Note on the description of join-distributive lattices by permutations

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Abstract. Let L be a join-distributive lattice with length n and width $(\mathrm{Ji}\, L) \le k$. There are two ways to describe L by $k-1$ permutations acting on an n-element set: a combinatorial way given by P. H. Edelman and R. E. Jamison in 1985 and a recent lattice theoretical way of the second author. We prove that these two approaches are equivalent. Also, we characterize join-distributive lattices by trajectories.

1. Introduction

For $x \neq 1$ in a finite lattice L, let x^* denote the join of upper covers of x. A finite lattice L is *join-distributive* if the interval $[x, x^*]$ is distributive for all $x \in L \setminus \{1\}$. The *join-width* of L, denoted by width $(\mathrm{Ji}\, L)$, is the largest k such that there is a k-element antichain of join-irreducible elements of L. As usual, S_n stands for the set of permutations acting on the set $\{1, \dots, n\}$. There are two known ways, Theorems 3.1 and 4.1, to describe a join-distributive lattice with join-width k and length n by $k-1$ permutations; our goal is to enlighten their connection; see Proposition 5.1. This connection exemplifies that Lattice Theory can be applied in Combinatorics and vice versa. Also, Proposition 6.1 gives a new characterization of join-distributive lattices.

2. Join-distributive lattices and related structures

The study of join-distributive lattices goes back to R. P. Dilworth [11], 1940. There were a lot of discoveries and rediscoveries of these lattices and equivalent combinatorial structures; see K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3], B. Monjardet [17], and M. Stern [18] for surveys. For more recent surveys and various definitions of these lattices, see also K. Adaricheva [2], N. Caspard and B. Monjardet [6], and G. Czédli [7, Proposition 2.1 and Remark 2.2]. Note that join-distributivity implies *semimodularity* (also called *upper semimodularity*); the origin of this result is the combination of M. Ward [19] (see also [11, page 771], where [19] is cited) and S. P. Avann [5] (see also P. H. Edelman [13, Theorem 1.1(E,H)], where [5] is recalled).

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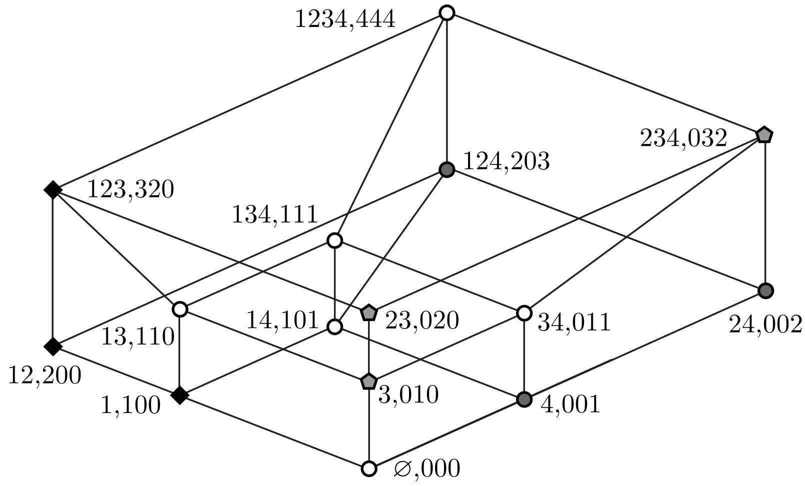

Figure 1. An example of $L_{\mathrm{EJ}}(\vec{\sigma})$ and $L_{\mathrm{C}}(\vec{\pi})$

