Some recent results on semimodular lattices*

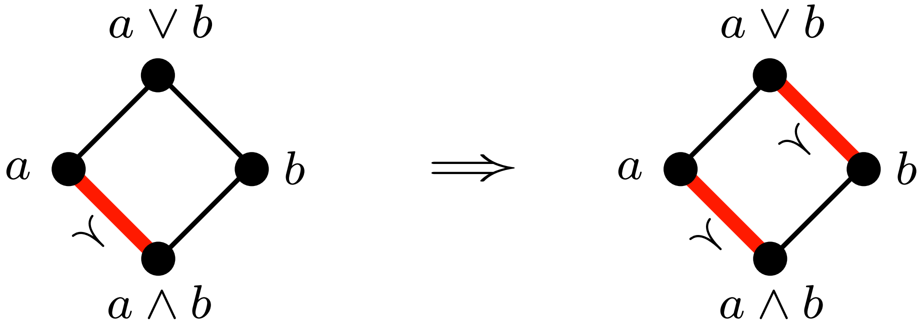
Gábor Czédli

AAA79, February 12, 2010, Olomouc

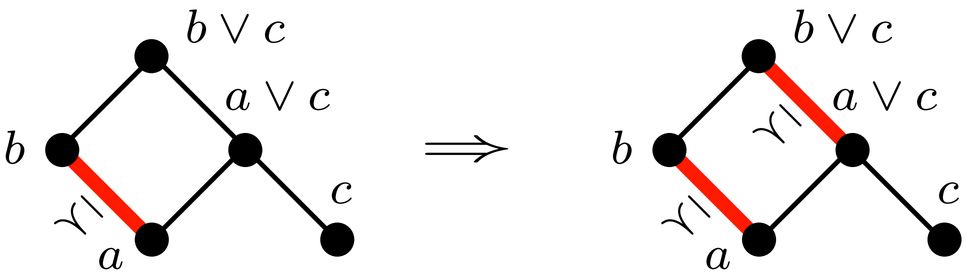
2010. február 10.

*www.math.u-szeged.hu/∼czedli/

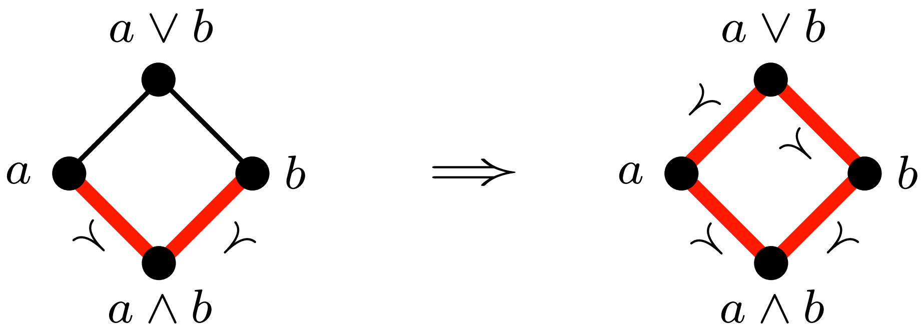
What is a semimodular lattice?



