Shifting Lemma and shifting lattice identities

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Dedicated to the memory of György Pollák (1929–2001)

Abstract. Gumm [6] used the Shifting Lemma with high success in congruence modular varieties. Later, some analogous diagrammatic statements, including the Triangular Scheme from [1] were also investigated. The present paper deals with the purely lattice theoretic underlying reason for the validity of these lemmas. The shift of a lattice identity, a special Horn sentence, is introduced. To any lattice identity λ and to any variable y occurring in λ we introduce a Horn sentence $S(\lambda, y)$. When $S(\lambda, y)$ happens to be equivalent to λ, we call it a *shift* of λ. When λ has a shift then it gives rise to diagrammatic statements resembling the Shifting Lemma and the Triangular Scheme. Some known lattice identities will be shown to have a shift while some others have no shift.

Gumm [6] has shown that every congruence modular algebra A satisfies the following property, called Shifting Lemma: for any $\alpha, \beta, \gamma \in \operatorname{Con} A$ if $\alpha\beta \leq \gamma$, $(x, y), (z, u) \in \alpha$, $(x, z), (y, u) \in \beta$ and $(z, u) \in \gamma$ then $(x, y) \in \gamma$. (Here and in the sequel the join resp. meet of two lattice elements, say μ and ν, will be denoted by $\mu + \nu$ resp. $\mu\nu$.) The Shifting Lemma is depicted in Figure 1.

Notice that the Shifting Lemma plays a crucial role in Gumm's way to develop modular commutator theory in [6]. Motivated by the Shifting Lemma, an analogous property under the name Triangular Scheme has been introduced by Chajda [1]. The Triangular Scheme is the following condition: for any congruences α, β, γ if $\alpha\beta \leq \gamma$, $(x, y) \in \gamma$, $(x, z) \in \beta$ and $(y, z) \in \alpha$ then $(x, z) \in \gamma$, cf. Figure 2. Every algebra with distributive congruence lattice satisfies the Triangular Scheme by [1] or by the rest of the present paper. It is an open problem if, for varieties, the Triangular Scheme implies congruence distributivity. However, the conjunction of the Triangular Scheme and the Shifting Lemma implies congruence distributivity for varieties; this was announced in Duda [4] and proved in [2]. Moreover, Duda

Date: February 21, 2003.

1991 *Mathematics Subject Classification*: Primary 06B99, secondary 08B10.

Key words and phrases: Congruence distributivity, shifting lemma, triangular lemma, triangular scheme, lattice identity, lattice Horn sentence.

The research of the first author was partially supported by the project J98:MSM153100011 of the Czech Government.

The research of the second and third authors was partially supported by the NFSR of Hungary (OTKA), grant no. T034137, T026243, T037877, and also by the Hungarian Ministry of Education, grant no. FKFP 0169/2001.

