Mal'cev functions on smalgebras

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Dedicated to László Megyesi on his sixtieth birthday

Abstract. Given a nine-element set A and a lattice L of permuting equivalences on A, it is shown that there exists a Mal'cev function $A^3 \to A$ that preserves all members of L. The same statement was previously known to hold for $|A| \le 8$ and to fail for $|A| \ge 25$, and it remains open for $10 \le |A| \le 24$.

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Introduction and the main result

Given a set A, a function $p : A^3 \to A$ is called a Mal'cev function on A if $p(x, y, y) = p(y, y, x) = x$ holds for all $x, y \in A$. If, in addition, $p(x_1, x_2, x_3) = p(x_{1\sigma}, x_{2\sigma}, x_{3\sigma})$ holds for all $x_1, x_2, x_3 \in A$ and any permutation σ then p is said to be commutative. If an algebra A has a Mal'cev function compatible with all congruences of A then A is known to be congruence permutable. A classical result of Mal'cev [5] asserts that the converse is also true when we consider a variety of algebras rather than a single algebra, and Gumm [3] points out that this is not the case for a single algebra. Remarkably enough, Pixley [6] proves that there is another congruence property, the arithmeticity, when the known Mal'cev characterization for varieties works for single finite algebras, and Gumm [3] shows that arithmeticity is, in some sense, the only congruence property where the passage from varieties to single algebras is possible.

For single algebras with a limited number of elements the situation is more pleasant. (The title of the paper refers to single small algebras, in short *smalgebras*.) Chajda [1] and later Chajda and Czédli [2] proved that if an algebra A has permuting congruences and $|A| \le 4$ resp. $|A| \le 8$ then there exists a Mal'cev function $A^3 \to A$ preserving all congruences of A. These proofs make heavy use of Pixley's ideas from [6]. In fact, [2] contains a bit stronger statement, namely

Theorem A. *([2]) Let A be a set with $|A| \leq 8$ and let L be a sublattice of the lattice of equivalences on A. Then the equivalences belonging to L permute (i.e., $\rho \circ \nu = \nu \circ \rho$ holds for all $\rho, \nu \in L$) iff there exists a commutative Mal'cev function on A which is compatible with every member of L.*

The authors have the feeling that numbers of the form k^m with integers $k \geq 2$ and $m \geq 2$ may play a distinguished role when investigating the existence of Mal'cev functions. This feeling is supported by the proofs presented here and in [2], in particular by Lemmas 6 and 7 in the present paper, by the fact that commutativity from Theorem A must surely be dropped when $|A|$ exceeds 2^3, cf. [2], and by Gumm's example showing that Mal'cev functions need not exist when $|A| \geq 25$, cf. [3] and [2]. This leads to the question if the smallest number for which Theorem A without commutativity fails is of the form k^m $(k, m \geq 2)$; this motivates the present investigation, which can also be of some interest in studying intersections of certain maximal clones on a finite set with less than ten elements.

We intend to prove the following result.

Theorem 1. *Let A be a set with $|A| \leq 9$, and let L be a sublattice of the lattice of all equivalences on A. Then the following two conditions are equivalent:*

(i) *the members of L permute;*

(ii) *there is a Mal'cev function on A which is compatible with each member of L.*

Lemmas and proofs

While (ii) $\Longrightarrow$ (i) is well-known, cf. e.g. Mal'cev [5], the converse implication follows less easily. Firstly, we recall six lemmas from [2]. Notice that the proofs of Lemmas 1, 3, 4, 5 and 6 did not use the condition $|A| \leq 8$. The original proof of Lemma 2 settles $|A| = 9$, which is sufficient for the present paper. (Note that a more or less straightforward modification of the original proof yields Lemma 2 for $|A| > 9$.) The general assumption in our lemmas is that A is a finite set and each permutable equivalence lattice on a set with less than $|A|$ elements permits a compatible Mal'cev function. We will often consider diamonds, i.e., five-element non-distributive modular (sub)lattices, their elements will be denoted by $\omega, \alpha, \beta, \gamma$ and ι such that $\omega < \alpha < \iota$, $\omega < \beta < \iota$ and $\omega < \gamma < \iota$.

