Kernels of tolerance relations

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Abstract. Algebras with 0 and their ideals in Gumm and Ursini's sense [11, 12] are considered. A variety $\mathcal{K}$ is called 0-*tolerance regular* if each tolerance relation α of any $A \in \mathcal{K}$ is uniquely determined by its kernel $[0]_\alpha = \{x \in A\colon \langle 0, x\rangle \in \alpha\}$. The main result, strengthening Agliano and Ursini [1], asserts that every 0-tolerance regular variety is congruence permutable. Tolerance kernels of single algebras are also considered.

Key words: ideal determined, variety, variety with 0, tolerance, kernel, ideal, congruence permutable.

Mathematics Subject Classification: Primary 08A30, Secondary 08B05, 08B99.

A part of this paper was written during the second and third authors' stay in Olomouc; they thank the Palacký University Olomouc for covering the expenses of their visit. The financial support from NATO Collaborative Research Grant LG 930 302 and that from NFSR (OTKA T7442) of Hungary are also gratefully acknowledged.

1. Introduction and basic definitions

Ideals of universal algebras were introduced by Ursini [12], cf. also Fichtner [9]. For definition, let $\mathcal{K}$ be a variety of algebras with a distinguished nullary operation 0 (or an equationally defined term 0) in its type. We say that $\mathcal{K}$ is a *variety with* 0; its members are called *algebras with* 0. In the sequel, $\mathcal{K}$ will always denote a variety with 0. Even without explicit mentioning all varieties and algebras in this paper are assumed to be with 0. A term $p(x_1, \ldots, x_m, y_1, \ldots, y_n)$ of $\mathcal{K}$ is called a $\mathcal{K}$-*ideal term* in the variables $y_1, \ldots, y_n$ if $\mathcal{K}$ satisfies the identity $p(x_1, \ldots, x_m, 0, \ldots, 0) \approx 0$. A nonvoid subset I of an algebra $A \in \mathcal{K}$ is called a $\mathcal{K}$-*ideal* of A if for every $\mathcal{K}$-ideal term $p(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in the last n variables and for all $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_n \in I$ we have $p(a_1, \ldots, a_m, b_1, \ldots, b_n) \in I$. When A does not belong to any specified variety, by a $\mathcal{K}$-*ideal term* resp. a $\mathcal{K}$-*ideal* we mean an $\mathbf{HSP}\{A\}$-ideal term resp. an $\mathbf{HSP}\{A\}$-ideal. $\mathbf{HSP}\{A\}$-ideal terms and $\mathbf{HSP}\{A\}$-ideals of A will also be called *ideal terms* and *ideals* even when A belongs to some variety $\mathcal{K}$. Note that, for $A \in \mathcal{K}$, every ideal of A is a $\mathcal{K}$-ideal of A. Notice that 0 belongs to every $\mathcal{K}$-ideal since the constant unary operation $c_0(y)$ with value 0 is a $\mathcal{K}$-ideal term in y. If $\mathcal{K}$ is the variety of all rings or lattices with zero, then $\mathcal{K}$-ideals are exactly the ideals in the usual sense. Following Agliano and Ursini [1], a nonempty subset C of an algebra A is called a *clot* if $0 \in C$ and for every term $q(x_1 \ldots, x_n)$ with $q(0, \ldots, 0) = 0$ and for all $c_1, \ldots, c_n \in C$ we have $q(c_1, \ldots, c_n) \in C$. For example, every ideal is a clot.

Given a compatible reflexive binary relation α of $A \in \mathcal{K}$ (i.e., a subalgebra of A^2 that includes the diagonal), the subset

$$[0]_\alpha = \{x \in A\colon\ \langle 0, x\rangle \in \alpha\}$$

is called the *kernel* of α. It is easy to see that $[0]_\alpha$ is an ideal of A. Kernels of congruences have been studied, e.g., in [1], [6], [11] and [12].