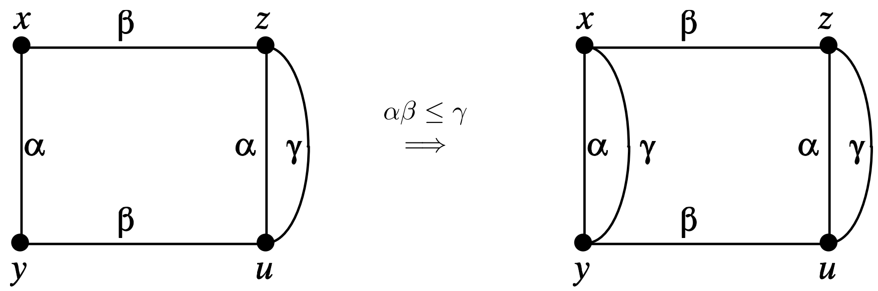


FIGURE 1

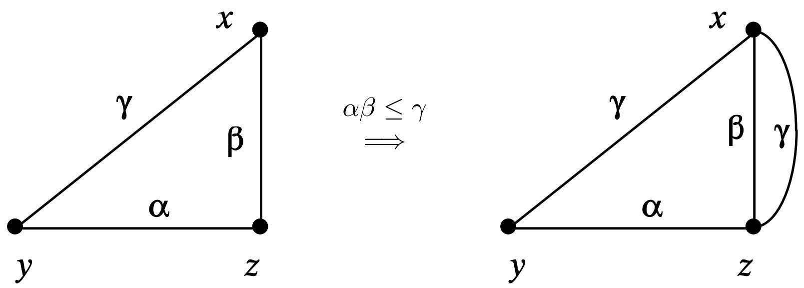


FIGURE 2

[3] found a diagrammatic condition which is equivalent to congruence distributivity for varieties; this condition is called Trapezoid Lemma in [2], where an easy proof of its equivalence with congruence distributivity is given.

Now, after [1], [2], Duda [3], [4] and mainly Gumm [6], no deep study of these lemmas would be reasonable in the present paper. Hence we restrict ourselves to an illustration, which slightly generalizes the well-known fact that congruence distributivity implies the Fraser–Horn property. The easy proof of the following assertion is left to the reader.

Assertion 1. *If both the Shifting Lemma and the Triangular Scheme hold in a direct product $A_1 \times A_2$ then $A_1 \times A_2$ satisfies the Fraser–Horn property, i.e., it has no skew congruences.*

The motivation for this paper is the following question: what is the purely lattice theoretic connection between the Shifting Lemma resp. the Triangular Scheme and modularity resp. distributivity?

Let

$$\lambda : \quad p(x_1, \ldots, x_n) \leq q(x_1, \ldots, x_n)$$

be a lattice identity. (Notice that by a lattice identity we always mean an inequality, i.e., we use $\leq$ but never $=$.) If y is a variable then let $S(\lambda, y)$ denote the Horn sentence

$$q(x_1, \dots, x_n) \le y \implies p(x_1, \dots, x_n) \le y.$$

If $y \notin \{x_1, \dots, x_n\}$ then λ is clearly equivalent to $S(\lambda, y)$. However, we are interested in the case when $y \in \{x_1, \dots, x_n\}$, say $y = x_i$ $(1 \le i \le n)$. Then $S(\lambda, x_i)$ is a consequence of λ. When $S(\lambda, x_i)$ happens to be equivalent to λ then $S(\lambda, x_i)$ will be called *a shift of* λ. If $S(\lambda, x_i)$ is equivalent to λ only within a lattice variety $\mathcal{V}$ then we say that $S(\lambda, x_i)$ *is a shift of* λ *in* $\mathcal{V}$.

As it will soon become clear, not every lattice identity has a shift. If an identity λ can be characterized by excluded (partial) sublattices then it is usually much easier to decide whether λ has a shift, but we also handle identities, n-distributivity and Fano identity, without such characterization.

First consider the distributive law

$$\text{dist:} \quad \beta(\alpha + \gamma) \le \beta\alpha + \beta\gamma.$$

Then $S(\text{dist}, \gamma)$ is $\beta\alpha + \beta\gamma \le \gamma \implies \beta(\alpha + \gamma) \le \gamma$, which is clearly equivalent to saying that

$$\alpha\beta \le \gamma \implies \beta(\alpha + \gamma) \le \gamma \tag{1}$$

is a shift of dist. Indeed, replacing γ by $\alpha\beta + \gamma$, (1) implies the identity $\beta(\alpha+\gamma) \le \alpha\beta + \gamma$, whence $\beta(\alpha + \gamma) \le \beta(\alpha\beta + \gamma)$. Using this second identity twice we obtain $\beta(\alpha + \gamma) \le \beta(\alpha\beta + \gamma) \le \beta\alpha + \beta\gamma$, the distributive law.

