observe that the restrictions of members of L to B_i permute. Indeed, if $\rho, \nu \in L$, $a, b, c \in B_i$, $a\rho c$ and $c\nu b$ then there is a $d \in A$ with $a\nu d\rho b$. If $d \notin B_i$ then $(c, d) \in \mu \circ \nu = \mu \cup \nu$ yields $c\nu d$, whence $a\nu b$ by transitivity. Therefore $a\nu b\rho b$, showing that the restrictions of ν and ρ to B_i permute. By the induction hypothesis on $|A|$ there is a commutative Mal'tsev function $p_i\colon B_i^3 \to B_i$ preserving the restrictions of members of L for each i, $1 \leqq i \leqq t$. Similarly, there is a Mal'tsev function $p_\mu\colon (A/\mu)^3 \to A/\mu$ preserving all the ρ/μ, $\mu \leqq \rho \in L$. Now let us fix an element $b_i \in B_i$ for each i, $1 \leqq i \leqq t$. For $x, y, z \in A$ let $B_k = B_k(x, y, z)$ be $p_\mu([x]\mu, [y]\mu, [z]\mu)$ and define $u = p(x, y, z)$ as follows:

(α) if x, y, z belong to the same μ-block B_j then $u = p_j(x, y, z)$ (note that $j = k$);

(β) if $|\{x, y, z\} \cap B_k| = 1$ then $u \in \{x, y, z\} \cap B_k$;

(γ) if $\{x, y, z\} \cap B_k = \emptyset$ then $u = b_k$.

Since p_μ is a commutative Mal'tsev function, $|\{x, y, z\} \cap B_k| = 2$ is impossible and it is easy to see that $p\colon A^3 \to A$ is a commutative Mal'tsev function. We do not have to use semicentrality to check that p preserves ρ if $\mu \leqq \rho$ or $\rho \leqq \mu$; the trivial details will be omitted. Now let $\rho \in L$, $\rho || \mu$, $x, x', y, z \in A$ and $x\rho x'$. We have to show that $p(x, y, z)\rho p(x', y, z)$. Suppose this is not the case. Since p preserves $\rho \circ \mu \in L$, we have $(p(x, y, z), p(x', y, z)) \in \rho \circ \mu = \rho \cup \cup \mu$ whence $p(x, y, z)\mu p(x', y, z)$. Therefore B_k in the definition of $p(x, y, z)$ and $p(x', y, z)$ is the same. If the same of (α), (β) and (γ) applies to both $p(x, y, z)$ and $p(x', y, z)$ then $p(x, y, z)\rho p(x', y, z)$. Moreover, if ($\alpha$) applies to one of $p(x, y, z)$ and $p(x', y, z)$ then it applies to the other as well. Thus we may assume that (β) applies to $p(x, y, z)$ and (γ) applies to $p(x', y, z)$. Then $p(x, y, z) = x$, $p(x', y, z) = b_k$ and $x' \notin B_k$. From $b_k \mu x \rho x'$ and $\mu \circ \rho = \mu \cup \rho$ we conclude $(b_k, x') \in \rho$. Then we obtain $p(x', y, z) = b_k \rho x = p(x, y, z)$ from $x' \rho x$ and transitivity; this is a contradiction. Q.e.d.

Whatever it is evident the following lemma offers a comfortable way to exploit the permutability of L.

LEMMA 3. *Let* $\mu, \rho \in L$, *let* B *and* C *be distinct* μ*-blocks and suppose that* $x\rho y$ *for some* $x \in B$, $y \in C$. *Then*

$$\mathrm{SP}(\mu, \rho)\colon \ (\forall b \in B)(\exists c \in C)(b\rho c) \text{ and } (\forall c \in C)(\exists b \in B)(b\rho c).$$

(The notation SP stands for "shifting principle" and gives an economic way of referring to the lemma.)

The proof is a trivial application of the fact that $\mu \circ \rho = \rho \circ \mu$.

We say that an equivalence is of pattern $i_1 + i_2 + \dots + i_t$ if it has t blocks and these blocks consists of $i_1, i_2, \dots, i_t$ elements.

