ON LATTICES WHOSE IDEALS ARE ALL TOLERANCE KERNELS

I. Chajda, G. Czédli and I. G. Rosenberg

Abstract. Lattices L with 0 are investigated such that each ideal of L is of the form $\{x\colon \langle x, 0\rangle \in \tau\}$ for some tolerance relation τ. We show that L has this property iff for any $b \in L$ and every unary lattice polynomial $p(x)$ with $p(0) = 0$ we have $p(b) \le b$. If, in addition, L is atomic then the ideal generated by any finite set of atoms in L is shown to be a Boolean sublattice of L.

It is known from Grätzer and Schmidt [5] and Hashimoto [6] that each ideal of a lattice L is the kernel $[0]_\Theta$ of some congruence $\Theta \in \mathrm{Con}(L)$ iff L is distributive. In this short note the analogous problem is investigated for *tolerances*, i.e. the compatible reflexive and symmetric relations, of L. The lattices we consider are always supposed to have a least element 0. Given a tolerance τ of a lattice L, in notation $\tau \in \mathrm{Tol}(L)$, the kernel $[0]_\tau := \{x \in L\colon \langle x, 0\rangle \in \tau\}$ is always an ideal. If each ideal of L is of this form then we say that the *ideals of L are tolerance kernels* or, in other words, that L has *tolerance determined ideals*. This property will be abbreviated by (TDI). Since $\mathrm{Con}(L) \subseteq \mathrm{Tol}(L)$, every distributive lattice has (TDI). The goal of the paper is to show that while the upper part of L with (TDI) can be arbitrary, the neighborhood of 0 shares many properties with distributive lattices.

The meet resp. join of a and b in a lattice will be denoted by ab resp. $a + b$. For general properties of lattices and lattice tolerances the reader is referred to Grätzer's book [4] and [1,2]. What we mention here is that (i) $\tau(a, b)$, the principal tolerance generated by $\langle a, b\rangle$, equals $\tau(ab, a+b)$; (ii) for every $\tau \in \mathrm{Tol}(L)$ the blocks of τ, i.e. the inclusion maximal subsets $X \subseteq L$ with $X^2 \subseteq \tau$, are convex sublattices of L; and (iii) $\langle a, b\rangle \in \tau$ iff there is a τ-block including $\{a, b\}$. First we show

Lemma 1. *If every principal ideal of L is a tolerance kernel then all ideals of L are tolerance kernels.*

Proof. For an ideal J of L let τ denote the tolerance generated by J^2. Clearly, $J \subseteq [0]_\tau$. To show the reverse inclusion, let $a \in [0]_\tau$. Since τ is the join of the principal tolerances $\tau(0, b)$, $b \in J$, and $\mathrm{Tol}(L)$ is an algebraic lattice, there are finitely many $b_i \in J$ such that $\langle 0, a\rangle \in \tau(0, b_1) + \ldots + \tau(0, b_n)$. For $c := b_1 + \ldots + b_n \in J$ we have

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1991 *Mathematics Subject Classification.* Primary 06B05, Secondary 06B10, 06B15.

Key words and phrases. Lattice, 0, tolerance, kernel, ideal, distributivity.

The financial support from NATO Collaborative Research Grant LG 930 302 to the first and third author is gratefully acknowledged.

The second author is grateful to the Hungarian National Foundation for Scientific Research (OTKA) grant no. T7442

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

$\tau(0, b_i) \subseteq \tau(0, c)$, whence $\langle 0, a\rangle \in \tau(0, c)$. By the assumption, the kernel of $\tau(0, c)$ is $(c]$. Hence we infer $a \in (c] \subseteq J$. $\square$

Characterizing the class of lattices with (TDI) seems to be difficult. The following assertion is only a partial solution, for describing all unary polynomials is not essentially easier than describing all tolerances. Yet, the following theorem will be quite useful not only when we present some examples of lattices without (TDI) but also in some proofs.

Theorem 1. *Let L be a lattice with 0. Then L has tolerance determined ideals if and only if for each $b \in L$ and every unary lattice polynomial $p(x)$ with $p(0) = 0$ we have $p(b) \leq b$.*

When we consider a *class* of lattices then it is useful to work with terms rather than polynomials. On the other hand, knowing more about the form of p in Theorem 1 could be useful. Thus, the following assertion is worth formulating.

