Distributivity via first meanders

LADISLAV BERAN [(*)] and GÁBOR CZÉDLI [(**)]

Abstract. We prove the following theorem: A lattice L having no infinite chains is distributive if and only if every filter of L is equal to the first meander of an ideal in L.

Given an ideal I of a lattice L, the *first meander* of I is the set

$$I^1 := \{a \in L : a \wedge b \in I \implies b \in I\}$$

(cf. [1, 2]). Since I^1 is a filter in L, $\varphi : I \to I^1$ is a mapping from the set $\mathrm{Id}(L)$ of all ideals in L into the set $\mathrm{Fi}(L)$ of all filters in L. In [3] the first author conjectured that the surjectivity of φ might characterize the distributivity of L. With the additional assumption that L has no infinite chains, our goal is to turn this conjecture to the following theorem.

Theorem 1. Let L be a lattice which has no infinite chains. Then L is distributive if and only if each filter of L is the first meander of an ideal of L.

This theorem was found by the first author, who derived it from Lemma 3 and gave a long original proof for Lemma 3. The present short proof of Lemma 3 is a joint work.

Now, for the reader's convenience, we collect some notions and statements from [1] and [2]. The first meander of a *filter* F is $F^1 := \{a \in L : a \vee b \in F \implies b \in F\}$. If X is an ideal or a filter then X^2, the *second meander* of X, is the meander of its first meander X^1. If F is a filter of L, $a \in L \setminus F$ and $x \in F$ for all $a < x \in L$ then a is called an *F-coatom*. The importance of F-coatoms is revealed by the following easy lemma, which was proved in [1].

Lemma 2 . Let L be a lattice without infinite chains and let F be a filter of L. Then

(i) for each $b \in L \setminus F$ there is an F-coatom a with $b \leq a$;

(ii) F^1 is the principal ideal $(i]$ where i is the meet of all F-coatoms.

After Rav [5], cf. also Chevalier [4], an ideal I of L is called *semiprime*, if

$$a \wedge b \in I \quad \& \quad a \wedge c \in I \implies a \wedge (b \vee c) \in I$$

for any $a, b, c \in L$. *Semiprime filters* are defined dually.

Lemma 3. Let L be a lattice without infinite chains, and let i be an element of L which is maximal with respect to the property that the ideal $(i]$ is not semiprime. Then there is no filter F in L with $F^1 = (i]$.

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Proof of Lemma 3. Suppose that there is an $F = [f)$ such that $F^1 = (i]$ and reason to a contradiction as follows. Let us fix elements $a, b, c \in L$ with $a \wedge b \le i$, $a \wedge c \le i$ and $a \wedge (b \vee c) \not\le i$. Clearly, $i \ne 1$. From the maximality of i we infer that, for any $x \in L$,

$$i \prec x \implies a \wedge (b \vee c) \le x. \tag{1}$$

Hence i is meet irreducible, for otherwise $i \prec x_1$, $i \prec x_2$ and $x_1 \parallel x_2$ would give $a \wedge (b \vee c) \le x_1 \wedge x_2 = i$. The meet irreducibility of i, Lemma 2(ii) and the fact that distinct F-coatoms are incomparable yield that i is the unique F-coatom in L. Hence we infer from Lemma 2(i) that $x \le i$ for all $x \in L \setminus F$, i.e., $L = (i] \cup [f)$. The $[f)$-coatom i does not belong to $[f)$, so this union is a disjoint one, i.e.

$$L = (i] \ \dot\cup \ [f). \tag{2}$$

Since $a \wedge (b \vee c) \not\le i$, we therefore have

$$a \in [f) \quad \text{and} \quad b \vee c \in [f). \tag{3}$$

Now $b \in [f)$ would imply $a \wedge b \in (i] \cap [f)$, a contradiction. Hence $b \in (i]$ and, similarly, $c \in (i]$, which is impossible by (2) and (3). Q.e.d.

Proof of Theorem 1. If L is distributive then, by the dual of [2, Thm. 2.4] (which is the "only if" part of the main result of [3], cf. Thm. 4 below), $F = F^2$ for each filter F of L, whence F is the first meander of the ideal F^1.

Now suppose that L is not distributive. Then $(x \vee y) \wedge (x \vee z) \ne x \vee (y \wedge z)$ for some $x, y, z \in L$. Hence $\big[(x \vee y) \wedge (x \vee z)\big)$ is a filter which is not semiprime. The dual of Lemma 3 implies that not every filter of L is the first meander of an ideal. Q. e. d.

We conclude the paper with recalling the main result of [3] in order to indicate how Theorem 1 makes the proof of its "if" part essentially shorter.

Theorem 4 ([3]). Let L be a lattice without infinite chains. Then L is distributive if and only if $I^2 = I$ holds for all ideals I of L.

Proof of the "if" part. If $I^2 = I$ holds for all ideals I, then every ideal is the first meander of the filter I^1, and L is distributive by the dual of Theorem 1.

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L. BERAN, Department of Algebra, Charles University, Sokolovská 83,
186 00 Prague 8, Czech Republic; *e-mail:* lberan@karlin.mff.cuni.cz

G. CZÉDLI, JATE Bolyai Institute, Szeged, Aradi vértanúk tere 1,
Hungary–6720; *e-mail:* czedli@math.u-szeged.hu