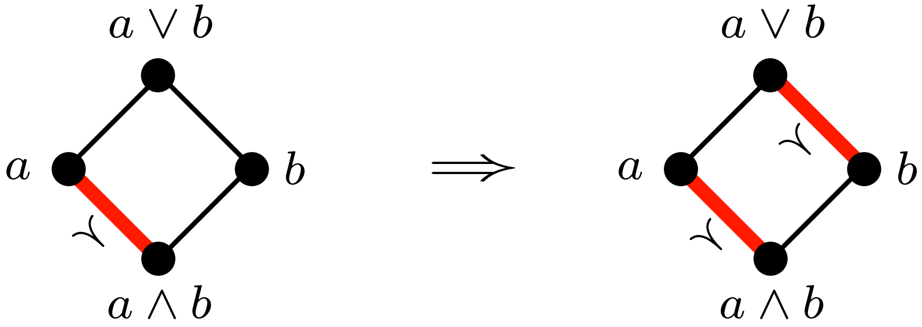
or, equivalently,



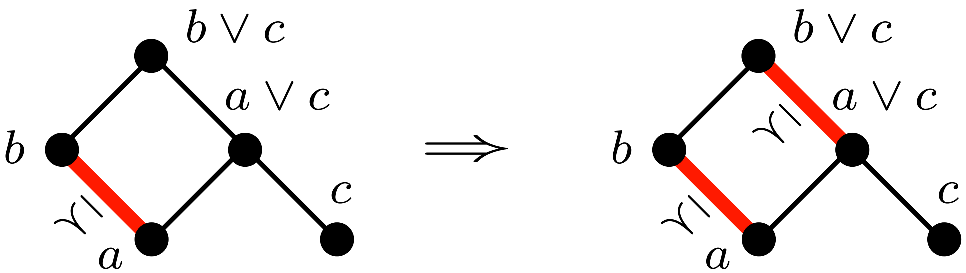
or, if length$(L) < \infty$,



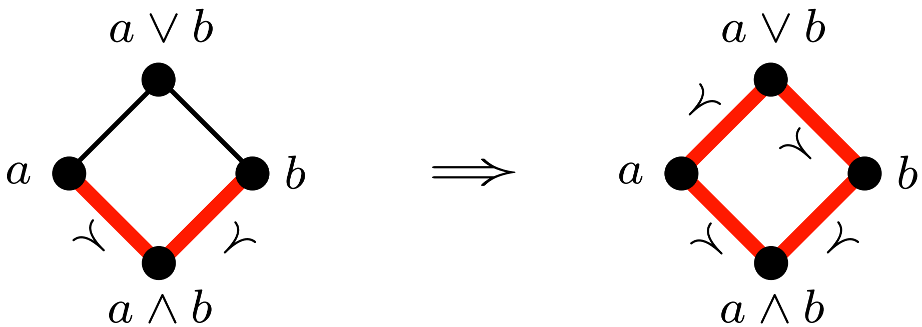
What is a semimodular lattice?



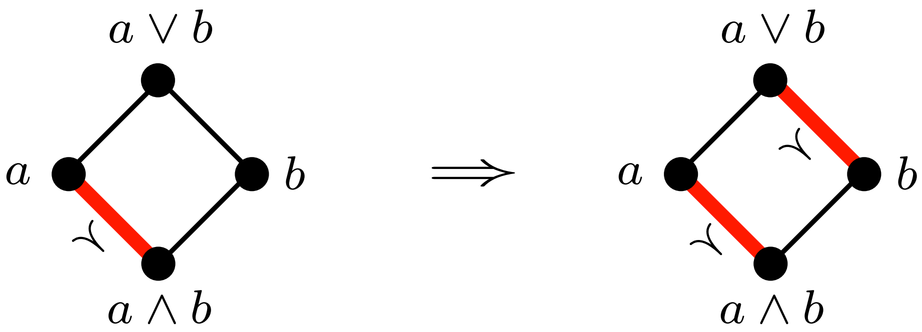
or, equivalently,



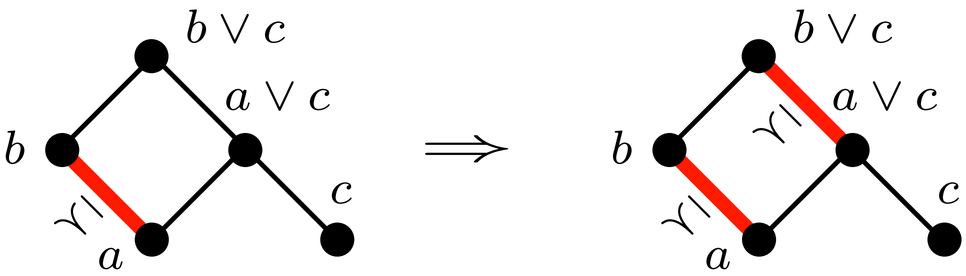
or, if length$(L) < \infty$,



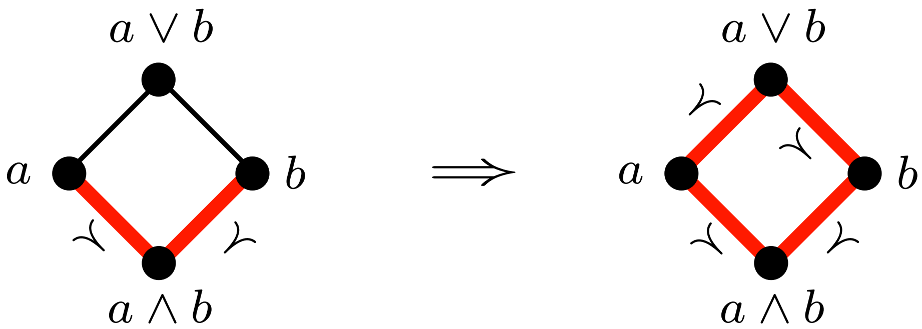
F



or, equivalently,



or, if $\text{length}(L) < \infty$,



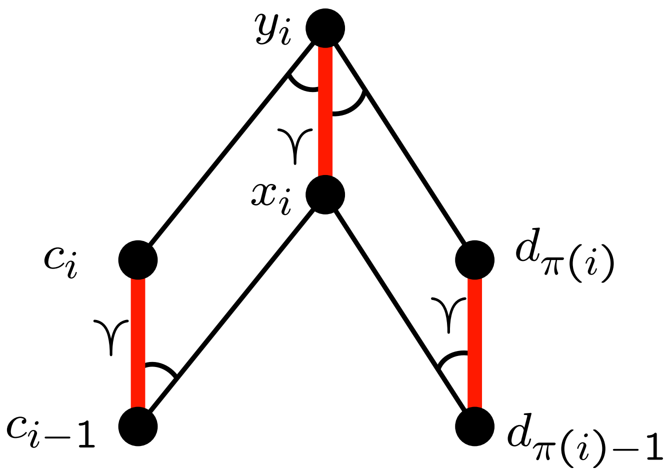
Finite length condition: always assumed. Then any two maximal chains have the same length. Moreover,

Theorem (Grätzer–Nation, 2009) Let $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}$ be maximal chains; L semimodular.