Lemma 1. *If there exists a $\mu \in L \setminus \{0\}$ such that $\mu \leq \omega$ holds for every diamond $\{\omega, \alpha, \beta, \gamma, \iota\}$ in L then we are done. (I.e., then there is an $A^3 \to A$ Mal'cev function which is compatible with all members of L).*

Let us call an equivalence $\mu \in L$ *semicentral* if $\mu \circ \nu = \mu \cup \nu$ (set theoretic union) holds for every $\nu \in L$. (Note that $\mu \circ \nu = \mu \vee \nu$ by permutability.) All references to the following lemma will use the fact that if $\mu \in L$ is not semicentral then $\nu \parallel \mu$ holds for some $\nu \in L$.

Lemma 2. *If there exists a semicentral $\mu \in L \setminus \{0, 1\}$ then we are done.*

Although the following assertion is evident, its notation, which comes from "shifting principle", gives an economic way of reference and of exploiting permutability.

Lemma 3. *Let $\mu, \rho \in L$, let B and C be distinct μ-blocks, and suppose $(B \times C) \cap \rho \neq \emptyset$. Then*

$$\mathrm{SP}(\mu,\rho): \qquad (\forall b \in B)(\exists c \in C)(b\ \rho\ c), \quad \text{and} \quad (\forall c \in C)(\exists b \in B)(b\ \rho\ c).$$

For positive integers $i_1 \geq i_2 \geq \cdots \geq i_t$ we say that an equivalence is of *pattern* $i_1 + i_2 + \cdots + i_t$ if it has exactly t blocks and these blocks consist of i_1, i_2, …, i_t elements. Blocks with more than one element are called *nontrivial blocks*.

Lemma 4. *If $L \setminus \{0,1\}$ has a member of pattern $j + 1 + \cdots + 1$ or a member of pattern $3 + 2 + 1 + \cdots + 1$ then we are done.*

Lemma 5. *If there are $\mu, \nu \in L$ such that*

- $\mu < \nu$,
- *ν has exactly two blocks, B and C,*
- *$|B| > 1$ and $|C| > 1$,*
- *C is a block of μ as well, and*
- *μ has a singleton block*

then we are done.

Lemma 6. *Let $M_3 = \{\omega, \alpha, b, \gamma, \iota\}$ be a diamond in L such that $|A/\omega| \leq 8$. Then the following three statements are true:*

(a) *Each block B of ι/ω consists of a square number of elements.*

(b) *If $|B| = 4$ then the restriction of any of α/ω, β/ω and γ/ω to B is of pattern $2 + 2$.*

(c) *If $|B| = 4$ and B is the only nontrivial block of ι/ω then the interval $[\omega, \iota]$ of L coincides with M_3.*

Let $\mathbf{Z}_3 = (\{0,1,2\}, +)$ be the cyclic group of order three. Denoting the elements of $\mathbf{Z}_3 \times \mathbf{Z}_3$ by xy or sometimes by (x, y), $x, y \in \mathbf{Z}_3$, we generalize the previous lemma as follows.

Lemma 7. *Let $M_3 = \{\omega, \alpha, b, \gamma, \iota\}$ be a diamond in L. Then (a), (b) and (c) of the previous lemma hold. Now let B be a nine-element block of ι/ω, then the following two statements are also valid:*

(d) *B is, up to a bijection, $\mathbf{Z}_3 \times \mathbf{Z}_3$, and we have*

$$\begin{aligned} xy\ \alpha\ x'y' &\iff x = x', \\ xy\ \beta\ x'y' &\iff y = y', \qquad \text{and} \\ xy\ \gamma\ x'y' &\iff x - y = x' - y'. \end{aligned}$$

(e) *If B is the only nontrivial block of ι/ω then the interval $[\omega, \iota]$ of L is either M_3 or $M_4 = M_3 \cup \{\delta\}$ where*

$$xy\ \delta\ x'y' \iff x + y = x' + y'.$$

Proof. The argument given for Lemma 6 in [2] proves (a), (b) and (c) of the present lemma as well. We can assume that $\omega = 0$, for otherwise A/ω and $\{\rho/\omega : \rho \in L,\ \omega \le \rho\}$ could be considered instead of A and L. Now Gumm [4, Lemma 2.3] and the fact that three-element loops are (isomorphic to) $\mathbf{Z}_3$ yield (d). (Notice that Gumm prefers δ to γ, but using the automorphism $x \mapsto -x$ for the second component of $\mathbf{Z}_3 \times \mathbf{Z}_3$ we can swap γ and δ.)