Corollary 1. *For any lattice L with 0 the following three conditions are equivalent:*

(i) *L has tolerance determined ideals,*
(ii) *for all $n > 1$, $b \in L$ and unary lattice polynomials $p_1, \dots, p_n$ of L, if $p_i(b) > b$ for all i and $p_1(0) \dots p_n(0) = 0$ then $p_1(b) \dots p_n(b) = b$,*
(iii) *for every $n \geq 1$ and each lattice term $\pi(x_1, \dots, x_n)$ the Horn sentence*

$$(\forall x_1, \dots, x_n, y) \quad \Big(\pi(x_1, \dots, x_n) = 0 \Longrightarrow \pi(x_1 + y, \dots, x_n + y) = y\Big)$$

holds in L.

Proof of Theorem 1. Suppose the (TDI) property. Since $\langle 0, b\rangle \in \tau(0, b)$ implies $\langle 0, p(b)\rangle \in \tau(0, b)$ and the kernel of $\tau(0, b)$ is $(b]$, we infer $p(b) \in (b]$, i.e. $p(b) \leq b$.

In order to show the converse, in virtue of Lemma 1 it suffices to show that if $\langle 0, a\rangle \in \tau(0, b)$ then $a \leq b$. Suppose $\langle 0, a\rangle \in \tau(0, b)$. Since $\tau(0, b)$ is the sublattice of L^2 generated by $\{\langle 0, b\rangle, \langle b, 0\rangle\} \cup \{\langle c, c\rangle \colon c \in L\}$, there are $c_3, \dots, c_n \in L$ and an n-ary lattice term g with

$$g(\langle 0, b\rangle, \langle b, 0\rangle, \langle c_3, c_3\rangle, \dots, \langle c_n, c_n\rangle) = \langle 0, a\rangle.$$

Considering the polynomial $p(x) := g(x, 0, c_3, \dots, c_n)$ we have $p(0) = g(0, 0, c_3, \dots, c_n) \leq g(0, b, c_3, \dots, c_n) = 0$. Hence $a = p(b) \leq b$. $\square$

Proof of Corollary 1. The implication (i) $\Longrightarrow$ (ii) comes easily from Theorem 1, for we can consider the polynomial $p(x) = p_1(x) \dots p_n(x)$.

To show (ii) $\Longrightarrow$ (iii) suppose (ii) holds but (iii) does not. Let $\pi(x_1, \dots, x_n)$ be a lattice term of the least possible length such that $\pi(a_1, \dots, a_n) = 0$ and $\pi(a_1 + b, \dots, a_n + b) \neq b$ for some $a_1, \dots, a_n, b \in L$. Since $\pi(a_1 + b, \dots, a_n + b) \geq \pi(b, \dots, b) = b$, we have

$$\pi(a_1 + b, \dots, a_n + b) > b. \tag{1}$$

Clearly, π is not a variable. Suppose first that π is of the form

$$\pi_1(x_1, \dots, x_n) + \pi_2(x_1, \dots, x_n).$$

Then $\pi_i(a_1, \dots, a_n) \leq \pi(a_1, \dots, a_n) = 0$, for $i = 1, 2$. Since the π_i are shorter than π we conclude $\pi(a_1+b, \dots, a_n+b) = \pi_1(a_1+b, \dots, a_n+b)+\pi_2(a_1+b, \dots, a_n+

$b) = b+b = b$, contradicting (1). Now let π be of the form $\pi_1(x_1, \dots, x_n)\pi_2(x_1, \dots, x_n)$ and consider the polynomials $p_i(y) = \pi_i(a_1 + y, \dots, a_n + y)$ $(i = 1, 2)$. Then $p_1(0)p_2(0) = \pi(a_1, \dots, a_n) = 0$. If we had $p_i(b) = b$ for some i then $\pi(a_1 + b, \dots, a_n + b) = p_1(b)p_2(b) \leq b$ would contradict (1). Hence we obtain from $p_i(b) = \pi_i(a_1 + b, \dots, a_n + b) \geq \pi_i(b, \dots, b) = b$ that $p_i(b) > b$ for $i = 1, 2$. Therefore (ii) applies and yields $\pi(a_1 + b, \dots, a_n + b) = p_1(b)p_2(b) = b$, which contradicts (1). This proves (ii) $\Longrightarrow$ (iii).