The set of all subsets of a set A, also called the *powerset of* A, will be denoted by $\mathrm{Pow}(A)$. Next, we recall some concepts and facts from P. H. Edelman [12, Theorem 3.3], P. H. Edelman and R. E. Jamison [14, Theorem 4.1], and R. E. Jamison-Waldner [16]; see also [3, Theorem 1.9 and Subsection 3.1], [7, Section 7 and Lemma 7.4], and D. Armstrong [4, Lemma 2.5] for secondary sources. For a finite set F, the pair $\langle F; \mathfrak{F}\rangle$ is an *antimatroid* if $\varnothing, F \in \mathfrak{F}$, $\mathfrak{F}$ is a join-subsemilattice of $\big(\mathrm{Pow}(F);\cup\big)$, and each $X \in \mathfrak{F}\setminus\{\varnothing\}$ contains an element a such that $X\setminus\{a\} \in \mathfrak{F}$. Note that $\mathfrak{F}$ is a lattice in this case. A pair $\mathfrak{G} = \langle G; \mathcal{L}\rangle$ is a *convex geometry* if $\langle G, \{G \setminus X : X \in \mathcal{L}\}\rangle$ is an antimatroid. Equivalently, if $\varnothing, G \in \mathcal{L}$, G is a finite set, $\mathcal{L}$ is a meet-subsemilattice of $\langle \mathrm{Pow}(G); \cap\rangle$, and, with the notation $\mathcal{L}(X) = \bigcap\{Y \in \mathcal{L} : X \subseteq Y\}$, the *anti-exchange property*

$$(x \notin A,\ y \notin A,\ x \neq y, \text{ and } x \in \mathcal{L}(A \cup \{y\})) \Rightarrow y \notin \mathcal{L}(A \cup \{x\})$$

holds for all $x, y \in G$ and $A \in \mathcal{L}$. Join-distributive lattices are characterized as lattices $\mathfrak{F}$, where $\langle F, \mathfrak{F}\rangle$ is an antimatroid, and also as lattices dual to $\mathcal{L}$, where $\langle G; \mathcal{L}\rangle$ is a convex geometry.

3. A combinatorial approach

For $n \in \mathbb{N} = \{1, 2, \dots\}$ and $k \in \{2, 3, \dots\}$, let $\vec{\sigma} = \langle \sigma_2, \dots, \sigma_k\rangle \in S_n^{k-1}$, where S_n^{k-1} denotes the $(k-1)$-fold Cartesian product $S_n \times \cdots \times S_n$. For convenience, $\sigma_1 \in S_n$ will denote the identity permutation. In the powerset join-semilattice $\langle \mathrm{Pow}(\{1, \dots, n\}); \cup\rangle$, consider the subsemilattice $L_{\mathrm{EJ}}(\vec{\sigma})$ generated by

$$\{\varnothing\} \cup \big\{\{\sigma_i(1), \dots, \sigma_i(j)\} : i \in \{1, \dots, k\},\ j \in \{0, \dots, n\}\big\}. \tag{3.1}$$

Since it contains $\varnothing$, $L_{\rm EJ}(\vec\sigma)$ is a lattice, the *Edelman-Jamison lattice* determined by $\vec\sigma$. Before pointing out how the definition of $L_{\rm EJ}(\vec\sigma)$ comes from P. H. Edelman and R. E. Jamison [14, Theorem 5.2], we present an example. Let $\sigma_2 = \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}$ and $\sigma_3 = \begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$. Then $\vec\sigma = \langle\sigma_2,\sigma_3\rangle \in S_4^2$, and $L_{\rm EJ}(\vec\sigma)$ is depicted in Figure 1. In the label of an element in $L_{\rm EJ}(\vec\sigma)$, only the part before the comma is relevant; to save space, subsets are denoted by listing their elements without commas. For example, 134,111 stands for the subset $\{1,3,4\}$ of $\{1,2,3,4\}$. The chain defined in (3.1), apart from its top $\{1,2,3,4\}$ and bottom $\varnothing$, corresponds to the black-filled small squares for $i=1$, the light grey-filled pentagons for $i=2$, and the dark grey-filled circles for $i=3$. Note that $L_{\rm EJ}(\vec\sigma)$ consists of all subsets of $\{1,2,3,4\}$ but $\{2\}$.