Theorem (Grätzer–Nation, 2009) Let $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}$ be maximal chains; L semimodular. Then $m = n$, and $\exists \pi \in S_n$ such that, for each i, there is a prime interval $[x_i, y_i]$ with $[c_{i-1}, c_i] \nearrow [x_i, y_i] \searrow [d_{\pi(i)-1}, d_{\pi(i)}]$.

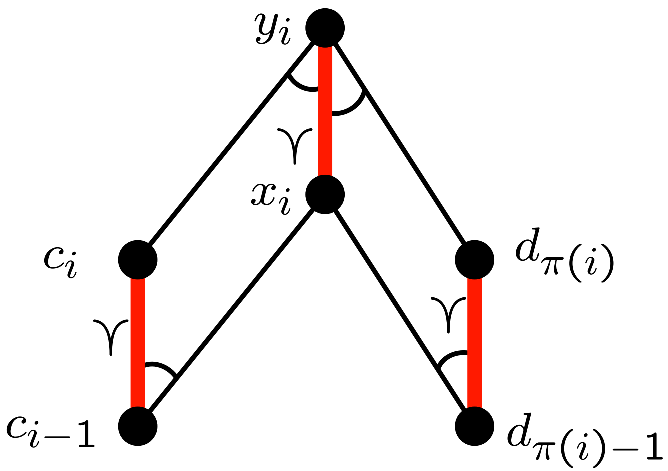
Theorem (Grätzer–Nation, 2009) Let $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}$ be maximal chains; L semimodular. Then $m = n$, and $\exists \pi \in S_n$ such that, for each i, there is a prime interval $[x_i, y_i]$ with $[c_{i-1}, c_i] \nearrow [x_i, y_i] \searrow [d_{\pi(i)-1}, d_{\pi(i)}]$. (Novelty: **up-and-down projectivities** between the the prime itervals of these chains.)

up-and-down projectivity:



Theorem (Grätzer–Nation, 2009) Let $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}$ be maximal chains; L semimodular. Then $m = n$, and $\exists \pi \in S_n$ such that, for each i, there is a prime interval $[x_i, y_i]$ with $[c_{i-1}, c_i] \nearrow [x_i, y_i] \searrow [d_{\pi(i)-1}, d_{\pi(i)}]$. (Novelty: **up-and-down projectivities** between the the prime itervals of these chains.)

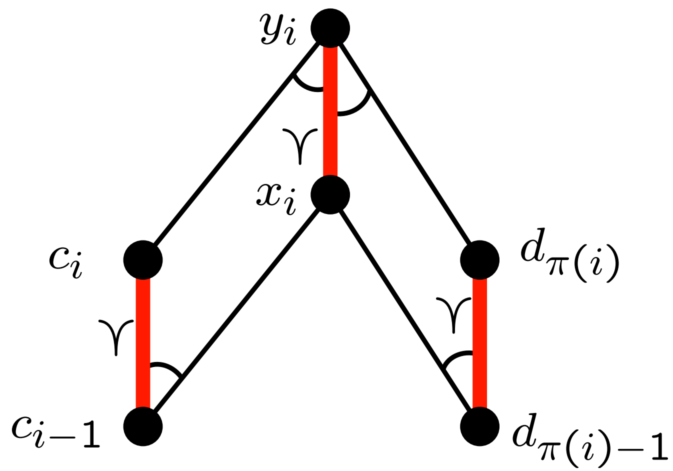
up-and-down projectivity:



Remark: The dual statement gives the Jordan–Hölder thm. for the *lattice* of subnormal groups of a finite group.

Theorem (Grätzer–Nation, 2009) Let $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}$ be maximal chains; L semimodular. Then $m = n$, and $\exists \pi \in S_n$ such that, for each i, there is a prime interval $[x_i, y_i]$ with $[c_{i-1}, c_i] \nearrow [x_i, y_i] \searrow [d_{\pi(i)-1}, d_{\pi(i)}]$. (Novelty: **up-and-down projectivities** between the the prime itervals of these chains.)

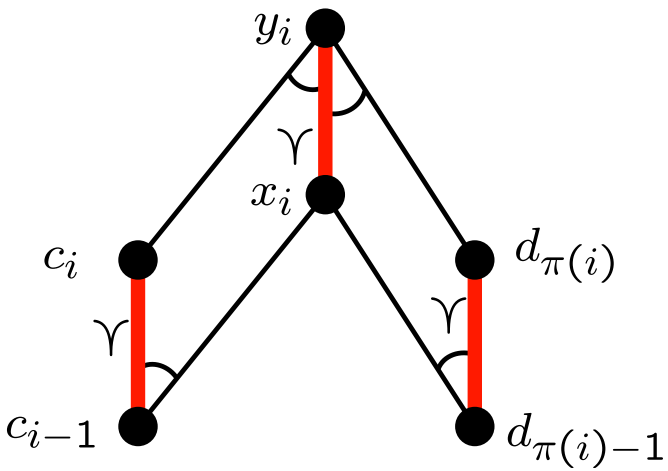
up-and-down projectivity:



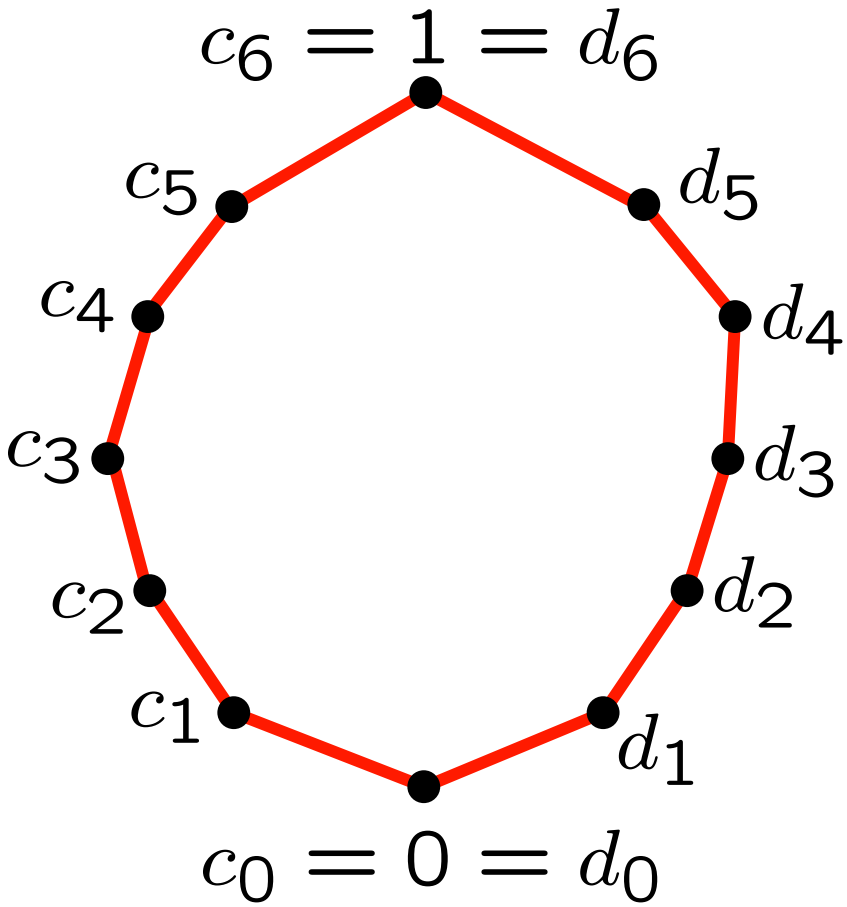
Remark: The dual statement gives the Jordan–Hölder thm. for the *lattice* of subnormal groups of a finite group. (H. Wielandt 1939: it is a lattice. Subnormal: transitive closure of $\triangleleft$)

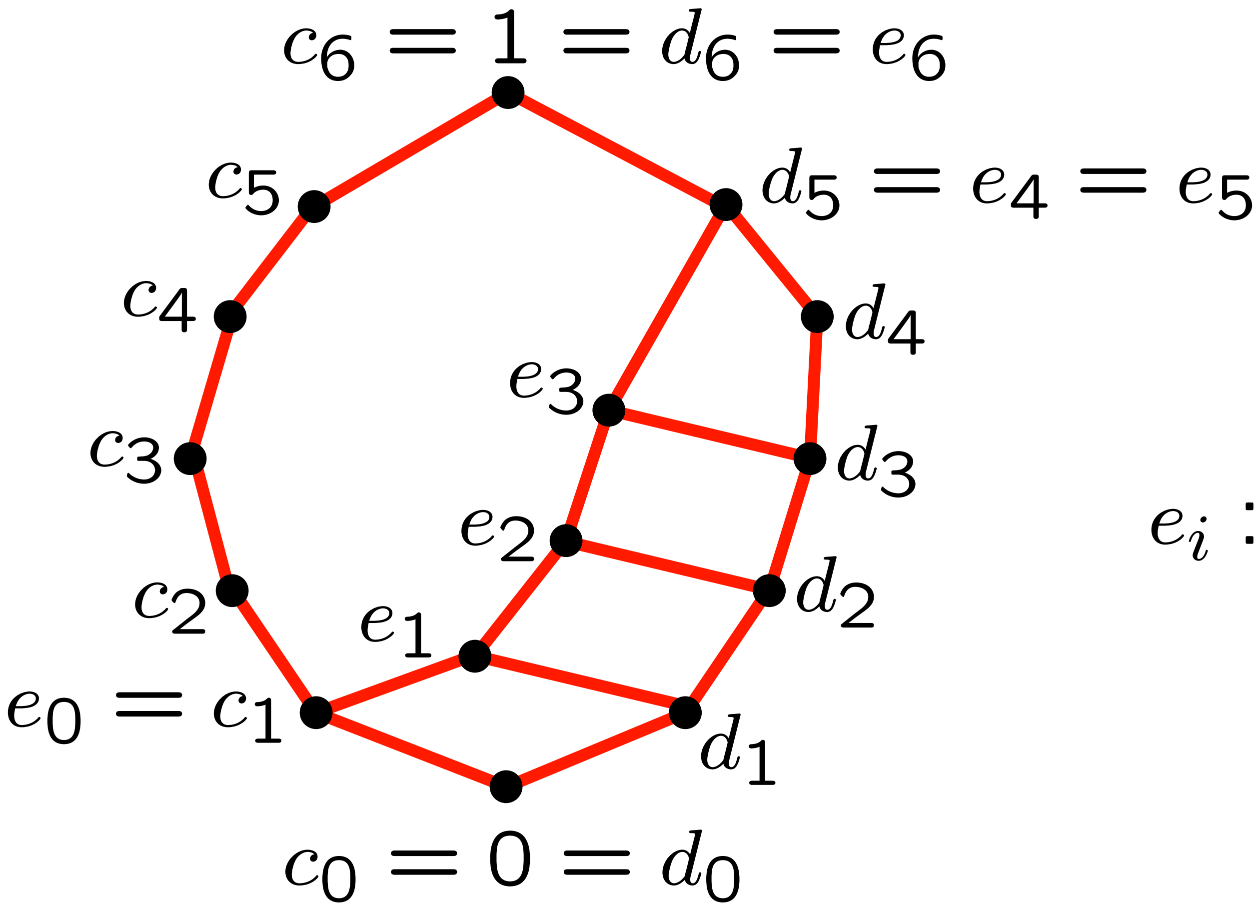
Theorem (Grätzer–Nation, 2009) Let $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}$ be maximal chains; L semimodular. Then $m = n$, and $\exists \pi \in S_n$ such that, for each i, there is a prime interval $[x_i, y_i]$ with $[c_{i-1}, c_i] \nearrow [x_i, y_i] \searrow [d_{\pi(i)-1}, d_{\pi(i)}]$. (Novelty: **up-and-down projectivities** between the the prime itervals of these chains.)