Now, to prove (e), we can assume that, in addition to $\omega = 0$, $\iota = 1$. Indeed, if $B \ne A$ then, by $SP(\iota, \ldots)$, the members of $L|_B = \{\rho|_B : \rho \in L\}$ permute, so we can work with B and $L|_B$ rather than A and L. We claim that

$$\alpha\ ,\ \ \beta \ \text{ and } \ \gamma \ \text{ are atoms in } \ L. \tag{1}$$

It suffices to show that α is an atom. Then so is β by symmetry, and we can argue for γ as follows. Let $(ab, cd) \in \gamma \setminus \nu$ for some $\nu \in L$ with $0 < \nu < \gamma$. Then $a \ne c$ or $b \ne d$. Let $a \ne c$; the other case is similar by α–β symmetry. Then $a0\ \alpha\ ab\ \nu\ cd\ \alpha\ c0$ and $a0\ \beta\ c0$ shows that $\beta \wedge (\alpha \vee \nu) \ne 0$. But β is an atom, so $\beta \le \alpha \vee \nu$. Using modularity and the description of M_3 let us compute: $\gamma = \gamma \wedge (\alpha \vee \beta) \le \gamma \wedge (\alpha \vee \alpha \vee \nu) = \gamma \wedge (\alpha \vee \nu) = \nu \vee (\alpha \wedge \gamma) = \nu \vee 0 = \nu$, whence $\nu = \gamma$ and γ is an atom.

Now, to show that α is an atom, suppose that $(xy, xz) \in \nu \le \alpha$ for some $\nu \in L \setminus \{0\}$ and $x, y, z \in \mathbf{Z}_3$, $y \ne z$. Then $\mathrm{SP}(\beta, \nu)$ and $\nu \le \alpha$ give $(yy, yz) \in \nu$. From $(00, yy) \in \gamma$, $\mathrm{SP}(\gamma, \nu)$ and $\nu \le \alpha$ we infer $(00, 0u) \in \nu$ where $u = z - y \ne 0$. Repeating the previous ideas, $\mathrm{SP}(\gamma, \nu)$ gives $\big(uu, (u, 2u)\big) \in \nu$, then $\mathrm{SP}(\beta, \nu)$ yields $\big(0u, (0, 2u)\big) \in \nu$. Hence $[00]\nu \supseteq \{00, 0u, (0, 2u)\} = [00]\alpha$, and $\mathrm{SP}(\beta, \nu)$ implies $\nu = \alpha$, proving (1).

Now we formulate the "dual" of (1):

$$\alpha\ ,\ \ \beta \ \text{ and } \ \gamma \ \text{ are dual atoms in } \ L. \tag{2}$$

Indeed, suppose that $\alpha < \nu < 1$ for some $\nu \in L$. Then ν collapses two α-blocks $[a0]\alpha$ and $[b0]\alpha$, $a \ne b$. Since $(a0, b0) \in \beta$, $\beta \wedge \nu \ne 0$. From (1) we obtain $\beta \le \nu$, and $1 = \alpha \vee \beta \le \nu \vee \nu = \nu$ is a contradiction. The treatment for the rest of (2) is similar.

Let $\widehat{M_4}$ denote the six-element (abstract) modular lattice of length 2. From (1), (2) and modularity we conclude that L is also of length 2. Hence we obtain that

$$\text{for any } \ \nu \in L \setminus M_3, \quad M_3 \cup \{\nu\} \cong \widehat{M_4}. \tag{3}$$

Now we claim that for any $\nu \in L$

$$\nu < \delta \Longrightarrow \nu = 0; \tag{4}$$

here we do not assume that $\delta \in L$. Suppose $0 < \nu < \delta$. Since δ-blocks consist of three elements, ν has a singleton block $\{xy\}$. Hence $[xy](\alpha \vee \nu) = [xy](\nu \circ \alpha) = [xy]\alpha \ne A$, albeit $\alpha \vee \nu = 1$ by (3). This shows (4).