If $\mathfrak{G}_i = \langle G;\mathcal{L}_i\rangle$ are convex geometries for $i\in\{1,2\}$, then so is

$$\mathfrak{G}_1 \vee \mathfrak{G}_2 = \langle\{X_1\cap X_2 : X_1\in\mathcal{L}_1 \text{ and } X_2\in\mathcal{L}_2\};\subseteq\rangle$$

by [14, Theorem 5.1]. Furthermore, by [14, Theorem 5.2], each convex geometry is of the form $\bigvee_{i\in I}\langle G;\mathcal{L}_i\rangle$ where the $\mathcal{L}_i$ are $(|G|+1)$-element chains. Note that if $|G| = n$ and $G = \{g_1,\dots,g_n\}$, then each $\mathcal{L}_i$ is of the form $\{\varnothing\}\cup\big\{\{x_{\pi(i)} : i\le j\} : j\in\{1,\dots,n\}\big\}$ for some permutation $\pi\in S_n$. By the duality principle of lattice theory and by the facts of the present paragraph, the following theorem is a straightforward consequence of [14, Theorem 5.2].

Theorem 3.1. *Up to isomorphism, join-distributive lattices of length n and join-width at most k are characterized as lattices $L_{\rm EJ}(\vec\sigma)$ with $\vec\sigma\in S_n^{k-1}$.*

4. A lattice theoretical approach

Next, we recall a related construction from [7]. Given $\vec\pi = \langle\pi_{12},\dots,\pi_{1k}\rangle \in S_n^{k-1}$, we let $\pi_{11} = \mathrm{id}$ and $\pi_{ij} = \pi_{1j}\circ\pi_{1i}^{-1}$ for $i,j\in\{1,\dots,k\}$. Here we compose permutations from right to left, that is, $(\pi_{1j}\circ\pi_{1i}^{-1})(x) = \pi_{1j}(\pi_{1i}^{-1}(x))$. Note that $\pi_{ii} = \mathrm{id}$, $\pi_{ij} = \pi_{ji}^{-1}$, and $\pi_{jt}\circ\pi_{ij} = \pi_{it}$ hold for all $i,j,t\in\{1,\dots,k\}$. By an *eligible $\vec\pi$-tuple* we mean a k-tuple $\vec x = \langle x_1,\dots,x_k\rangle \in \{0,1,\dots,n\}^k$ such that $\pi_{ij}(x_i+1)\ge x_j+1$ holds for all $i,j\in\{1,\dots,k\}$ such that $x_i<n$. Note that an eligible $\vec\pi$-tuple belongs to $\{0,1,\dots,n-1\}^k\cup\{\langle n,\dots,n\rangle\}$ since $x_j=n$ implies $x_i=n$. The set of eligible $\vec\pi$-tuples is denoted by $L_{\rm C}(\vec\pi)$. It is a poset with respect to the componentwise order: $\vec x\le\vec y$ means that $x_i\le y_i$ for all $i\in\{1,\dots,k\}$. It was not hard to show in [7, first paragraph of Proof 6.2] that $L_{\rm C}(\vec\pi)$ is a meet-subsemilattice of the k-th direct power $\{0\prec\dots\prec n\}^k$ of the chain $\{0\prec 1\prec\dots\prec n\}$ and that $\langle n,\dots,n\rangle\in L_{\rm C}(\vec\pi)$. Therefore, $L_{\rm C}(\vec\pi)$ is a lattice, the *$\vec\pi$-coordinatized lattice*. Its construction is motivated by G. Czédli and E. T. Schmidt [8, Theorem 1], see also M. Stern [18], who proved that there is a surjective cover-preserving join-homomorphism $\varphi\colon\{0\prec\dots\prec n\}^k\to L$, provided L is semimodular. Then, as it is easy to verify, $u\mapsto\bigvee\{x:\varphi(x)=u\}$ is a meet-embedding of L into $\{0\prec\dots\prec n\}^k$.