up-and-down projectivity:

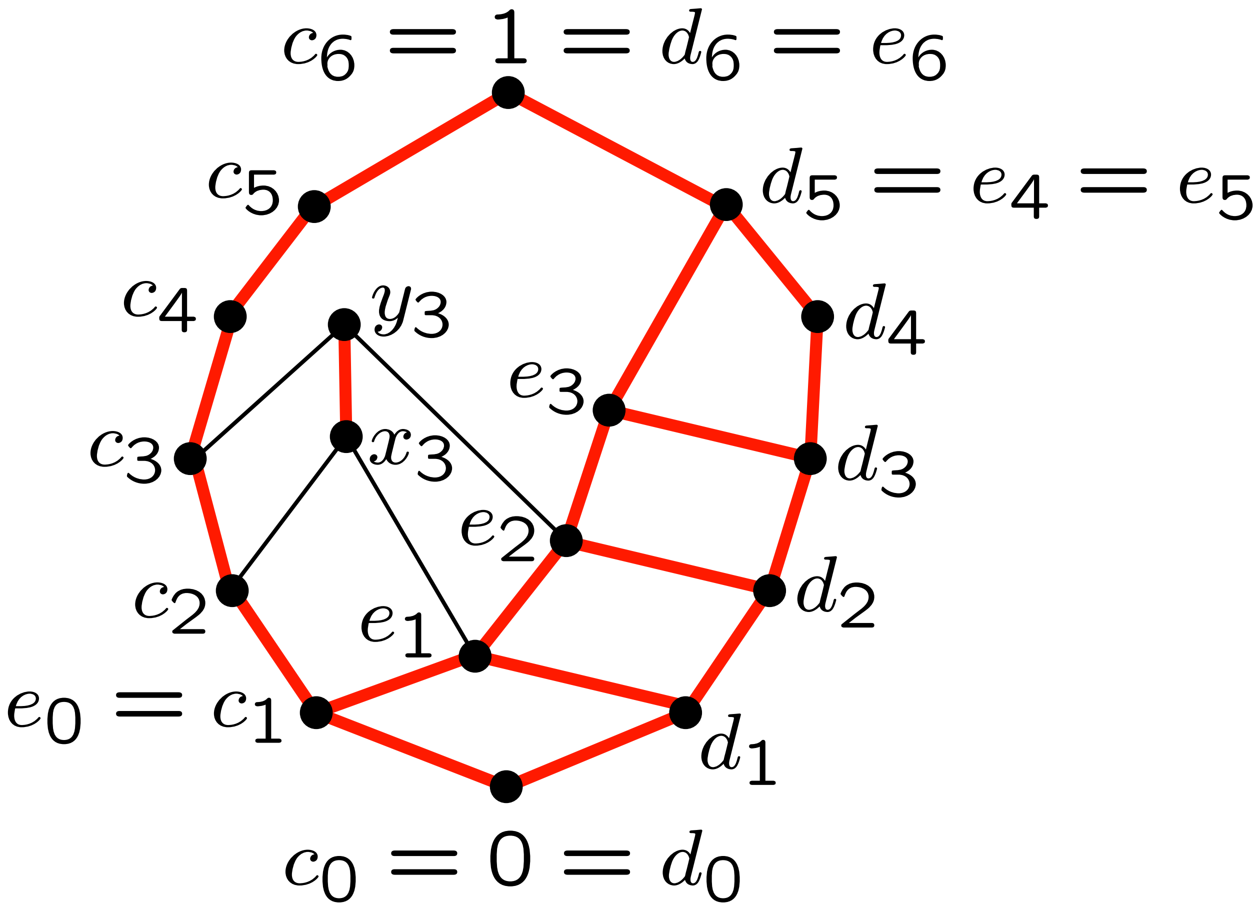


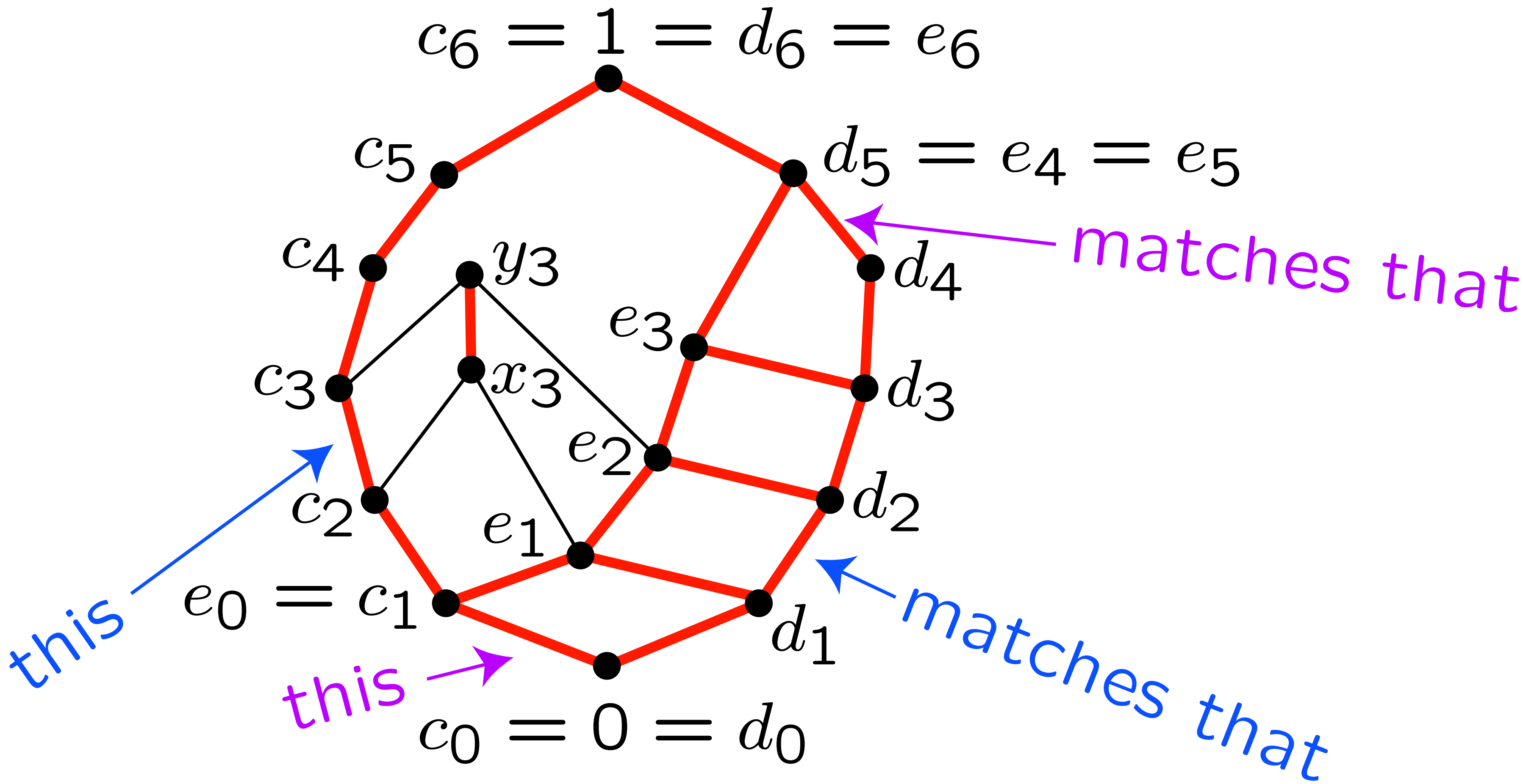
Remark: The dual statement gives the Jordan–Hölder thm. for the *lattice* of subnormal groups of a finite group. (H. Wielandt 1939: it is a lattice. Subnormal: transitive closure of $\triangleleft$) **Proof:**