It is easy to check that for each $xy \in A$,

$$[xy]\delta = \{xy\} \cup \Big(A \setminus \big([xy]\alpha \cup [xy]\beta \cup [xy]\gamma\big)\Big). \tag{5}$$

Finally, let $\nu \in L \setminus M_3$. Since $\nu \wedge \alpha = \nu \wedge \beta = \nu \wedge \gamma = 0$ by (3), we obtain $\nu \le \delta$ from (5), and (4) gives $\nu = \delta$. This proves the lemma. ◇

An element $a \in A$ will be called *separated* (with respect to L) if $[a]\nu$ is a singleton for all $\nu \in L \setminus \{1\}$. Given an equivalence $\nu \in L$ and a subset $X \subseteq A$, X is said to be ν*-closed* if $[y]\nu \subseteq X$ for every $y \in X$.

Lemma 8. *If A has a separated element then we are done.*

Proof. Let $z \in A$ be a separated element, and let $B = A \setminus \{z\}$. Since $L|_B = \{\nu|_B : \nu \in L\}$ is a lattice of permuting equivalences over B and $|B| < |A|$, there is a Mal'cev function $q : B^3 \to B$ which is compatible with $L|_B$. Let us define $p : A^3 \to A$ by the following properties: p extends q, $p(a,b,z) = p(a,z,b) = p(z,a,b) = z$ for all $a, b \in B$, and $p(x,z,z) = p(z,x,z) = p(z,z,x) = x$ for all $x \in A$. Then p is a Mal'cev function, which is compatible with L. $\diamond$

Armed with the previous lemmas, the proof of Theorem 1 runs as follows. We can assume that L includes a diamond, for otherwise Lemma 1 is applicable with $\mu = 1$. Let us fix a diamond $M_3 = \{\omega, \alpha, \beta, \gamma, \iota\}$ in L for which ω is minimal. By Lemmas 6 and 7 we do not have too many possibilities for M_3. Moreover, if we disregard from those settled by Lemma 4 (for ι or ω) or Lemma 5 (for ι and ω) then it is easy to list the rest, and eleven cases remain. These cases are depicted in Figures 1–11. These figures indicate the ι-blocks by closed polygons and the ω-blocks by closed curves, however, singleton blocks are never indicated. Whenever the elements of A are labelled, we always assume

$$\{(a,d),(b,c)\} \subseteq \alpha, \quad \{(a,c),(b,d)\} \subseteq \beta \quad \text{and} \quad \{(a,b),(c,d)\} \subseteq \gamma.$$

Hence our figures determine α, β and γ, provided a, b, c and d occur as labels. Some ι-blocks are denoted by capital letters. Equivalences will often be given by partitions, so formulas like $\rho = \big\{\{a,b,c,d,e\}, \{f,g,h,i\}\big\}$ should not cause any confusion.

Now *Case* 1, cf. Figure 1, is clearly settled by Lemma 7, for the Mal'cev function $p(x,y,z) = x - y + z$ on the Abelian group $\mathbf{Z}_3 \times \mathbf{Z}_3$ preserves the group congruences α, β, γ and δ described in the lemma.