To give an example, let $\pi_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, $\pi_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$, and let $\vec{\pi} = \langle \pi_{12}, \pi_{13} \rangle \in S_4^2$. Then Figure 1 also gives $L_{\mathrm{C}}(\vec{\pi})$; the eligible $\vec{\pi}$-tuples are given after the commas in the labels. For example, 23,020 in the figure corresponds to $\langle 0, 2, 0 \rangle$. Note that if $\mu_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ and $\mu_{13} = \pi_{13}$, then $L_{\mathrm{C}}(\vec{\pi}) \cong L_{\mathrm{C}}(\vec{\mu})$. The problem of characterizing those pairs of members of S_n^{k-1} that determine the same lattice is not solved yet if $k \geq 3$. For $k = 2$ the solution is given in G. Czédli and E. T. Schmidt [10]; besides $L_{\mathrm{C}}(\vec{\pi}) \cong L_{\mathrm{C}}(\vec{\mu})$ above, see also [7, Example 5.3] for the difficulty. The next theorem was motivated and proved by the second author [7] in a purely lattice theoretical way.

Theorem 4.1. *Up to isomorphism, join-distributive lattices of length n and join-width at most k are characterized as the $\vec{\pi}$-coordinatized lattices $L_{\mathrm{C}}(\vec{\pi})$ with $\vec{\pi} \in S_n^{k-1}$.*

5. The two constructions are equivalent

For $\langle \gamma_2, \ldots, \gamma_k \rangle \in S_n^{k-1}$, we let $\langle \gamma_2, \ldots, \gamma_k \rangle^{-1} = \langle \gamma_2^{-1}, \ldots, \gamma_k^{-1} \rangle$.

Proposition 5.1. *For every $\vec{\sigma} \in S_n^{k-1}$, $L_{\mathrm{EJ}}(\vec{\sigma})$ is isomorphic to $L_{\mathrm{C}}(\vec{\sigma}^{-1})$.*

In some vague sense, Figure 1 reveals why $L_{\mathrm{EJ}}(\vec{\sigma})$ could be of the form $L_{\mathrm{C}}(\vec{\pi})$ for some $\vec{\pi}$. Namely, for $x \in L_{\mathrm{EJ}}(\vec{\sigma})$ and $i \in \{1, \ldots, k\}$, we can define the i-th coordinate of x as the length of the intersection of the ideal $\{y \in L_{\mathrm{EJ}}(\vec{\sigma}) : y \leq x\}$ and the chain given in (3.1). However, the proof is more complex than this initial idea.

Proof. Denote $\vec{\sigma}^{-1}$ by $\vec{\pi} = \langle \pi_{12}, \ldots, \pi_{1k} \rangle$. Note that $\pi_{11} = \sigma_1^{-1} = \mathrm{id} \in S_n$. For $U \in L_{\mathrm{EJ}}(\vec{\sigma})$ and $i \in \{1, \ldots, k\}$, let $U(i) = \max\bigl\{j : \{\sigma_i(1), \ldots, \sigma_i(j)\} \subseteq U\bigr\}$, where $\max \varnothing$ is defined to be 0. We assert that the map

$$\varphi \colon L_{\mathrm{EJ}}(\vec{\sigma}) \to L_{\mathrm{C}}(\vec{\pi}), \text{ defined by } U \mapsto \langle U(1), \ldots, U(k) \rangle,$$

is a lattice isomorphism. To prove that $\varphi(U)$ is an eligible $\vec{\pi}$-tuple, assume that $i, j \in \{1, \ldots, k\}$ such that $U(i) < n$. Then $\sigma_i(U(i) + 1) \notin U$ yields $\sigma_i(U(i) + 1) \notin \{\sigma_j(1), \ldots, \sigma_j(U(j))\}$. However, $\sigma_i(U(i) + 1) \in \{1, \ldots, n\} = \{\sigma_j(1), \ldots, \sigma_j(n)\}$, and we conclude that $\sigma_i(U(i) + 1) = \sigma_j(t)$ holds for some $t \in \{U(j) + 1, \ldots, n\}$. Hence

$$\begin{aligned} \pi_{ij}(U(i) + 1) &= (\pi_{1j} \circ \pi_{i1})(U(i) + 1) = \pi_{1j}\bigl(\pi_{i1}(U(i) + 1)\bigr) \\ &= \pi_{1j}\bigl(\pi_{1i}^{-1}(U(i) + 1)\bigr) = \sigma_j^{-1}\bigl(\sigma_i(U(i) + 1)\bigr) \\ &= \sigma_j^{-1}\bigl(\sigma_j(t)\bigr) = t \geq U(j) + 1. \end{aligned}$$