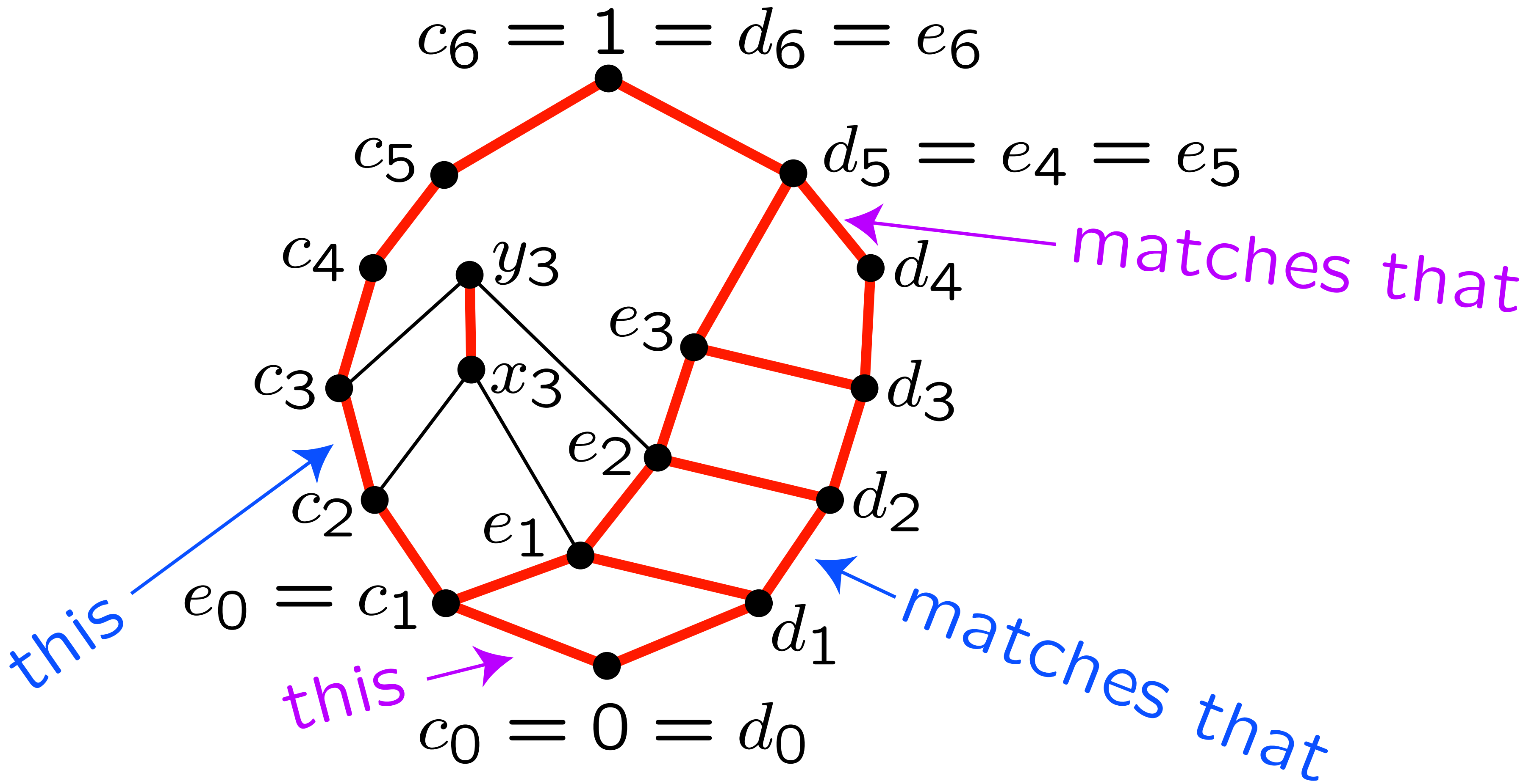

$c_6 = 1 = d_6$
c_5
d_5
c_4
d_4
c_3
d_3
c_2
d_2
c_1
d_1
$c_0 = 0 = d_0$



$$e_i := c_1 \vee d_i$$


$c_6 = 1 = d_6 = e_6$
c_5
$d_5 = e_4 = e_5$
c_4
y_3
d_4
e_3
c_3
x_3
d_3
e_2
c_2
d_2
e_1
$e_0 = c_1$
d_1
$c_0 = 0 = d_0$


$c_6 = 1 = d_6 = e_6$
c_5
$d_5 = e_4 = e_5$
matches that
c_4
y_3
d_4
e_3
c_3
x_3
d_3
e_2
c_2
e_1
d_2
this
$e_0 = c_1$
matches that
d_1
this
$c_0 = 0 = d_0$


$c_6 = 1 = d_6 = e_6$
c_5
$d_5 = e_4 = e_5$
matches that
c_4
y_3
d_4
e_3
c_3
x_3
d_3
e_2
c_2
d_2
e_1
this
$e_0 = c_1$
d_1
matches that
this
$c_0 = 0 = d_0$

Q.e.d.

Each finite lattice is a join-homomorphic image of a finite boolean lattice.

Each finite lattice is a join-homomorphic image of a finite boolean lattice. But $\mathbf{H}$ does not preserve semimodularity.

Each finite lattice is a join-homomorphic image of a finite boolean lattice. But $\mathbf{H}$ does not preserve semimodularity. The useful notion is $\mathbf{H}^{\vee}_{\prec}$ (cover-preserving $\vee$-homorphic images). Grätzer and Knapp (2007): $\mathbf{H}^{\vee}_{\prec}$ preserves semimodularity!