To show (iii) $\Longrightarrow$ (i) let $b \in L$ and let $p(y)$ be a polynomial with $p(0) = 0$. Then $p(y) = \pi(y, a_2, \dots, a_n)$ for some lattice term π and elements $a_2, \dots, a_n \in L$. Since $\pi(0, a_2, \dots, a_n) = p(0) = 0$, we infer from (iii) that $p(b) = \pi(b, a_2, \dots, a_n) \leq \pi(0 + b, a_2 + b, \dots, a_n + b) = b$, whence (i) follows by Theorem 1. □

Example 1. The well-known five-element non-distributive lattices $N_5 = \{0, b, c_1, c_2, 1\}$ with $0 < b < c_2 < 1 > c_1 > 0$ and $M_3 = \{0, b, c_1, c_2, 1\}$ with incomparable b, c_1, c_2, and L_1 on Figure 1 do not have (TDI); this is witnessed by the respective polynomials $p(x) := (x + c_1)c_2$, $p(x) := (x + c_1)c_2$, and $p(x) := ((u + c_3)c_4 + c_5)c_6$ where $u := (x + c_1)(x + c_2)$.

Let S be a sublattice of L. If $0 = 0_L$ belongs to S then S is called a *0-sublattice* of L. It is clear from Corollary 1 that, considering 0 as a fundamental operation, *the class of lattices with (TDI) is a quasivariety.* In particular, if S is a 0-sublattice of L and S does not have (TDI) then L does not have it either. Combining this with Example 1 we have managed to capture an evident 'distributivity near 0' like property of lattices with (TDI) in the following

Observation 1. *If L has (TDI) then neither N_5 nor M_3 is a 0-sublattice of L.*

The lattice L_1 in Figure 1 indicates that the converse of Observation 1 does not hold. Now it is easy to derive a connection among distributivity, (TDI) and the condition in Theorem 1.

Observation 2. *For any lattice L with 0 the following three conditions are equivalent:*

(a) *L is distributive,*
(b) *if $a < b \in L$ and $p(x)$ is a lattice polynomial such that $p(a) = a$ then $p(b) \leq b$,*
(c) *every principal dual ideal of L has (TDI).*

Proof. (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (a) come from Theorem 1 and Observation 1, respectively. Suppose (a) and let p be a lattice polynomial with $p(a) = a$. Then $p(x) = \pi(x, c_2, \dots, c_n)$ for some term π. An easy induction on the length of π shows that the identity $\pi(y_1, \dots, y_n) + z = \pi(y_1 + z, \dots, y_n + z)$ holds in L. Therefore, for the polynomial $\hat{p}(x) := \pi(x, c_2 + a, \dots, c_n + a)$, we obtain $\hat{p}(a) = \hat{p}(a + a) = \pi(a + a, c_2 + a, \dots, c_n + a) = p(a) + a = a + a = a$. By distributivity $[a)$ has (TDI), and we infer $\hat{p}(b) \leq b$ from Theorem 1. Therefore $p(b) \leq \hat{p}(b) \leq b$, proving (b). □

The quasivariety of lattices with 0 as a fundamental operation and (TDI) is very large but it is not a variety; the latter follows from the following observation because L_2 is a homomorphic image of L.

Observation 3. *Suppose L_1 is a bounded lattice with (TDI) and L_2 is an arbitrary lattice with 0. Let L be the Hall–Dilworth gluing (cf. [3] or Grätzer [4, p. 31, Ex.*

20-22]) of L_1 and L_2 which identifies the dual ideal $\{1_1\}$ of L_1 with the ideal $\{0_2\}$ of L_2. Then L has (TDI), too.

Proof. Let $b \in L$. If $b \in L_2$ then clearly (or cf. [4]) there is a tolerance of L with exactly two blocks: $(b]$ and $[0_2)$. Hence $(b]$ is a tolerance kernel. Suppose $b \in L_1$. By assumption, there is a $\tau \in \mathrm{Tol}(L_1)$ whose kernel is $(b]$. Now $\rho := \tau \cup \{\langle x, x\rangle\colon\ x \in L_2\} \in \mathrm{Tol}(L)$ and $(b]$ is the kernel of ρ. Hence the observation follows from Lemma 1. $\square$

The lattice L_1 in Example 1, where the lower component is the eight element Boolean lattice, indicates that Observation 3 is not valid for arbitrary Hall–Dilworth gluings.