LEMMA 4. *If L has a member of pattern $j+1+1+\cdots+1$ where $1<j<$ $<n\leqq 8$ or $3+2+1+1+\cdots+1$ where $5\leqq n\leqq 8$ then we are done.*

PROOF. We will show that Lemma 2 is applicable. Assume that $\mu\in L$ is of pattern $j+1+\cdots+1$ and let B be the j-element block of μ. We claim that μ is semicentral. Indeed, if $(x,y)\in\mu\circ\rho=\rho\circ\mu$ but $(x,y)\notin\mu$ then, e.g., $x\notin B$ and $z\rho z\mu y$ holds for some $z\in A$. Since $[x]\mu$ is a singleton, $\mathrm{SP}(\mu,\rho)$ yields $(x,y)\in\rho$.

Now let μ be of pattern $3+2+1+\cdots+1$. Assume that μ is not semicentral. Let $B=\{a,b,c\}$ and $C=\{d,e\}$ be the nontrivial μ-blocks. We can consider a $\nu\in L$ and $x,y\in A$ with $(x,y)\in(\mu\circ\nu)\backslash(\mu\cup\nu)$. If $|\{x,y\}\cap$ $\cap(B\cup C)|=1$, say $x\in B$, then $\mathrm{SP}(\mu,\nu)$ yields $(x,y)\in(B\cup\{y\})^2\subseteqq\nu$, a contradiction. Therefore $x\in B$ and $y\in C$ (or conversely). If $\nu|_C=1_C$ then $(x,y)\in(B\cup C)^2\subseteqq\nu$ by $\mathrm{SP}(\mu,\nu)$. Therefore $(d,e)\notin\nu$. Using $\mathrm{SP}(\mu,\nu)$ we have $B=\{z\in B: z\nu d\}\cup\{z\in B: z\nu e\}$ and we conclude that $\mu\cap\nu$ is of pattern $2+1+\cdots+1$. Therefore $\mu\cap\nu\in L$ is semicentral and Lemma 2 applies.

LEMMA 5. *If there are $\mu,\nu\in L$ such that*

$\mu<\nu$,
ν has exactly two blocks B and C,
$|B|>1$, $|C|>1$,
C is a block of μ and
there is a $b\in B$ with $[b]\mu=\{b\}$

then we are done.

PROOF. We intend to show that ν is semicentral. Assume that $\nu\circ\rho\neq$ $=\nu\cup\rho$ for some $\rho\in L$. Then there are $x,y\in A$ with $(x,y)\in\rho\backslash\nu$. By $\mathrm{SP}(\nu,\rho)$, there is a $c\in C$ with $b\rho c$. From $\mathrm{SP}(\mu,\rho)$ we conclude that $b\rho z$ holds for all $z\in C$. I.e., $C^2\subseteqq\rho$. Therefore $\mathrm{SP}(\nu,\rho)$ yields $\rho=1$, a contradiction. Q.e.d.

LEMMA 6. *Let $M_3=\{\omega,\alpha,\beta,\gamma,\iota\}$ be a diamond in L. Then every nontrivial block of ι/ω consists of four elements. The restriction of any of α/ω, β/ω and γ/ω to a four-element block of ι/ω has two two-element blocks. If ι/ω has only one nontrivial block (in particular, if $|A/\omega|<8$) then the interval $[\omega,\iota]$ of L coincides with M_3.*

PROOF. Since the ρ/ω (where $\omega\leqq\rho\in L$) permute, we can assume that $\omega=0$. Let B be a nontrivial ι/ω-block. Since M_3 is simple and the restriction map of M_3 to the equivalence lattice of B is a lattice homomorphism, $\{0_B,\alpha|_B,\beta|_B,\gamma|_B,1_B=\iota|_B\}$ is a diamond, too. It follows from Gumm [3, Lemma 3.2] and $|A/\omega|\leqq 8$ that $|B|=4$ and any of $\alpha|_B$, $\beta|_B$ and $\gamma|_B$ has two two-element blocks. We infer from Lemma 3 that beside $\alpha|_B$, $\beta|_B$ and $\gamma|_B$ no nontrivial equivalence on B permute with $\alpha|_B$, $\beta|_B$ and $\gamma|_B$ simultaneously. Thus $[0,\iota]=M_3$, provided B is the only nontrivial block of ι/ω. Q.e.d.