Although $S(\text{dist}, \gamma)$ and, rather, (1) are not lattice identities, they have two conspicuous advantages over distributivity. Firstly, if we want to test the distributivity of an n-element lattice in the most straightforward way then we have to evaluate both sides of $\beta(\alpha+\gamma) \le \beta\alpha+\beta\gamma$ for n^3 triplets. But to test $S(\text{dist}, \gamma)$ resp. (1) we have to evaluate $\beta(\alpha + \gamma)$ for those triplets for which $\beta\alpha + \beta\gamma$ resp. $\alpha\beta$ is below γ. Secondly, $S(\text{dist}, \gamma)$ or (1) makes it clear that the Triangular Scheme holds when the congruence lattice is distributive. (In fact, the Triangular Scheme is equivalent to congruence distributivity provided the algebra in question has permutable congruences.)

Practically the same is true for the modular law

$$\text{mod:} \quad \alpha(\beta + \alpha\gamma) \le \alpha\beta + \alpha\gamma.$$

Now $S(\text{mod}, \gamma)$: $\alpha\beta + \alpha\gamma \le \gamma \implies \alpha(\beta + \alpha\gamma) \le \gamma$, which is clearly equivalent to

$$\alpha\beta \le \gamma \implies \alpha(\beta + \alpha\gamma) \le \gamma. \tag{2}$$

To show that (2) implies modularity it suffices to observe that (2) fails in the pentagon (five element nonmodular lattice) when $\beta \parallel \gamma < \alpha \parallel \beta$. Again, $S(\text{mod}, \gamma)$ and (2) are easier to test from a computational point of view, they evidently imply the Shifting Lemma, and, in fact, the satisfaction of (2) is equivalent to the Shifting Lemma provided the algebra has 3-permutable congruences.

The examples above show the advantage of shifts of lattice identities: they are easier to test and they give rise to congruence diagrammatic statements which could be quite useful. In the rest of the paper we consider some concrete lattice identities, and we give their shifts or show that no shift exists.

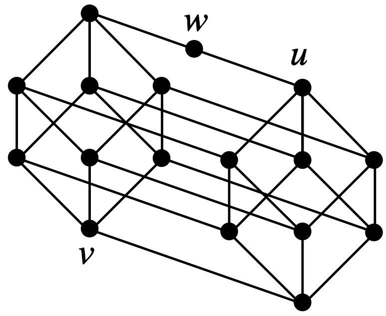


FIGURE 3

Following Huhn [8] and [9], a lattice L is said to be n-distributive ($n \geq 1$) if the identity

$$\mathrm{dist}_n: \quad \beta \sum_{i=0}^{n} \alpha_i \leq \sum_{j=0}^{n} \left(\beta \sum_{i \in \{0,\dots,n\} \setminus \{j\}} \alpha_i \right)$$

holds in L. (Notice that in his earlier papers Huhn assumed modularity in the definition but later he dropped this assumption.) Clearly, dist_1 is the usual distributivity.

Theorem 1. *$S(dist_n, \alpha_0)$ is a shift of $dist_n$ in the variety of modular lattices. However, if $n \geq 2$ then $dist_n$ has no shift (in the variety of all lattices).*

Proof. Let L be a modular lattice such that dist_n fails in L. Then, by Huhn [8] and [9], L contains an n-diamond[1], i.e., there are pairwise distinct elements $u, v, a_0, \dots, a_{n+1}$ in L such that for any n-element subset $H \subseteq \{0, 1, \dots, n+1\}$ and $k \in \{0, \dots, n+1\} \setminus H$ we have

$$a_k \sum_{i \in H} a_i = u \quad \text{and} \quad a_k + \sum_{i \in H} a_i = v.$$

Notice that these equations mean that any $n+1$ elements of $\{a_0, \dots, a_{n+1}\}$ are the atoms of a Boolean sublattice with bottom u and top v. Now the substitution $\alpha_i = a_i$, $i = 0, \dots, n$, and $\beta = a_{n+1}$ shows that $S(\mathrm{dist}_n, \alpha_0)$ fails in L.