This proves that $\varphi(U)$ is an eligible $\vec{\pi}$-tuple, and φ is a map from $L_{\mathrm{EJ}}(\vec{\sigma})$ to $L_{\mathrm{C}}(\vec{\pi})$. Since $L_{\mathrm{EJ}}(\vec{\sigma})$ is generated by the set given in (3.1), we conclude

$$U = \bigcup_{i=1}^{k} \{\sigma_i(1), \dots, \sigma_i(U(i))\}.$$

This implies that U is determined by $\langle U(1), \dots, U(k)\rangle = \varphi(U)$, that is, φ is injective. To prove that φ is surjective, let $\vec{x} = \langle x_1, \dots, x_k\rangle$ be a $\vec{\pi}$-eligible tuple, that is, $\vec{x} \in L_{\mathrm{C}}(\vec{\pi})$. Define

$$V = \bigcup_{i=1}^{k} \{\sigma_i(1), \dots, \sigma_i(x_i)\}. \tag{5.1}$$

(Note that if $x_i = 0$, then $\{\sigma_i(1), \dots, \sigma_i(x_i)\}$ denotes the empty set.) For the sake of contradiction, suppose $\varphi(V) \neq \vec{x}$. Then, by the definition of φ, there exists an $i \in \{1, \dots, k\}$ such that $x_i < n$ and $\sigma_i(x_i + 1) \in V$. Hence, there is a $j \in \{1, \dots, k\}$ such that $\sigma_i(x_i + 1) \in \{\sigma_j(1), \dots, \sigma_j(x_j)\}$. That is, $\sigma_i(x_i + 1) = \sigma_j(t)$ for some $t \in \{1, \dots, x_j\}$. Therefore,

$$\begin{aligned} \pi_{ij}(x_i + 1) &= \pi_{1j}(\pi_{i1}(x_i + 1)) = \pi_{1j}(\pi_{1i}^{-1}(x_i + 1) = \sigma_j^{-1}(\sigma_i(x_i + 1)) \\ &= \sigma_j^{-1}(\sigma_j(t)) = t \leq x_j, \end{aligned}$$

which contradicts the $\vec{\pi}$-eligibility of $\vec{x}$. Thus $\varphi(V) = \vec{x}$ and φ is surjective.

We have shown that φ is bijective. For $\vec{x} \in L_{\mathrm{C}}(\vec{\pi})$, $\varphi^{-1}(\vec{x})$ is the set V given in (5.1). Thus φ and φ^{-1} are monotone, and φ is a lattice isomorphism. □

Remark 5.2. Since there is no restriction on $(n, k) \in \mathbb{N} \times \{2, 3, \dots\}$ in Theorems 3.1 and 4.1, one might have the feeling that, for a given n, the join-width of a join-distributive lattice of length n can be arbitrarily large. This is not so because, up to isomorphism, there are only finitely many join-distributive lattices of length n; this folkloric fact follows trivially from (6.1), given later.

Remark 5.3. Clearly, Proposition 5.1 and Theorem 3.1 imply Theorem 4.1. Thus, we obtain a new, combinatorial proof of Theorem 4.1. Similarly, Proposition 5.1 and Theorem 4.1 imply Theorem 3.1. Hence, we obtain a new, lattice theoretical proof of Theorem 3.1 and that of P. H. Edelman and R. E. Jamison [14, Theorem 5.2].

Comparison. We can compare Theorems 3.1 and 4.1, and the corresponding original approaches, as follows. In case of Theorem 3.1, the construction of the lattice $L_{\mathrm{EJ}}(\vec{\sigma})$ is very simple, and a join-generating subset is also given.

In case of Theorem 4.1, the elements of the lattice $L_{\mathrm{C}}(\vec{\pi})$ are exactly given by their coordinates, the eligible $\vec{\pi}$-tuples. Moreover, the meet operation is easy, and we have a satisfactory description of the optimal meet-generating subset, since it was proved in [7, Lemma 6.5] that the *set of meet-irreducible elements* of $L_{\mathrm{C}}(\vec{\pi})$ is $\mathrm{Mi}(L_{\mathrm{C}}(\vec{\pi})) = \big\{\langle \pi_{11}(i)-1, \dots, \pi_{1k}(i)-1\rangle : i \in \{1, \dots, n\}\big\}$.