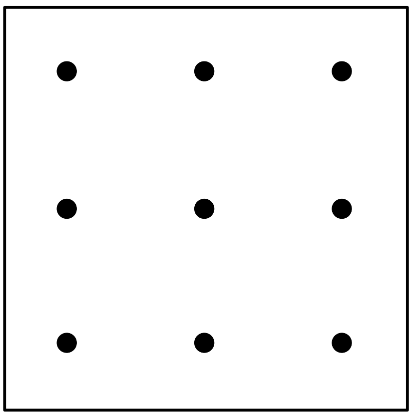

Figure 1

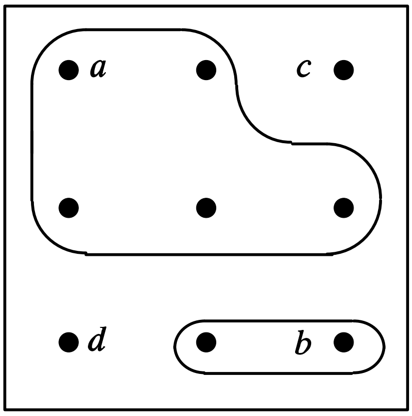

Figure 2

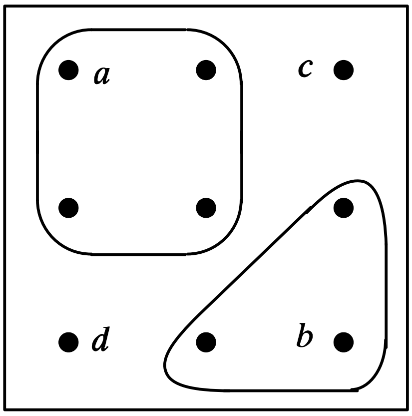

Figure 3

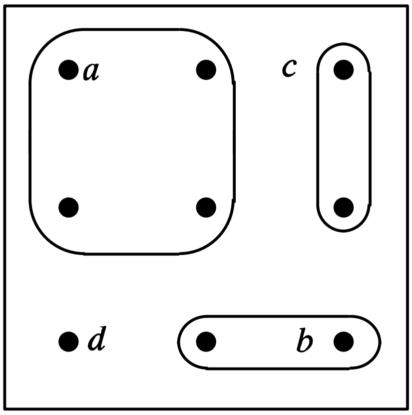

Figure 4

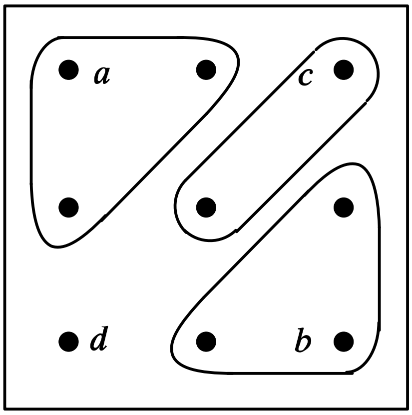

Figure 5

In Cases 2, 3, 4 *and* 5 we are going to show that for any other diamond $M_3' = \{\omega', \alpha', \beta', \gamma', \iota'\}$ in L we have $\omega \leq \omega'$; then Lemma 1 applies with $\mu = \omega$. Suppose $\omega \not\leq \omega'$. By the minimality of ω, $\omega' \parallel \omega$, so we can choose a pair $(x, y) \in \omega' \setminus \omega$. Using $\mathrm{SP}(\alpha, \omega')$ or $\mathrm{SP}(\beta, \omega')$ we can assume that $x = d$. If $y \in [a]\omega$ then $\mathrm{SP}(\omega, \omega')$ yields $[d]\omega' \supseteq [a]\omega \cup \{d\}$, and we infer from $\mathrm{SP}(\beta, \omega')$ that ω' has at most $|[c]\omega| + 1 \leq 3$ blocks, which contradicts Lemma 6. Similarly, $y \in [b]\omega$ implies $[d]\omega' \supseteq [b]\omega \cup \{d\}$ by $\mathrm{SP}(\omega, \omega')$, whence ω' has at most $|[c]\omega| + 1 \leq 3$ blocks by $\mathrm{SP}(\alpha, \omega')$. Hence $[d]\omega' \setminus \{d\} \subseteq [c]\omega$. Repeating the previous argument with $\mathrm{SP}(\omega, \omega')$ and $\mathrm{SP}(\alpha, \omega')$ we obtain that $[d]\omega' = [c]\omega \cup \{d\}$ and ω' has at most $|[b]\omega| + 1$ blocks. This settles Cases 2 and 4 by Lemma 6. Moreover, in Cases 3 and 5, ω' has exactly four blocks: $[d]\omega'$ and the ω'-blocks of elements of $[b]\omega$. Clearly, $\omega \wedge \omega'$ is of pattern $2 + 1 + \cdots + 1$, and Lemma 4 applies.