Recall that an algebra A is said to be *congruence permutable* if $\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in \mathrm{Con}(A)$. As usual, a variety $\mathcal{K}$ is said to have a property if all of its members have this property. If a property of single algebras includes ideals, then (even without explicit mentioning) the corresponding property for $\mathcal{K}$ includes $\mathcal{K}$-ideals instead of ideals, of course. A classical theorem of A. I. Mal'cev asserts that a variety $\mathcal{K}$ is congruence permutable iff there is a *Mal'cev term* in $\mathcal{K}$, i.e. a ternary term p such that the identities $p(x,x,y) \approx y$ and $p(x,y,y) \approx x$ hold in $\mathcal{K}$. If $[0]_{\alpha\circ\beta} = [0]_{\beta\circ\alpha}$ holds for all $\alpha, \beta \in \mathrm{Con}(A)$, then A is called *(congruence) permutable at* 0. When $[0]_\alpha = [0]_\beta$ implies $\alpha = \beta$ for any $\alpha, \beta \in \mathrm{Con}(A)$, A is called 0-*regular*. If $\alpha \mapsto [0]_\alpha$ is a bijection from $\mathrm{Con}(A)$ to the set of ideals of A, then A is said to be *ideal determined*. A famous theorem of Gumm and Ursini [11] asserts that a variety $\mathcal{K}$ is ideal determined iff $\mathcal{K}$ is permutable at 0 and 0-regular.

Motivated by this theorem, other compatible reflexive relations have also been studied from similar aspects, cf. e.g. [1] and [5]. Compatible reflexive symmetric binary relations are called *tolerances*; cf. [4] for basic facts about them. The tolerances of A form an algebraic lattice, which is denoted by $\mathrm{Tol}(A)$. If $\mathrm{Tol}(A) = \mathrm{Con}(A)$, then A is said to be *tolerance trivial*. A is called a 0-*tolerance regular algebra* if, for all $\alpha, \beta \in \mathrm{Tol}(A)$, the equality $[0]_\alpha = [0]_\beta$ implies $\alpha = \beta$. When $\alpha \mapsto [0]_\alpha$ is a bijection from $\mathrm{Tol}(A)$ resp. from the set of compatible reflexive relations of A to the set of ideals resp. clots of A, then A is called an *ideal tolerance-determined* resp. *clot determined algebra*. Agliano and Ursini have proved that every clot determined variety is congruence permutable, cf. [1, Thm. 2.7]. Notice that 0-tolerance regularity is a much weaker condition than being clot determined; first because "regular" is weaker than "determined", and secondly because it is a condition only on tolerances rather than all compatible reflexive relations. Hence the following theorem, the main achievement of the paper, seems to be an essential improvement of the above-mentioned result.

II. Results and proofs

Theorem 1. *If a variety with 0 is 0-tolerance regular, then it is congruence permutable.*

Proof. Let $\mathcal{K}$ be a 0-tolerance regular variety. For $A \in \mathcal{K}$ and $R \subseteq A^2$ the tolerance relation generated by R will be denoted by $T(R)$. As usual, we will write $T(a,b)$ instead of $T(\{\langle a,b\rangle\})$. Consider the free algebra $A = F_{\mathcal{K}}(x,y)$ with two free generators x and y. Set $\alpha = T(x,y)$, $I = [0]_\alpha$ and $\beta = T(\{0\} \times I)$. Observe that $\{0\} \times I \subseteq \alpha$ implies $\beta \subseteq \alpha$, so we obtain $I \subseteq [0]_\beta \subseteq [0]_\alpha = I$. The 0-tolerance regularity of $\mathcal{K}$ gives $\alpha = \beta$, whence

$\langle x,y\rangle \in \beta$. Now we need the following easy description of generated tolerances:

(1) $\langle a,b\rangle \in T(\{\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle\})$ iff there are $m\ge 0$, elements $e_1\dots,e_m$, and a $(2n+m)$ -ary term r such that $a=r(a_1,\dots,a_n,b_1,\dots,b_n,e_1,\dots,e_m)$ and $b=r(b_1,\dots,b_n,a_1,\dots,a_n,e_1,\dots,e_m)$.