6. Characterization by trajectories

For a lattice L of finite length, the set $\{[a,b] : a \prec b,\ a,b \in L\}$ of prime intervals of L will be denoted by $\mathrm{PrInt}(L)$. For $[a,b],[c,d] \in \mathrm{PrInt}(L)$, we say that $[a,b]$ and $[c,d]$ are *consecutive* if $\{a,b,c,d\}$ is a covering square, that is, a 4-element cover-preserving boolean sublattice of L. The transitive reflexive closure of the consecutiveness relation on $\mathrm{PrInt}(L)$ is an equivalence, and the blocks of this equivalence relation are called the *trajectories* of L; this concept was introduced for some particular semimodular lattices in G. Czédli and E. T. Schmidt [9]. For distinct $[a,b],[c,d] \in \mathrm{PrInt}(L)$, these two prime intervals are *comparable* if either $b \le c$, or $d \le a$.

Proposition 6.1. *For a semimodular lattice L, the following three conditions are equivalent.*

(i) *L is join-distributive.*
(ii) *L is of finite length, and for every trajectory T of L and every maximal chain C of L, $|\mathrm{PrInt}(C) \cap T| = 1$.*
(iii) *L is of finite length, and no two distinct comparable prime intervals of L belong to the same trajectory.*

Since we intend to bring the lattice theoretical and the combinatorial approaches closer and to enlighten their connection, we give two proofs.

Lattice theoretical proof of Proposition 6.1. Since any two comparable prime intervals belong to the set of prime intervals of an appropriate maximal chain C, (ii) trivially implies (iii).

Assume (i). By a *cover-preserving diamond* of L we mean a cover-preserving sublattice that is (isomorphic to) the five element, non-distributive, modular lattice M_3. Referencing [5] and [19], we have already mentioned that join-distributivity implies semimodularity. Clearly, it also implies that L contains no cover-preserving diamond. Thus, [7, Lemma 3.3] yields (ii).

Next, assume (iii). Since any two prime intervals of a cover-preserving diamond would belong to the same trajectory, L contains no such diamond. By [7, Corollary 4.4 and Proposition 6.1(A)], (i) holds. □

Combinatorial proof of Proposition 6.1. (ii) ⇒ (iii) is trivial, as before.

Assume (i). We can also assume that $L = \mathfrak{F}$, where $\langle\{1,\dots,n\};\mathfrak{F}\rangle$ is an antimatroid. We assert that, for any $X, Y \in \mathfrak{F}$,

$$X \prec Y \quad \text{iff} \quad X \subset Y \text{ and } |Y \setminus X| = 1. \tag{6.1}$$

The "if" part is clear. Suppose, for a contradiction, that $X \prec Y$ and x and y are distinct elements in $Y \setminus X$. Pick a sequence $Y = Y_0 \supset Y_1 \supset \cdots \supset Y_t = \varnothing$ in $\mathfrak{F}$ with $|Y_{i-1} \setminus Y_i| = 1$ for $i \in \{1,\dots,t\}$, and a j such that $|Y_j \cap \{x,y\}| = 1$. This is a contradiction proving (6.1), since $X \cup Y_j \in \mathfrak{F}$ but $X \subset X \cup Y_j \subset Y$.

Armed with (6.1), assume that $\{A = B \wedge C, B, C, D = B \cup C\}$ is a covering square in $\mathfrak{F}$. Note that A and $B \cap C$ can be different; however, $A \subseteq B \cap C$.