In Cases 6 *and* 7 we can assume that h is not separated, for otherwise Lemma 8 is

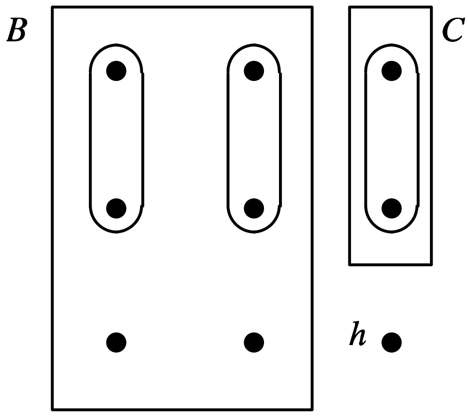

Figure 6

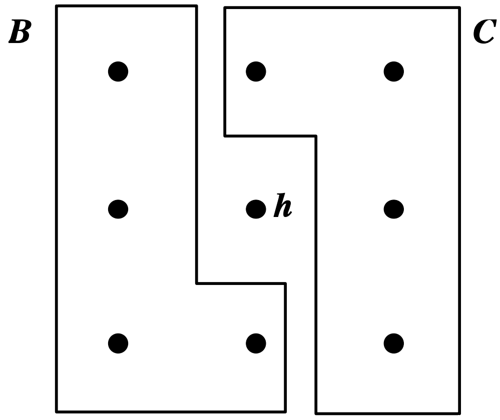

Figure 7

applicable. Hence there is a $\nu \in L \setminus \{1\}$ such that $[h]\nu \cap (B \cup C) \neq \emptyset$. We infer from $\mathrm{SP}(\iota, \nu)$ that $[h]\nu \cap B \neq \emptyset$ implies $[h]\nu \supseteq B \cup \{h\}$ and $[h]\nu \cap C \neq \emptyset$ implies $[h]\nu \supseteq C \cup \{h\}$. Hence $B \cup \{h\}$ or $C \cup \{h\}$ is a block of ν, and Lemma 5 applies for ι and $\iota \vee \nu$.

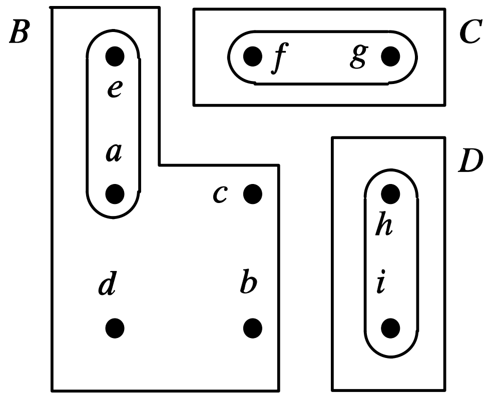

Figure 8

In Case 8 we claim that, for any $\rho \in L \setminus \{1\}$,

$$\text{if } B \text{ is not } \rho\text{-closed then } [a]\rho = B \cup C \text{ or } [a]\rho = B \cup D. \tag{6}$$

Indeed, suppose the contrary. Since the role of C and D is symmetric, $\rho \cap (B \times C) \neq \emptyset$ can be assumed. From $\mathrm{SP}(\iota, \rho)$ we conclude $[b]\rho \cap C \neq \emptyset$, whence $\mathrm{SP}(\omega, \rho)$ yields $[b]\rho \supseteq \{b, f, g\}$. Resorting to $\mathrm{SP}(\iota, \rho)$ again we obtain $B \cup C \subseteq [b]\rho = [a]\rho$. This inclusion cannot be proper, for otherwise $\mathrm{SP}(\iota, \rho)$ would lead to $\rho = 1$.

Now let us observe that

$$\text{if } B \text{ is not } \rho\text{-closed for some } \rho \in L \setminus \{1\} \text{ then we are done.} \tag{7}$$

Indeed, by (6) we can suppose $[a]\rho = B \cup C$. Then $\iota \vee \rho = \{B \cup C, D\}$, and Lemma 5 applies for ω and $\iota \vee \rho$.

Now Case 8 will be settled rapidly. By Lemma 2 we may suppose that ι is not semicentral and, by (7), this is witnessed by some ρ ($\rho \parallel \iota$, $\rho \in L$) such that B is ρ-closed. Then $\iota \vee \rho = \{B, C \cup D\}$ is either semicentral and Lemma 2 applies or B is not ν-closed for some $\nu \in L \setminus \{1\}$ and we invoke (7).