Note that (1) is just Lemma 1.7 in [4]; the reader can also prove it directly. Since $\mathrm{Tol}(A)$ is an algebraic lattice, there is a finite subset $\{c_1(x,y),\dots,c_n(x,y)\}$ of I such that

$$\langle x,y\rangle \in T(\{0\}\times\{c_1(x,y),\dots,c_n(x,y)\}).$$

By (1) there is a $(2n+m)$-ary term r and there are binary terms e_i such that

$$x=r(0,\dots,0,c_1(x,y),\dots,c_n(x,y),e_1(x,y),\dots,e_m(x,y)),$$
$$y=r(c_1(x,y),\dots,c_n(x,y),0,\dots,0,e_1(x,y),\dots,e_m(x,y)).$$

For simplicity, let us consider the term $g(x_1,\,x_2,\dots,x_{2n+2})=r(x_1,\,x_2,\dots,x_{2n},\,e_1(x_{2n+1},\,x_{2n+2}),\dots,e_m(x_{2n+1},\,x_{2n+2}))$. Then we have

$$x=g(0,\dots,0,c_1(x,y),\dots,c_n(x,y),x,y),\qquad (2)$$
$$y=g(c_1(x,y),\dots,c_n(x,y),0,\dots,0,x,y).$$

We claim that the terms c_i satisfy

$$c_i(x,x)\approx 0 \qquad \text{for } i=1,\dots,n. \qquad (3)$$

Indeed, $\langle 0,c_i(x,y)\rangle \in \alpha = T(x,y)$. Hence, for each i, the description (1) provides us with $u_j(x,y)\in A$ and a term s such that $0=s(x,y,u_1(x,y),\ \dots,\ u_k(x,y))$ and $c_i(x,y)=s(y,x,u_1(x,y),\ \dots,\ u_k(x,y))$. Therefore, using the fact that equations for the free generators are valid identities in $\mathcal{K}$, we obtain

$$c_i(x,x)\approx s(x,x,u_1(x,x),\ \dots,\ u_k(x,x))\approx 0,$$

showing (3). Now define

$$p(x,y,z)=g(c_1(y,z),\dots,c_n(y,z),c_1(x,y),\dots,c_n(x,y),x,z).$$

From (2) and (3) we infer

$$p(x,x,y)\approx g(c_1(x,y),\dots,c_n(x,y),0,\dots,0,x,y)\approx y$$

and

$$p(x,y,y)\approx g(0,\dots,0,c_1(x,y),\dots,c_n(x,y),x,y)\approx x,$$

i.e., p is a Mal'cev term. Thus $\mathcal{K}$ is congruence permutable. $\diamond$

Corollary 2. *The following four conditions are equivalent for a variety $\mathcal{K}$ with 0.*

(a) $\mathcal{K}$ is 0-tolerance regular;

(b) $\mathcal{K}$ is ideal determined and congruence permutable;

(c) $\mathcal{K}$ is ideal determined and tolerance trivial;

and

(d) $\mathcal{K}$ is congruence permutable and 0-regular.

Proof. Since 0-regularity is an evident consequence of 0-tolerance regularity, the implication (a) $\Longrightarrow$ (d) follows from Theorem 1. By [2] or [4, Thm. 4.11], tolerance triviality and congruence permutability for varieties are equivalent conditions. This gives (d) $\Longrightarrow$ (a) and (b) $\iff$ (c). Since permutability at 0 trivially follows from congruence permutability, the mentioned result from Gumm and Ursini [11] yields (b) $\iff$ (d). $\diamond$

Note that there are known Mal'cev characterizations of the equivalent conditions of Corollary 2; indeed, [11] resp. Agliano and Ursini [1, Thm. 2.7] gives an appropriate Mal'cev condition equivalent to (d) resp. (b). Notice also that the five element non-modular lattice is 0-tolerance regular but not ideal determined. (Here and in the sequel, the description of lattice tolerances by their blocks, cf. [7] or [4, Corollary to Thm. 2.16] or [8], makes the verification of some examples easier.) Hence much less can be stated about tolerance kernels in case of single algebras than in case of varieties.