Let S_1, S_2, …, S_n be 0-sublattices of L. We say that these 0-sublattices are *independent* if

$$s_j \sum_{\substack{i=1 \\ i\neq j}}^{n} s_i = 0$$

for all $s_1 \in S_1$, …, $s_n \in S_n$ and $1 \le j \le n$. Note that when $\{a_1, \dots, a_n\}$ is an independent set of elements, cf. Grätzer [4, p. 167], then the intervals $[0, a_i]$ are independent 0-sublattices in our sense. For distributive lattices the subsequent statements become more or less trivial, but now we have to work a bit more.

Proposition 1. *Let L be a lattice with (TDI) and let A_1, A_2, …, A_n be independent 0-sublattices of L such that*

$$a_1a_1' + a_2a_2' + \ldots + a_na_n' = (a_1 + a_2 + \ldots a_n)(a_1' + a_2' + \ldots a_n') \tag{2}$$

for all $a_1, a_1' \in A_1$, …, $a_n, a_n' \in A_n$. Then $S = \{a_1 + \ldots + a_n\colon\ a_1 \in A_1, \dots, a_n \in A_n\}$ is the sublattice generated by $A_1 \cup \ldots \cup A_n$ and $S \cong A_1 \times \ldots \times A_n$.

Proof. Consider the map $\psi\colon\ A_1 \times \ldots \times A_n \to S$, $\langle a_1, \dots, a_n\rangle \mapsto a_1 + \ldots + a_n$. Clearly, ψ is surjective and join-preserving. By (2), it is also meet-preserving. Hence S, as a homomorphic image of $A_1 \times \ldots \times A_n$, is a sublattice. Clearly, S is generated by $A_1 \cup \ldots \cup A_n$. Now we show that ψ is injective. Let $a_i, a_i' \in A_i$ $(i = 1, 2, \dots, n)$ satisfy $a_1 + \ldots + a_n = a_1' + \ldots + a_n'$. Set $u = a_1 \ldots + a_n$ and notice that $a_1 + a_1' \le u$. Consider the lattice polynomial $p(x) = (a_1 + a_1')(x + a_2 + a_3 + \ldots + a_n)$. The independence of A_1, …, A_n yields $p(0) = 0$. Hence Theorem 1 applies and we obtain $a_1 \ge p(a_1) = (a_1 + a_1')u = a_1 + a_1'$. This gives $a_1' \le a_1$. By symmetry $a_1 \le a_1'$ and $a_1 = a_1'$. By the same token, $a_i = a_i'$ for all i, and ψ is injective. $\square$

Corollary 2. *Suppose L is a lattice with (TDI). Let A_1 and A_2 be independent 0-sublattices of L such that $A_2 = \{0, b\}$ where $b > 0$. If for all $a_1, a_1' \in A_1$*

$$a_1a_1' + b = (a_1 + b)(a_1' + b), \tag{3}$$

then $S = \{a_1 + a_2\colon\ a_1 \in A_1,\ a_2 \in A_2\}$ is the sublattice generated by $A_1 \cup A_2$ and $S \cong A_1 \times A_2$.

Proof. We derive (2) from (3). Apart from symmetry the only nontrivial case of (2) is $a_1a_1' + 0b \overset{?}{=} (a_1 + 0)(a_1' + b)$, i.e. $a_1a_1' \overset{?}{\ge} a_1(a_1' + b)$. Set $p(x) = (a_1 + x)(x + b)$. From $p(0) = 0$ and Theorem 1 we obtain $a_1' \ge p(a_1') = (a_1 + a_1')(a_1' + b)$. Hence $a_1a_1' \ge a_1(a_1 + a_1')(a_1' + b) = a_1(a_1' + b)$, proving the inequality in question. $\square$

Example 2. The lattices L_2, L_3 and L_4 on Figures 2–4 have (TDI).

The long but easy consideration which, based on [2], yields (TDI) for L_2, L_3 and L_4 will be omitted. $A_1 = (c]$ on Figure 2 indicates that (3) in Corollary 2 cannot be removed.