Now let $n \geq 2$. We define a lattice L such that dist_n fails but all the "shift candidates" $S(\mathrm{dist}_n, \beta)$, $S(\mathrm{dist}_n, \alpha_0)$, …, $S(\mathrm{dist}_n, \alpha_n)$ hold in L. Take the finite Boolean lattice with $n+2$ atoms, pick an atom v, let u be the complement of v and insert a new element w in the prime interval $[u, 1]$. This way we obtain L, which is depicted in Figure 3 when $n = 2$. Letting $\{\alpha_0, \dots, \alpha_n\}$ be the set of covers of v and $\beta = w$ we see that dist_n fails in L. Clearly, $S(\mathrm{dist}_n, \beta)$ holds in any lattice. Now,

[1]This is the current terminology. Huhn called an equivalent notion as an $(n-1)$-diamond.

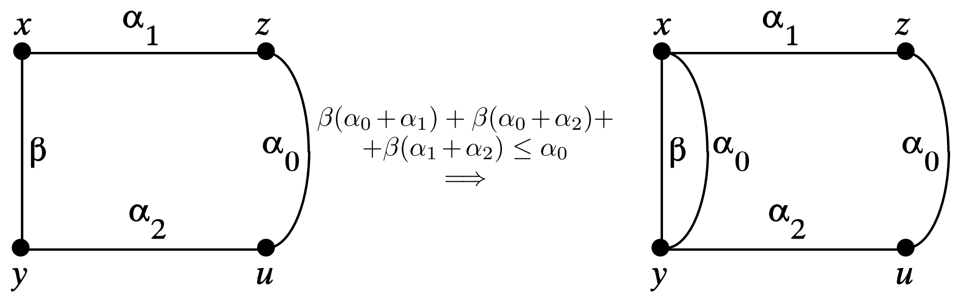


FIGURE 4

by way of contradiction, assume that $S(\mathrm{dist}_n, \alpha_0)$ fails for some $\beta, \alpha_0, \dots, \alpha_n \in L$. Then we have

$$p \not\leq q, \quad q \leq \alpha_0, \quad p \not\leq \alpha_0, \tag{3}$$

$$(\forall i)\ \beta \not\leq \alpha_i, \tag{4}$$

$$(\forall j)\ \alpha_j \not\leq \sum_{i\in\{0,\dots,n\}\setminus\{j\}} \alpha_i, \tag{5}$$

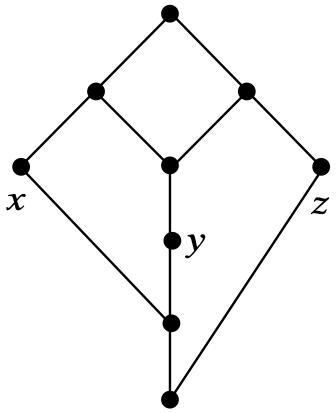
$$w \in \{\beta, \alpha_0, \dots, \alpha_n\}. \tag{6}$$

Indeed, (5) follows from (3), and (6) follows from (3) and the fact that $p \leq q$ in the Boolean lattice $L \setminus \{w\}$. If $w = \alpha_k$, $0 \leq k \leq n$, then either the interval $[v, 1]$ contains some α_i and $1 = \alpha_k + \alpha_i$ contradicts (5) (this is where $n \geq 2$ is used) or all the α_i belong to $[0, w] = [0, \alpha_k]$, which contradicts (5) again. Hence (6) yields $\beta = w$. In what follows, $\stackrel{d}{=}$ will refer to distributivity applied for elements of the sublattice $L \setminus \{w\}$. If $\sum_{i\in\{0,\dots,n\}} \alpha_i \neq 1$ then, for any $H \subseteq \{0, \dots, n\}$, $\beta \sum_{i\in H} \alpha_i = u \sum_{i\in H} \alpha_i$, and using the above-mentioned distributivity clearly gives $p = q$, contradicting (3). Hence $\sum_{i\in\{0,\dots,n\}} \alpha_i = 1$ and $p = \beta = w$. Since