By (6.1), there exist $u, x \in D$ such that $B = D \setminus \{u\}$ and $C = D \setminus \{x\}$. These elements are distinct since $B \neq C$. Hence $x \in B$ and, by $A \subseteq C$, $x \notin A$. Using (6.1) again, we obtain $A = B \setminus \{x\}$. We have seen that whenever $[A, B]$ and $[C, D]$ are consecutive prime intervals, then there is a common x such that $A = B \setminus \{x\}$ and $C = D \setminus \{x\}$. This implies that for each trajectory T of $\mathfrak{F}$, there exists an $x_T \in \{1, \dots, n\}$ such that $X = Y \setminus \{x_T\}$ holds for all $[X, Y] \in T$. Clearly, this implies that

$$|C \cap T| \leq 1 \tag{6.2}$$

holds for every trajectory T and every maximal chain C. To prove that (6.2) is actually an equality, pick a prime interval $[A, A \cup \{x_T\}]$ of T. The maximal chain C has a unique prime interval of the form $[B, B \cup \{x_T\}]$. Since $\mathfrak{F}$ is $\cup$-closed, $[A \cup B, A \cup B \cup \{x_T\}]$ is a prime interval of $\mathfrak{F}$. Let $E_0 = A \prec E_1 \prec \cdots \prec E_t = A \cup B$ be a maximal chain of $\mathfrak{F}$ in the interval $[A, A \cup B]$. Using (6.1) or semimodularity, it follows that $[E_{i-1}, E_{i-1} \cup \{x_T\}]$ and $[E_i, E_i \cup \{x_T\}]$ form a pair of consecutive prime intervals for $i \in \{1, \dots, t\}$. This implies that $[A, A \cup \{x_T\}]$ and $[A \cup B, A \cup B \cup \{x_T\}]$ belong to the same trajectory, which is T. We obtain similarly that $[B, B \cup \{x_T\}]$ and $[A \cup B, A \cup B \cup \{x_T\}]$ belong to the same trajectory. Hence, $[B, B \cup \{x_T\}] \in T$, and $|C \cap T| \geq 1$. This, together with (6.2), proves that $\mathfrak{F}$ satisfies (ii), and so does L.

Next, in order to prove that (iii) implies (i), assume (iii). Let $n = \operatorname{length} L$. Obviously, as in the previous proof, L contains no cover-preserving diamond. H. Abels [1, Theorem 3.9(a⇒b)] implies that L is a cover-preserving join-subsemilattice of a finite distributive lattice D. Thus if $x \in L \setminus \{1\}$, then the interval $[x, x^*]_L$ of L is a cover-preserving join-subsemilattice of D. Let $a_1, \dots, a_t$ be the covers of x in L, that is, the atoms of $[x, x^*]_L$. If we had, say, $a_1 \leq a_2 \vee \cdots \vee a_t$, then we would get a contradiction in D as follows: $a_1 = a_1 \wedge (a_2 \vee \cdots \vee a_t) = (a_1 \wedge a_2) \vee \cdots \vee (a_1 \wedge a_t) = x \wedge \cdots \wedge x = x$. Thus $a_1, \dots, a_t$ are independent atoms in $[x, x^*]_L$. Therefore, it follows from G. Grätzer [15, Theorem 380] and the semimodularity of $[x, x^*]_L$ that the sublattice S generated by $\{a_1, \dots, a_t\}$ in L is the 2^t-element boolean lattice. In particular, $\operatorname{length} S = t = \operatorname{length}\big([x, x^*]_L\big)$ since $\{x, x^*\} \subseteq S \subseteq [x, x^*]_L$. Since the embedding is cover-preserving, the length of the interval $[x, x^*]_D$ in D is also t. Hence $|\operatorname{Ji}([x, x^*]_D)| = t$ by [15, Corollary 112], which clearly implies $|[x, x^*]_D| \leq 2^t$. Now from $[x, x^*]_L \subseteq [x, x^*]_D$ and $2^t = |S| \leq |[x, x^*]_L| \leq |[x, x^*]_D| \leq 2^t$ we conclude $[x, x^*]_L = [x, x^*]_D$. This implies that $[x, x^*]_L$ is distributive. Thus, (i) holds. □

Corollary 6.2. *If no two distinct comparable prime intervals of a semimodular lattice L of finite length belong to the same trajectory, then L is finite.*

Proof. Join-distributive lattices are finite. Hence, Proposition 6.1 applies. □

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