Proposition 2. *Let L be a lattice with (TDI). If C_1, C_2, $\dots$, C_n are chains in L such that $0 \in C_1 \cap \ldots \cap C_n$ and C_1, C_2, $\dots$, C_n are independent then $S = \{a_1 \ldots + a_n\colon\ a_1 \in C_1, \ldots, a_n \in C_n\}$ is the sublattice generated by $C_1 \cup \ldots \cup C_n$ and $S \cong C_1 \times \ldots \times C_n$.*

Proof. First we show that for all $I, J \subseteq \{1, 2, \ldots, n\}$ and $a_\ell \in C_\ell$ $\;(\ell \in I \cup J)$

$$I \cap J = \emptyset \quad\Longrightarrow\quad \Big(\sum_{i\in I} a_i\Big)\Big(\sum_{j \in J} a_j\Big) = 0. \tag{4}$$

Here the empty sum means 0, so (4) is evident when I or J is empty. We prove (4) via induction on $|I \cup J|$. For $|I \cup J| = 2$, (4) follows from the independence of C_1, $\dots$, C_n.

Now let $2 < k \leq n$ and suppose that (4) holds for any choice of fewer than k elements. Let I and J be disjoint nonempty subsets of $\{1, \ldots, n\}$ with $|I \cup J| = k$. Since the role of I and J is symmetric, we may assume that $|I| \geq 2$. Fix a $t \in I$ and consider the polynomial $p(x) = \Big(x + \sum_{i \in I\setminus\{t\}} a_i\Big) \sum_{j\in J} a_j$. By the induction hypothesis we have $p(0) = 0$. From Theorem 1 we infer $a_t \geq p(a_t) = \Big(\sum_{i\in I} a_i\Big)\Big(\sum_{j\in J} a_j\Big)$. Hence $\Big(\sum_{i\in I} a_i\Big)\Big(\sum_{j\in J} a_j\Big) = a_t\Big(\sum_{i\in I} a_i\Big)\Big(\sum_{j\in J} a_j\Big)$, which is 0 since, by the induction hypothesis, $a_t\Big(\sum_{j\in J} a_j\Big) = 0$. This proves (4).

Armed with (4), now we show (2). Assume that $a_1, a_1' \in C_1$, $\dots$, $a_n, a_n' \in C_n$. Let $I = \{i\colon\ a_i \geq a_i',\ \ 1 \leq i \leq n\}$ and $J = \{1, \ldots, n\} \setminus I$. Set

$$p(x) = \Big(x + \sum_{i \in I}(a_i + a_i')\Big)\ \Big(x + \sum_{j\in J}(a_j + a_j')\Big).$$

We obtain from (4) that $p(0) = 0$. Denoting $a_1a_1' + \ldots + a_na_n' + \sum_{i\in I}(a_i + a_i')$ resp. $a_1a_1' + \ldots + a_na_n' + \sum_{j\in J}(a_j + a_j')$ by u resp. v, we obtain from Theorem 1 that $a_1a_1' + \ldots + a_na_n' \geq p(a_1a_1' + \ldots + a_na_n') = uv$. Since $a_ia_i' + a_i + a_i' = a_i$ for $i \in I$ and $a_ia_i' = a_i$ for $i \notin I$, we obtain $u = a_1 + \ldots + a_n$. Similarly, $a_ia_i' + a_i + a_i' = a_i'$ for $i \in J$ and $a_ia_i' = a_i'$ for $i \notin J$, and we get $v = a_1' + \ldots + a_n'$. Thus $a_1a_1' + \ldots + a_na_n' \geq (a_1 + \ldots + a_n)(a_1' + \ldots + a_n')$. The converse inequality in (2), i.e. $\leq$, is immediate. Hence the statement follows from Proposition 1. $\square$

For two-element chains consisting of atoms and 0 much more can be stated. A lattice L is called *atomic* if for each $a \in L \setminus \{0\}$ the ideal $(a]$ contains an atom. For example, every finite lattice is atomic.