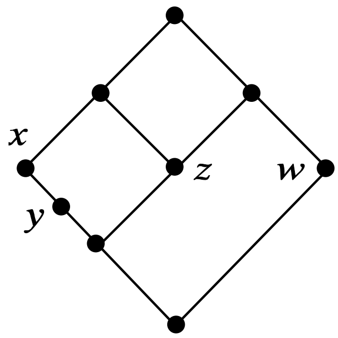
$$q = \sum_{j\in\{0,\dots,n\}} \beta \sum_{i\in\{0,\dots,n\}\setminus\{j\}} \alpha_i \geq \sum_{j\in\{0,\dots,n\}} u \sum_{i\in\{0,\dots,n\}\setminus\{j\}} \alpha_i \stackrel{d}{=} u \sum_{j\in\{0,\dots,n\}} \alpha_j = u$$

and $q \leq p \not\leq q$, we have $q = u$. Then (3) gives $\alpha_0 = u$ and (5) gives a contradiction again, either because $[v, 1]$ contains some α_i and $\alpha_0 + \alpha_i = 1$ or because $[0, \alpha_0]$ contains all the α_i. $\square$

Now, to show once again how a shift leads to a diagrammatic statement, we visualize dist_2. The following statement clearly follows from the preceding part of the paper. It is worth mentioning that when congruence lattices of all algebras of a given variety are considered then each of dist_n is equivalent to the usual distributivity by Nation [11]; hence the following statement is totally uninteresting for varieties instead of single algebras.



Q_0



Q_1

Corollary 1. *(A) Let A be an algebra with modular congruence lattice $\mathrm{Con}\, A$. If $\mathrm{Con}\, A$ is 2-distributive then the diagrammatic statement depicted in Figure 4 holds in A.*

(B) If A is congruence permutable then $\mathrm{Con}\, A$ is 2-distributive if and only if the diagrammatic statement depicted in Figure 4 holds in A.

The next group of lattice identities we consider is taken from McKenzie [10]. These identities are as follows:

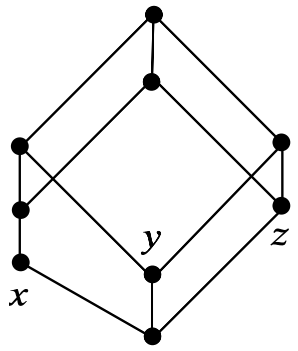
$$\begin{array}{lrcl} \zeta_0: & (x+y(z+xy))(z+xy) & \leq & y+(x+z(x+y))(y+z), \\ \zeta_1: & x(xy+z(w+xyz)) & \leq & xy+(z+w)(x+w(x+z)), \\ \zeta_2: & (x+y)(x+z) & \leq & x+(x+y)(x+z)(y+z), \\ \zeta_3: & (x+yz)(z+xy) & \leq & z(x+yz)+x(z+xy), \text{and} \\ \zeta_4: & y(z+y(x+yz)) & \leq & x+(x+y)(z+x(y+z)). \end{array}$$

Notice that ζ_3 is Gedeonová's p-modularity, cf. [5].

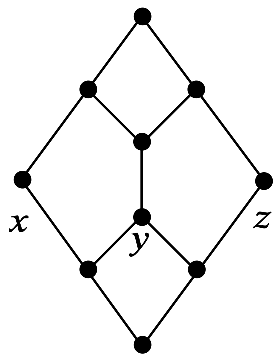
Theorem 2. *$S(\zeta_0, y)$, $S(\zeta_1, y)$, $S(\zeta_2, x)$, and $S(\zeta_3, y)$ are shifts of ζ_0, ζ_1, ζ_2 and ζ_3, respectively. On the other hand, ζ_4 has no shift.*

Proof. Consider the lattices $Q_0, \dots, Q_4$ given by their Hasse diagram. For $i = 0, \dots, 4$ McKenzie [10] proved that Q_i is a projective splitting lattice with conjugate identity ζ_i. As a consequence, for an arbitrary lattice L, ζ_i holds in L if and only if Q_i is not (isomorphic to) a sublattice of L; for $i = 3$ this was previously proved by Gedeonová [5].