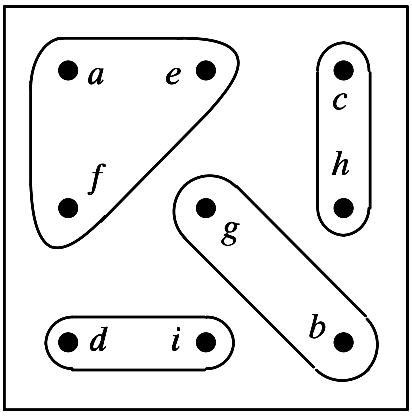

Figure 9

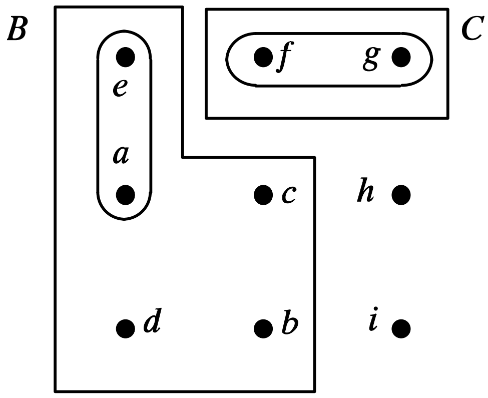

Figure 10

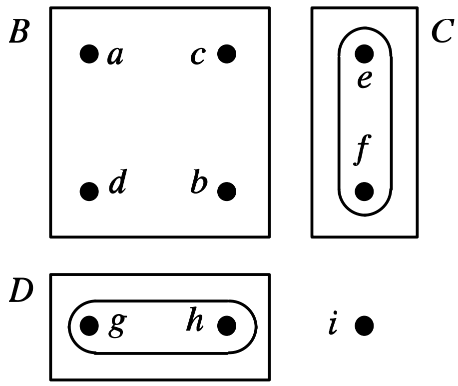

Figure 11

Now we are left with Cases 9, 10 *and* 11. In virtue of Lemma 1 and the fact that Cases 1, ..., 8 have been settled we can assume that L includes two diamonds $M_3 = \{\omega, \alpha, \beta, \gamma, \iota\}$ and $M_3' = \{\omega', \alpha', \beta', \gamma', \iota'\}$ such that $\omega \parallel \omega'$ and both M_3 and M_3' belong to Cases 9, 10 and 11. Apart from M_3–M_3' symmetry, this gives rise to six possibilities, which will be handled separately. In what follows, figures 9, 10 and 11 will describe M_3 while the elements $a', b', \ldots$ and subsets $B', C', \ldots$ of A together with the corresponding figure refer to M_3'.

Cases 9–10 *and* 9–11: when M_3 is of type 9 and M_3' is of type 10 or 11. Let us take a pair $(x, y) \in \omega' \setminus \omega$. Apart from labelling, $(x, y) = (a, b)$ or $(x, y) = (b, c)$. When $(x, y) = (a, b) \in \omega'$ then $\mathrm{SP}(\omega, \omega')$ cannot hold, for ω' has two-element nontrivial blocks only. Hence $(x, y) = (b, c) \in \omega'$, and $\mathrm{SP}(\beta, \omega')$ leads to a contradiction, for $(b, c) \notin \beta$ and there are (numerous) singleton ω'-blocks.

Cases 10–10 *and* 10–11, when M_3 is of type 10 and M_3' is of type 10 or 11. By Lemma 4, $\omega \wedge \omega' = 0$ can be assumed. If $(x, y) \in \omega' \setminus 0$ then $\{x, y\} \subseteq B$ or $\{x, y\} = \{h, i\}$, for otherwise $\mathrm{SP}(\iota, \omega')$ would enlarge ω'. Hence if $(h, i) \in \omega'$ then $\iota \wedge \omega'$ is of pattern $2 + 1 + \cdots + 1$ and Lemma 4 applies. Therefore both two-element ω'-blocks are included in B. So either $|[a]\omega'| = 2$ or $|[e]\omega'| = 2$, which contradicts $\mathrm{SP}(\omega, \omega')$ and $\omega \wedge \omega' = 0$.

Case 9–9, when both M_3 and M_3' are of type 9. We focus our attention at the three-element blocks $[a]\omega$ and $[a']\omega'$.