Theorem 2. *Let L be a lattice whose ideals are tolerance kernels and let $a_1, a_2, \ldots, a_n$ be distinct atoms of L. If the principal ideal $P = (a_1 + \ldots + a_n]$ is atomic then P is the 2^n-element Boolean lattice.*

Proof. Let $\mathrm{H}(n)$ denote the statement of the theorem and let $[a_1, \ldots, a_n]$ denote the sublattice generated by $\{a_1, \ldots, a_n\}$. We will show $\mathrm{H}(n)$ via induction on n.

Since H(1) is evident, let $n \geq 2$ and assume H(k) for all $k < n$. Let c denote $a_n(a_1 + \ldots + a_{n-1})$. Clearly, c is either a_n or 0, and c belongs to the ideal $(a_1 + \ldots + a_{n-1}]$. By H($n-1$), $(a_1 + \ldots + a_{n-1}]$ has exactly $n-1$ atoms, so these atoms are $a_1, \ldots, a_{n-1}$. Hence $c = 0$. Since the role of $a_1, \ldots, a_n$ in the statement is symmetric, we obtain that the chains $C_i = \{0, a_i\} = (a_i]$, $i = 1, \ldots, n$, are independent. From Proposition 2 we obtain that $[a_1, \ldots, a_n]$ is isomorphic to the n-th direct power of the two-element lattice. Therefore $[a_1, \ldots, a_n]$ is the 2^n-element Boolean lattice.

Now we claim that for any atom $b \in L$

$$b \leq a_1 + \ldots + a_n \Longrightarrow b \in \{a_1, \ldots, a_n\}. \tag{5}$$

If $b \leq a_1 + \ldots + a_{n-1}$ then (5) holds by H($n-1$). Thus let $b \not\leq a_1 + \ldots + a_{n-1}$ and consider the polynomial $p(x) = b(a_1 + \ldots + a_{n-1} + x)$. Clearly $p(0) = 0$ and so by Theorem 1 we obtain $a_n \geq p(a_n) = b(a_1 + \ldots + a_n) = b$. This proves (5).

Now we are in the position to show $[a_1, \ldots, a_n] = (a_1 + \ldots + a_n]$. The $\subseteq$ inclusion being trivial, assume that $0 < c < a_1 + \ldots + a_n$. Set $I := \{j\colon\ a_j \leq c,\ 1 \leq j \leq n\}$. Since $(a_1 + \ldots + a_n]$ is atomic, $b \leq c$ for some atom b. By (5) $b = a_j$ for some $1 \leq j \leq n$. Thus I is nonempty. Moreover, $I \neq \{1, \ldots, n\}$ since otherwise $c \geq a_1 + \ldots + a_n$. Choose the notation so that $I = \{1, \ldots, i\}$ and set $f = a_{i+1} + \ldots + a_n$. We claim that $cf = 0$. Indeed, $cf \in (a_{i+1} + \ldots + a_n]$ and from H($n-i$) we obtain $cf = \sum_{k \in K} a_k$ for some $K \subseteq \{i+1, \ldots, n\}$. Here $K = \emptyset$ because for each $k \in K$ we have both $c \not\geq a_k$ (due to $k > i$) and $c \geq a_k$ (due to $c \geq cf$). This proves the claim. Set $e = a_1 + \ldots + a_i$ and consider the polynomial $p(x) = c(x + f)$. Clearly $p(0) = 0$ and therefore from Theorem 1 we obtain $e \geq p(e) = c(a_1 + \ldots + a_n) = c$. By the definition of I also $c \geq a_1 + \ldots + a_i = e$ and so $c = e \in [a_1, \ldots, a_n]$.

Note that the converse of Theorem 2 does not hold, this is witnessed by L_1 on Figure 1. $C_1 = \{c_0, c_1, \ldots\}$ and $C_2 = \{d_0, d_1, \ldots\}$ on Figures 3 and 4 indicate that Theorem 2 cannot be generalized for chains with more that two elements.

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Algebra and Geometry, Palacký University Olomouc, Tomkova 39, 779 00 Olomouc, Czech Republic

E-mail address: chajda@risc.upol.cz

JATE Bolyai Institute, Aradi vértenúk tere 1, H-6720 Szeged, Hungary

E-mail address: czedli@math.u-szeged.hu

Math. and Stat., Université de Montréal, C.P. 6128, Succ. Centre-ville Montréal, Québec, Canada H3C 3J7

E-mail address: rosenb@ere.umontreal.ca