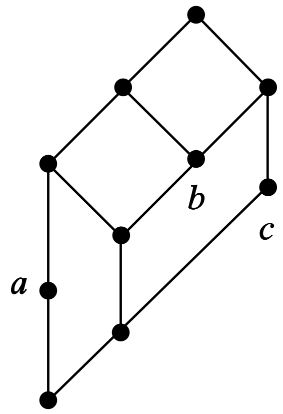
(Since it is not so easy to extract this well-known consequence from [10], perhaps a short hint is helpful. By definitions, for any lattice variety $\mathcal{V}$ either ζ_i holds in $\mathcal{V}$ or $Q_i \in \mathcal{V}$. Now suppose that ζ_i fails in a lattice L. Then $Q_i \in \mathbf{H}\,\mathbf{S}\,\mathbf{P}\{L\} = \mathbf{P}_s\,\mathbf{H}\,\mathbf{S}\,\mathbf{P}_u\{L\}$. Splitting lattices are subdirectly irreducible, so $Q_i \in \mathbf{H}\,\mathbf{S}\,\mathbf{P}_u\{L\}$. Since Q_i is projective, $Q_i \in \mathbf{S}\,\mathbf{P}_u\{L\}$, i.e., Q_i can be embedded into an ultrapower of L. But Q_i is finite, its embeddability can be expressed by a first order formula, so applying Loś' theorem we conclude that Q_i is embeddable into L.)



Q_2



Q_3



Q_4

Now if the shift of ζ_i, $0 \leq i \leq 3$, (i.e., $S(\zeta_2, x)$ for $i = 2$ and $S(\zeta_i, y)$ for $2 \neq i \leq 3$) held but ζ_i failed in a lattice L then Q_i would be a sublattice of L and the elements $x, y, \ldots$ indicated in the diagram of Q_i would refute the satisfaction of the shift of ζ_i in L.

It follows from definitions (or by substituting $(x, y, z) = (a, b, c)$) that ζ_4 fails in Q_4. So, to prove that $\zeta_4 : p_4 \leq q_4$ has no shift, it suffices to show that all the "shift candidates" $S(\zeta_4, x)$, $S(\zeta_4, y)$ and $S(\zeta_4, z)$ hold in Q_4. If $x, y, z \in Q_4$ with $\{x, y, z\} \neq \{a, b, c\}$ then the sublattice $[x, y, z]$ is distinct from Q_4, so it has no sublattice isomorphic to Q_4, hence ζ_4 and therefore the shift candidates hold in $[x, y, z]$. Hence it suffices to test substitutions with $\{x, y, z\} = \{a, b, c\}$; six cases. It turns out that $(x, y, z) = (a, b, c)$ is the only case when $p_4 \not\leq q_4$, so it is quite easy to see that all the shift candidates hold in Q_4. □

Theorem 2 raises the problem of characterizing splitting lattices whose conjugate identities have shifts.

All the previous lattice identities have known characterizations by excluded (partial) sublattices (at least in the variety of modular lattices) and, except for distributivity, our proofs were based on these characterizations. (Even in the second half of the proof of Theorem 1 the construction was motivated by Huhn' characterization for the modular case.) It would be interesting but probably difficult to avoid the use of excluded sublattices. The Fano identity (cf., e.g., Herrmann and Huhn [7]):

$$\chi_2 : \quad (x+y)(z+t) \le (x+z)(y+t) + (x+t)(y+z)$$

has no similar known characterization; yet, we have the following statement.

Theorem 3. *The Fano identity has no shift — not even in the variety of modular lattices.*

Proof. Suppose that χ_2 has a shift in the variety of modular lattices. Since the role of its variables is symmetric, we can assume that this shift is

$$S(\chi_2, x) : \quad (x+z)(y+t) + (x+t)(y+z) \le x \implies (x+y)(z+t) \le x.$$

Let L be the subspace lattice of the real projective plane. Then L is a modular lattice with length 3. It contains $0 = 0_L = \emptyset$, the atoms are the projective points (as singleton subspaces), the coatoms are the projective lines, and the full plane is $1 = 1_L$. It follows from Herrmann and Huhn [7] that χ_2 fails in L. We intend to show that $S(\chi_2, x)$ holds in L and this will imply our theorem. We will use the modular law in its classical form

$$x \le z \implies (x+y)z = x + yz$$

and also in the form of shearing identity

$$x(y+z) \overset{s}{=} x(y(x+z)+z) = x(y(x+z) + z(x+y)).$$

First we show that χ_2 and therefore $S(\chi_2, x)$ hold for $x, y, z, t \in L$ when $\{x, y, z, t\}$ is not an antichain. By symmetry, it is enough to treat two cases.