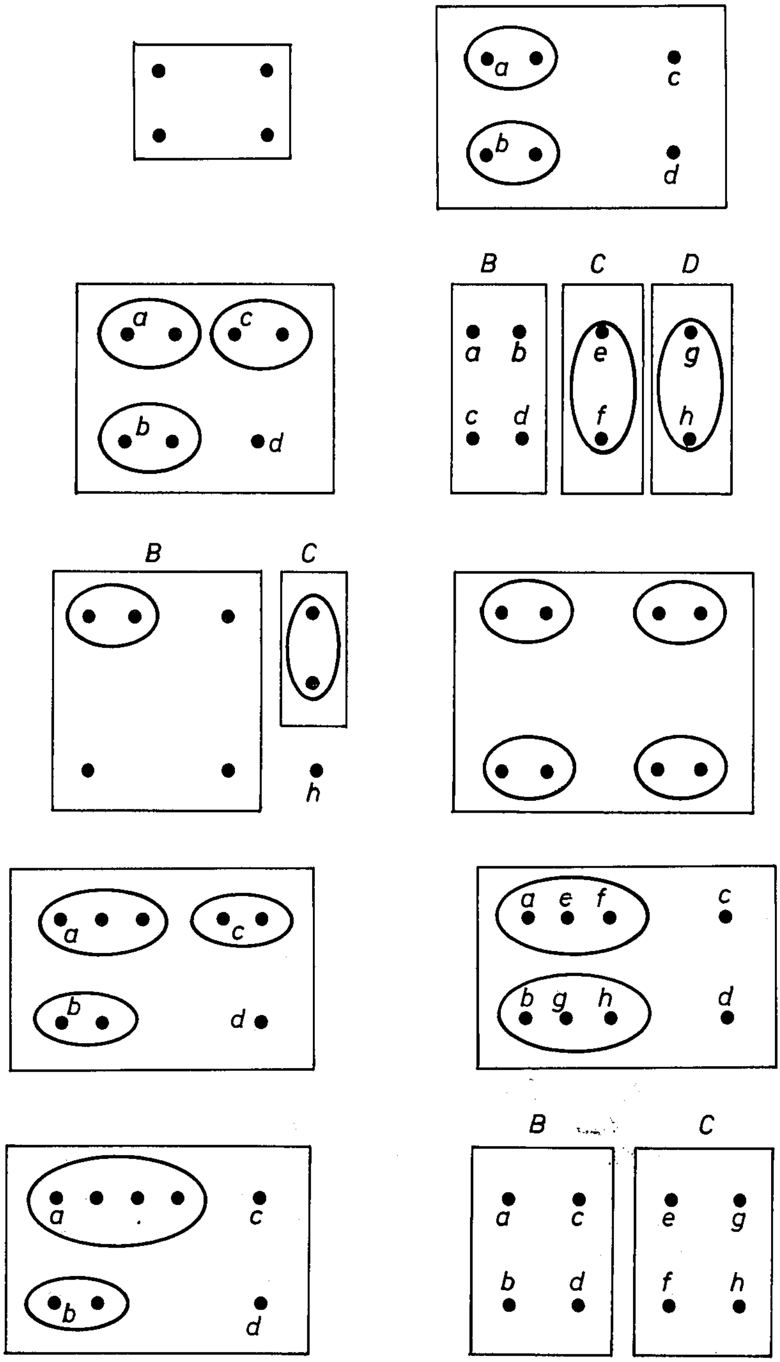
In virtue of Lemma 1 we have to prove our theorem only for those cases when L includes a diamond $M_3=\{\omega,\alpha,\beta,\gamma,\iota\}$. L can include more than one diamond but $M_3=\{\omega,\alpha,\beta,\gamma,\iota\}$ will always denote a fixed diamond for which ω is minimal. It is well-known in the theory of modular lattices that if a modular lattice M has a diamond whose bottom is $x \in M$ then there is an interval $[x, y]$ of length two which includes a diamond. (Having no simple reference at hand we refer to the far more general Freese [1, Thm. 1.7].) Therefore we always assume that our fixed diamond M_3 with minimal ω also satisfies $\omega -\!\!< \alpha -\!\!< \iota$, $\omega -\!\!< \beta -\!\!< \iota$ and $\omega -\!\!< \gamma -\!\!< \iota$. By Lemma 6 we do not have too many possibilities for M_3. Moreover, if Lemma 4 or Lemma 5 applies for ω and/or ι then we are done. Now it is easy to check that we are left with ten cases only; they are depicted on Figs. 1–10. On these figures, the nontrivial ι-blocks are denoted by rectangles while the nontrivial ω-blocks, if there is any, are encircled. When some or all of the elements of A are labelled, we always assume that $(a, d), (b, c) \in \alpha$, $(b, d), (a, c) \in \beta$ and $(c, d), (a, b) \in \gamma$; this convention generally determines α, β and γ in virtue of Lemma 6. Sometimes ι-blocks are labelled with capital letters.