In the sequel, $\tau(a)$ will stand for $T(a,0)$, the tolerance generated by $\langle a,0\rangle$. Given an algebra A with 0, if for all $a,b\in A$ there exists a $c\in A$ with $\tau(c)=\tau(a)\circ\tau(b)=\tau(a)\vee\tau(b)$ (in $\mathrm{Tol}(A)$), then A is called *strongly* 0*-tolerance principal*. For example, using the results of [2] and [3], it is not too hard to show that distributive lattices with 0 are strongly 0-tolerance principal. To present an example of a different nature, let $C=\{0,a,1\}$ be a three element chain, and define $L=(C\times C)\cup\{b\}$ where $\langle 1,a\rangle\prec b\prec\langle 1,1\rangle$. Then L is not strongly 0-tolerance principal, for $\tau(\langle 1,a\rangle)\vee\tau(\langle 1,a\rangle)\neq\tau(\langle 1,a\rangle)\circ\tau(\langle 1,a\rangle)$. Finally, we formulate

Proposition 3. *Let A be a strongly 0-tolerance principal algebra. Then the following two conditions are equivalent:*

(i) every ideal of A is a congruence kernel;

(ii) every ideal of A is a tolerance kernel.

Proof. For $S\subseteq A$ let $I(S)$ denote the ideal generated by S. As usual, we will write $I(s_1,\dots,s_n)$ instead of $I(\{s_1,\dots,s_n\})$. Let us consider the condition

(iii) $I(s_1,\dots,s_n)=[0]_{\tau(s_1)\circ\dots\circ\tau(s_n)}$ holds for all $n>0$ and all $s_1,\dots,s_n\in A$.

Before showing that (i), (ii) and (iii) are equivalent, two easy properties of A are worth formulating. Firstly,

$(*)$ for all $a\in A$, $\tau(a)\in\mathrm{Con}(A)$;

indeed, for an appropriate $c\in A$, $\tau(c)=\tau(a)\vee\tau(a)=\tau(a)\circ\tau(a)$ gives the transitivity of $\tau(a)=\tau(c)$. Secondly, a straightforward induction shows that

(∗∗) for all $a_1, \ldots, a_n \in A$ there exists a $c \in A$ such that $\tau(c) = \tau(a_1) \circ \tau(a_2) \circ \ldots \circ \tau(a_n) = \tau(a_1) \vee \ldots \vee \tau(a_n)$ (in $\mathrm{Tol}(A)$).

The implication (i) $\Longrightarrow$ (ii) is trivial.

Suppose (ii) and let $s_1, \ldots, s_n \in A$. Then $I(s_1, \ldots, s_n) = [0]_\alpha$ for some $\alpha \in \mathrm{Tol}(A)$. From $s_i \in [0]_\alpha$ we conclude $\alpha \geq \tau(s_1) \vee \ldots \vee \tau(s_n)$ in $\mathrm{Tol}(A)$, whence $I(s_1, \ldots, s_n) \supseteq [0]_{\tau(s_1)\vee\ldots\vee\tau(s_n)} = [0]_{\tau(s_1)\circ\ldots\circ\tau(s_n)}$. The converse inclusion follows from $s_i \in [0]_{\tau(s_i)} \subseteq [0]_{\tau(s_1)\circ\ldots\circ\tau(s_n)}$. This proves (ii) $\Longrightarrow$ (iii).

Now suppose (iii). By (∗) and (∗∗), every finitely generated ideal is a congruence kernel. Let J be an arbitrary ideal of A, and let H denote the set ot all finite subsets of J. For $X \in H$ the ideal $I(X)$ is a congruence kernel, hence it is the kernel of the congruence Θ_X generated by $\{0\} \times I(X)$. Set $\Theta = \bigvee_{X \in H} \Theta_X$. Since J is the union of its finitely generated subideals, $J \subseteq [0]_\Theta$. Conversely, let $a \in [0]_\Theta$. Since the Θ_X $(X \in H)$ form a directed system, $\langle 0, a\rangle \in \Theta_X$ holds for some $X \in H$, and we obtain $a \in I(X) \subseteq J$. Hence $J = [0]_\Theta$, proving (iii) $\Longrightarrow$ (i). ◇

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