If $[a]\omega = [a']\omega'$ then $\omega \wedge \omega'$ is one of the following patterns: $3+1\cdots+1$, $3+2+1+\cdots+1$ and $3 + 2 + 2 + 2$. However, the last one is impossible by $\omega \neq \omega'$, and the first two are settled by Lemma 4.

If $|[a]\omega \cap [a']\omega'| = 2$ then we can assume that $a = a'$, $e' = e$ and $f' = c$, i.e., $[a']\omega' = \{a, e, c\}$. It follows from $\mathrm{SP}(\alpha, \omega')$ that ω and ω' have no two-element block in common. Thus $\omega \wedge \omega'$ is of pattern $2 + 1 + \cdots + 1$, and Lemma 4 applies.

If $|[a]\omega \cap [a']\omega'| = 1$ then let $x \in [a']\omega' \setminus [a]\omega$ and let y be the unique element of $[x]\omega \setminus \{x\}$. We have $(x, y) \notin \omega'$, for otherwise $\mathrm{SP}(\omega, \omega')$ would enlarge ω'. Then, however, $\mathrm{SP}(\omega, \omega')$ yields $|[y]\omega' \cap [a]\omega| = 2$, contradicting $|[y]\omega'| < |[a']\omega'| = 3$.

If $[a]\omega \cap [a']\omega' = \emptyset$ then there is an $x \in [a]\omega$ such that $|[x]\omega' \cap [a]\omega| = 1$ and $|[x]\omega'| = 2$. Let $y \in [x]\omega' \setminus \{x\}$, working with $[a]\omega$ and $[y]\omega$ we get a contradiction by $\mathrm{SP}(\omega, \omega')$.

Case 11–11, when both M_3 and M_3' are of type 11. By Lemma 4 we can assume that $\omega \wedge \omega' = 0$. First we show that

$$B = B' \quad \textit{or} \quad B \cap B' = \emptyset. \tag{8}$$

Indeed, suppose the contrary. Then there are elements $x_1 \in B \cap B'$, $x_2 \in B \setminus B'$ and $x_3 \in B' \setminus B$. Then $|[x_3]\iota| = 2$, for otherwise $\mathrm{SP}(\iota, \iota')$ would enlarge ι'. Let $x_4 \in [x_3]\iota \setminus \{x_3\}$. Then $(x_3, x_4) \in \omega$, and $\mathrm{SP}(\iota, \iota')$ gives $(x_2, x_4) \in \iota'$. Hence $x_4 \notin B'$, and $\mathrm{SP}(\iota', \omega)$ clearly leads to a contradiction. This proves (8).

Now let us assume that $|[i]\iota'| > 1$. From $\mathrm{SP}(\iota, \iota')$ we easily infer that $|[i]\iota'| \neq 2$, whence $[i]\iota' = B'$. By (8) we can assume that $B' = \{i, e, f, g\}$. Then $h \notin B'$, and $\mathrm{SP}(\iota', \omega)$ leads to a contradiction. Therefore $|[i]\iota'| = 1$, i.e., $i = i'$.

By Lemma 8 we can assume that i is not separated. Hence there is a $\nu \in L \setminus \{1\}$ with $|[i]\nu| > 1$. Suppose first that $B \cap B' = \emptyset$, i.e., $B' = \{e, f, g, h\}$. Since the role of M_3 and M_3' is symmetric, $[i]\nu \cap B \neq \emptyset$ can be assumed. Then $[i]\nu \supseteq B$ by $\mathrm{SP}(\iota, \nu)$. Since $[i]\nu \cap B' \neq \emptyset$ would similarly imply $[i]\nu \supseteq B'$ albeit $\nu \neq 1$, $[i]\nu = B \cup \{i\}$ and Lemma 5 applies for $\nu \vee \iota' = \big\{\{a, b, c, d, i\}, \{e, f, g, h\}\big\}$ and $\iota \vee \iota' = \big\{\{a, b, c, d\}, \{i\}, \{e, f, g, h\}\big\}$. Secondly, let $B \cap B' \neq \emptyset$, then $B = B'$ by (8). Now, taking $\omega \wedge \omega' = 0$ into account, B is the only nontrivial block of $\iota \wedge \iota'$, and Lemma 4 applies. This proves Case 11–11, and also Theorem 1.

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