In Case 1 (cf. Fig. 1) we can equip A with an Abelian group structure so that A be of exponent two and $\mathrm{Con}(A) = L$. Then $p(x, y, z) = x + y + z$ is a commutative Mal'tsev function compatible with L.

In Cases 2, 3, 7, 8 and 9 we are going to show that for any other diamond $\{\omega', \alpha', \beta', \gamma', \iota'\}$ in L we have $\omega \leqq \omega'$. (Then Lemma 1 is applicable with $\mu = \omega$.) Suppose this is not the case, i.e., $\omega \| \omega'$. We intend to show that ω' must have less than four blocks, which contradicts Lemma 6. Take an $(x, y) \in \omega' \backslash \omega$. Using $\mathrm{SP}(\gamma, \omega')$ or $\mathrm{SP}(\beta, \omega')$ we may assume that $x = d$. If $y \in [a]\omega$ then $\mathrm{SP}(\omega, \omega')$ yields $([a]\omega \cup \{d\})^2 \subseteqq \omega'$ and, by using $\mathrm{SP}(\beta, \omega')$, we can see that ω' has at most $|[c]\omega| \leqq 2$ further blocks beside $[a]\omega'$. Similarly, if $y \in [b]\omega$ then $\mathrm{SP}(\omega, \omega')$ yields $([b]\omega \cup \{d\})^2 \subseteqq \omega'$ and, by $\mathrm{SP}(\gamma, \omega')$, ω' has at most $|[c]\omega| + 1 \leqq 3$ blocks. Now suppose $x \in [c]\omega$. Then, by $\mathrm{SP}(\omega, \omega')$, $\{d\} \cup [c]\omega \subseteqq [d]\omega'$. If $|[a]\omega| < 3$ or $|[b]\omega| < 3$ then, by $\mathrm{SP}(\beta, \omega')$, ω' has at most three blocks. Therefore ω' may have four blocks only in Case 8 and, apart from labelling, these blocks are $\{a, b\}$, $\{e, g\}$, $\{f, h\}$ and $\{c, d\}$. By Lemma 6, $\rho = abeg; fhcd \in [\omega', \iota'] \subseteqq L$. (Here and often in the sequel an equivalence relation is denoted by the list of its nontrivial blocks separated by semicolons.) Hence $\mathrm{SP}(\rho, \omega)$ leads to a contradiction.

To settle Case 4, assume that ι is not semicentral. Then there is a $\rho \in L \backslash \{1\}$ such that $(x, y) \in \rho \backslash \iota$. If $\rho \subseteqq B^2 \cup (C \cup D)^2$ then Lemma 2 applies for $\iota + \rho = abcd; efgh$, which is semicentral. Indeed, if we had, e.g., $(a, e) \in \nu \backslash (\iota + \rho)$ for some $\nu \in L \backslash \{1\}$ then $\mathrm{SP}(\omega, \nu)$ would give $[a]\nu \supseteqq \{a, e, f\}$, $\mathrm{SP}(\iota, \nu)$ would yield $[a]\nu \supseteqq B \cup C$ and $\mathrm{SP}(\nu, \rho)$ would lead to a contradiction since $[g]\rho \cap C \neq \emptyset$ and $[h]\rho \cap C \neq \emptyset$ by $\mathrm{SP}(\omega, \rho)$. Therefore $(x, y) = (a, e) \in \rho$ can be assumed. Then $[a]\rho \supseteqq B \cup C$ like in case of ν before. Hence $[a]\rho = B \cup C$ as otherwise $\mathrm{SP}(\iota, \rho)$ would lead to $\rho = 1$. Now either Lemma 4 applies for ρ or Lemma 5 applies for $\omega < \rho$.

The treatment for Case 5 starts with assuming that ι is not semicentral. Then $\rho \circ \iota \neq \rho \cup \iota$ for some $\rho \in L \backslash \{0, 1\}$. If $[h]\rho = \{h\}$ then $B \cup C$ is the only nontrivial block of $\rho + \iota$ and Lemma 4 applies. Observe that $[h]\rho \cap$



Figs. 1–10

$\cap B \neq \emptyset$ implies $(B \cup \{h\})^2 \subseteqq \rho$ and $[h]\rho \cap C \neq \emptyset$ implies $(C \cup \{h\})^2 \subseteqq \rho$ by $\mathrm{SP}(\iota, \rho)$, but only one of these two possibilities can occur as $\rho \neq 1$. Therefore if $[h]\rho \neq \{h\}$ then Lemma 5 applies for ι and $\iota + \rho$.