Case 1: $x \le y$, then

$$(x+y)(z+t) = y(z+t) \overset{s}{=} y(z(y+t) + t(y+z)) \le z(y+t) + t(y+z) \le$$
$$(x + z(y+t)) + (x + t(y+z)) = (x+z)(y+t) + (x+t)(y+z).$$

Case 2: $x \le z$, then

$$(x+y)(z+t) = x + y(z+t) \overset{s}{=} x + y(z(y+t) + t(y+z)) \le$$
$$z(y+t) + x + t(y+z) = (x+z)(y+t) + (x+t)(y+z).$$

Let $\{x, y, z, t\}$ be an antichain in L. Thus each of x, y, z and t is a point or a line.

If x is a line then we infer $x+z = 1$ from $z \not\le x$ and the premise of $S(\chi_2, x)$ gives $x \ge (x+z)(y+t) = y+t \ge y$, a contradiction. Therefore x is a point. If z is a line then $x+z = 1$ again and we can derive the same contradiction. Hence z is point and, by z–t symmetry, so is t. Similarly, if y is a line then $x \ge (x+z)(y+t) = x+z \ge z$, therefore y is a point.

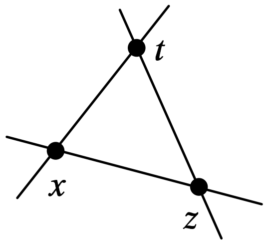


FIGURE 5

We have seen that x, y, z and t are pairwise distinct points. Let us consider the "triangle" xzt, cf. Figure 5. The premise of $S(\chi_2, x)$ says $(x+z)(y+t) \leq x$, which is possible only when $y \leq x+t$ (i.e., y is on the line through x and t). Similarly, $(x+t)(y+z) \leq x$ forces $y \leq x+z$. Hence $y \leq (x+t)(x+z) = x$, a contradiction. ☐

Acknowledgment. Numerous helpful comments of the referee, including a substantial abridgement of the proof of Theorem 3 and raising problems that led to the "second part" of Theorems 1 and 2, are highly appreciated.

REFERENCES

[1] Chajda and E. K. Horváth, *A triangular scheme for congruence distributivity,* Acta Sci. Math. (Szeged) **68** (2002), 29–35.

[2] I. Chajda, G. Czédli and E. K. Horváth, *Trapezoid Lemma and congruence distributivity,* Math. Slovaka, to appear.

[3] J. Duda, *The Upright Principle for congruence distributive varieties,* Abstract of a seminar lecture presented in Brno, March, 2000.

[4] J. Duda, *The Triangular Principle for congruence distributive varieties,* Unpublished abstract from March, 2000.

[5] E. Gedeonová, *Jordan–Hölder theorem for lines,* Mat. Časopis Sloven. Akad. Vied. **22** (1972), 177-198.

[6] H. P. Gumm, *Geometrical methods in congruence modular algebras,* Mem. Amer. Math. Soc. **45** (1983), no. 286, viii+79 pp.

[7] C. Herrmann and A. P. Huhn, *Zum Begriff der Charakteristik modularer Verbände,* Math. Z. **144** (1975), 185–194.

[8] A. P. Huhn, *Schwach distributive Verbände I,* Acta Sci. Math. (Szeged) **33** (1972), 297-305.

[9] A. P. Huhn, *On Grätzer's problem concerning automorphisms of a finitely presented lattice,* Algebra Universalis **5** (1975), 65–71.

[10] R. McKenzie, *Equational bases and non-modular lattice varieties,* Trans. Amer. Math. Soc. **174** (1972), 1–43.

[11] J. B. Nation, *Varieties whose congruences satisfy certain lattice identities,* Algebra Universalis **4** (1974), 78–88.

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