In Case 6 we may assume by Lemma 1 that there exists another diamond $M_3' = \{\omega', \alpha', \beta', \gamma', \iota'\}$ with $\omega \nleqq \omega'$. We choose this M_3' so that ω' be minimal. Like in case of M_3 we may assume that $\omega' -\!\!< \alpha' -\!\!< \iota'$, $\omega' -\!\!< \beta' -\!\!< \iota'$ and $\omega' -\!\!< \gamma' -\!\!< \iota'$. Since $\omega' \| \omega$ and the previous cases have been handled, we may suppose that ω' is also of pattern $2+2+2+2$. As $\omega' \| \omega$, they can have 0, 1 or 2 blocks in common. However, if they had exactly one block in common then Lemma 4 would apply to $\omega' \cap \omega$; if they had two blocks, say $\{a, e\}$ and $\{b, f\}$, in common then $\mathrm{SP}(\alpha, \omega')$ would lead to a contradiction. Therefore, by $\mathrm{SP}(\omega, \omega')$, we may assume that the situation is as depicted on Figure 11, where the horizontal lines indicate ω'. Since the role of α', β' and γ' is symmetric, we assume that $\alpha' = abcd; efgh$, $\beta' = abef; cdgh$ and $\gamma' = abgh; cdef$. Let $\mathbf{Z}_2 = \{0, 1\}$ denote the two-element Abelian group. We consider A the (support of) $\mathbf{Z}_2^3$ as indicated on Figure 11. Since $\mathrm{Con}\,(\mathbf{Z}_2^3)$ admits a commutative Mal'tsev function $p(x, y, z) = x + y + z$, it suffices to show that $L \subseteqq \mathrm{Con}\,(\mathbf{Z}_2^3)$. If $0 < \rho \leqq \omega$ for $\rho \in L$ then $\rho = \omega$ by $\mathrm{SP}(\alpha', \rho)$. I.e., ω is an atom in L. So is ω', for the role of M_3 and M_3' is symmetric. If $\rho \in L$ is in $[\omega, \iota] = [\omega, 1]$ or $[\omega', \iota'] = [\omega', 1]$ then $\rho \in \mathrm{Con}\,(\mathbf{Z}_2^3)$ by Lemma 6 and $M_3, M_3 \subseteqq \mathrm{Con}\,(\mathbf{Z}_2^3)$. Suppose $\rho \in L \backslash \{0\}$ but $\omega \nleqq \rho$, $\omega' \nleqq \rho$. Then $\rho \cap \omega = \rho \cap \cap \omega' = 0$. If $\rho \leqq \omega + \omega'$ then a standard argument with the height function of L yields that $\{0, \omega, \omega', \rho, \omega + \omega'\}$ is a diamond, which contradicts the minimality of ω. Hence $\rho \nleqq \omega + \omega'$, whence $x \rho y$ holds for some $x \in \{a, b, e, f\}$ and $y \in \in \{c, d, g, h\}$. We can suppose $x = a$ by $\mathrm{SP}(\omega, \rho)$ and $\mathrm{SP}(\omega', \rho)$. Since the possibilities $a\rho d$, $a\rho c$, $a\rho g$ and $a\rho h$ are quite analogous, we detail the case $a\rho d$ only. Then using $\mathrm{SP}(\omega, \rho)$ and $\mathrm{SP}(\omega', \rho)$ we derive $\rho \supseteqq ad; bc; fg; eh$. If $\rho = ad; bc; fg; eh$ then $\rho \in \mathrm{Con}\,(\mathbf{Z}_2^3)$. So suppose $\rho \supset ad; bc; fg; eh$. Since $\rho \cap \cap \omega = \rho \cap \omega' = 0$, it follows either $a\rho f$ or $b\rho e$. By $\mathrm{SP}(\omega, \rho)$ both hold. Hence $\rho \supseteqq adfg; bceh$. Since $\rho \neq 1$, $\rho = adfg; bceh \in \mathrm{Con}\,(\mathbf{Z}_2^3)$.

In Case 10, the restriction map to any block of ι is injective, for it does not collapse $\omega = 0$ and ι. Therefore $\alpha = ad; bc; eh; fg$, $\beta = bd; ac; eg; fh$ and $\gamma = cd; ab; ef; gh$ can be assumed. We consider A as $\mathbf{Z}_2^3$ exactly the same way as before. We intend to show $L \subseteqq \mathrm{Con}\,(\mathbf{Z}_2^3)$. Evidently, $M_3 = \{0, \alpha, \beta, \gamma, \iota\} \subseteqq \mathrm{Con}\,(\mathbf{Z}_2^3)$. To show $[0, \iota] = M_3$ assume that $0 < \rho < \iota$, $\rho \in L \backslash M_3$. Applying Lemma 6 to $\{\mu|_B : \mu \in [0, \iota]\}$ and $\{\mu|_C : \mu \in [0, \iota]\}$ we derive that the restriction of ρ to either block of ι coincides with the restriction of a member of M_3. E.g., $\rho|_B = \alpha|_B$ but $\rho|_C \neq \alpha|_C$. Then $\rho|_C \neq \iota|_C$ implies $0 < \rho \cap \alpha < \alpha$ while $\rho|_C = \iota|_C$ yields $\alpha < \rho < \iota$, both contradicting $0 -\!\!< \alpha -\!\!< \iota$. Having seen that $[0, \iota] \subseteqq \mathrm{Con}\,(\mathbf{Z}_2^3)$ let us assume that $\rho \nleqq \iota$, $\rho \in L \backslash \{1\}$. Then, e.g., $a\rho e$. Now $\mathrm{SP}(\gamma, \rho)$ gives $b\rho f$ and $\mathrm{SP}(\alpha, \rho)$ gives $ae; bf; cg; dh \subseteqq \rho$. If we have equality then $\rho \in \mathrm{Con}\,(\mathbf{Z}_2^3)$. If $ae; bf; cg; dh \subset \rho$ then $\rho \cap \alpha \neq 0$ or $\rho \cap \beta \neq 0$ or $\rho \cap \gamma \neq 0$. E.g., suppose $\rho \cap \alpha \neq 0$. As α is an atom, $\rho \geqq \alpha$. Hence

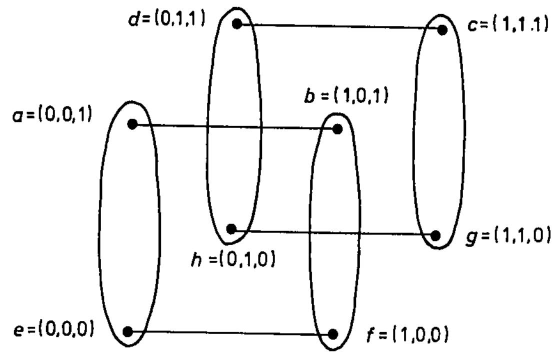


Fig. 11

$\rho \geqq aedh; bfcg$. I.e., $\rho = 1$ or $\rho = aedh; bfcg$, whence $\rho \in \mathrm{Con}(\mathbf{Z}_2^3)$. Q.e.d.

REFERENCES

[1] FREESE, R., The variety of modular lattices is not generated by its finite members, *Trans. Amer. Math. Soc.* **255** (1979), 277–300. *MR* **81g**:06003

[2] GRÄTZER, G., *General lattice theory*, Pure and Applied Mathematics, 75, Academic Press, Inc., New York – London, 1978, *MR* **80c**:06001b

[3] GUMM, H.-P., Is there a Mal'cev theory for single algebras?, *Algebra Universalis* **8** (1978), 320–329. *MR* **57** #12343

[4] GUMM, H.-P., Algebras in permutable varieties: geometrical properties of affine algebras, *Algebra Universalis* **9** (1979), 8–34. *MR* **80d**:08010

[5] MAL'CEV, A. I., On the general theory of algebraic systems, *Mat. Sbornik N. S.* **35** (77) (1954), 3–20 (in Russian). *MR* **16**–440

[6] PIXLEY, A. F., Completeness in arithmetical algebras, *Algebra Universalis* **2** (1972), 179–196. *MR* **48** #208

(Received March 12, 1990)

DEPARTMENT OF ALGEBRA AND GEOMETRY
PALACKY UNIVERSITY OLOMOUC
TOMKOVA 38
779 00 OLOMOUC
CZECH REPUBLIC

JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZETE
ARADI VÉRTANUK TERE 1
H–6720 SZEGED
HUNGARY