Each finite lattice is a join-homomorphic image of a finite boolean lattice. But $\mathbf{H}$ does not preserve semimodularity. The useful notion is $\mathbf{H}^{\vee}_{\prec}$ (cover-preserving $\vee$-homorphic images). Grätzer and Knapp (2007): $\mathbf{H}^{\vee}_{\prec}$ preserves semimodularity! They proved some converse for planar lattices. Soon afterwards, planarity was dropped:

Theorem (Czédli–Schmidt 2008) Finite semimodular lattices are characterized as $\mathbf{H}^{\vee}_{\prec}$ images of finite distributive lattices.

Proof (Almost trivial)

Each finite lattice is a join-homomorphic image of a finite boolean lattice. But $\mathbf{H}$ does not preserve semimodularity. The useful notion is $\mathbf{H}^{\vee}_{\prec}$ (cover-preserving $\vee$-homorphic images). Grätzer and Knapp (2007): $\mathbf{H}^{\vee}_{\prec}$ preserves semimodularity! They proved some converse for planar lattices. Soon afterwards, planarity was dropped:

Theorem (Czédli–Schmidt 2008) Finite semimodular lattices are characterized as $\mathbf{H}^{\vee}_{\prec}$ images of finite distributive lattices.

Proof (Almost trivial) Let L be finite semimodular. $k := w(J(L))$

Each finite lattice is a join-homomorphic image of a finite boolean lattice. But $\mathbf{H}$ does not preserve semimodularity. The useful notion is $\mathbf{H}^{\vee}_{\prec}$ (cover-preserving $\vee$-homorphic images). Grätzer and Knapp (2007): $\mathbf{H}^{\vee}_{\prec}$ preserves semimodularity! They proved some converse for planar lattices. Soon afterwards, planarity was dropped:

Theorem (Czédli–Schmidt 2008) Finite semimodular lattices are characterized as $\mathbf{H}^{\vee}_{\prec}$ images of finite distributive lattices.

Proof (Almost trivial) Let L be finite semimodular. $k := w(J(L))$ (max. size of antichains). We show that L is an $\mathbf{H}^{\vee}_{\prec}$-image of the direct product of k chains.

Dilworth $\Longrightarrow$ $J(L)$ is the union of k chains.

Dilwoth $\Longrightarrow J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \ldots, C_k$ of L. Let

Dilworth $\Longrightarrow$ $J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \dots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \dots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Dilworth $\Longrightarrow$ $J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \ldots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \ldots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism.

Dilworth $\Longrightarrow$ $J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \dots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \dots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism. Since $J(L) \subseteq C_1 \cup \cdots \cup C_k$, φ is surjective.

Dilworth $\Longrightarrow J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \dots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \dots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism. Since $J(L) \subseteq C_1 \cup \cdots \cup C_k$, φ is surjective.

Next, assume that $\vec{u} \preceq \vec{v}$. Then, apart from indexing, $\vec{u} = (a, c_2, \dots, c_k)$ and $\vec{v} = (b, c_2, \dots, c_k)$, where $a \preceq b$ in L. T

Dilworth $\Longrightarrow J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \ldots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \ldots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism. Since $J(L) \subseteq C_1 \cup \cdots \cup C_k$, φ is surjective.

Next, assume that $\vec{u} \preceq \vec{v}$. Then, apart from indexing, $\vec{u} = (a, c_2, \ldots, c_k)$ and $\vec{v} = (b, c_2, \ldots, c_k)$, where $a \preceq b$ in L. This relation is preserved by $x \vee c_2 \vee \cdots \vee c_k$, so $\varphi(\vec{u}) \preceq \varphi(\vec{v})$. $\surd$

Dilworth $\Longrightarrow J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \ldots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \ldots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism. Since $J(L) \subseteq C_1 \cup \cdots \cup C_k$, φ is surjective.

Next, assume that $\vec{u} \preceq \vec{v}$. Then, apart from indexing, $\vec{u} = (a, c_2, \ldots, c_k)$ and $\vec{v} = (b, c_2, \ldots, c_k)$, where $a \preceq b$ in L. This relation is preserved by $x \vee c_2 \vee \cdots \vee c_k$, so $\varphi(\vec{u}) \preceq \varphi(\vec{v})$. $\surd$

The original proof was longer, due to an important lemma of separate interest.

Dilworth $\Longrightarrow$ $J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \ldots, C_k$ of L. Let

$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \ldots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism. Since $J(L) \subseteq C_1 \cup \cdots \cup C_k$, φ is surjective.

Next, assume that $\vec{u} \preceq \vec{v}$. Then, apart from indexing, $\vec{u} = (a, c_2, \ldots, c_k)$ and $\vec{v} = (b, c_2, \ldots, c_k)$, where $a \preceq b$ in L. This relation is preserved by $x \vee c_2 \vee \cdots \vee c_k$, so $\varphi(\vec{u}) \preceq \varphi(\vec{v})$. $\surd$

The original proof was longer, due to an important lemma of separate interest. Sm lattices are better uderstood as D/Θ, so we should understand what is Θ.

Dilworth $\Longrightarrow J(L)$ is the union of k chains. Extend these chains to maximal chains $C_1, \ldots, C_k$ of L. Let

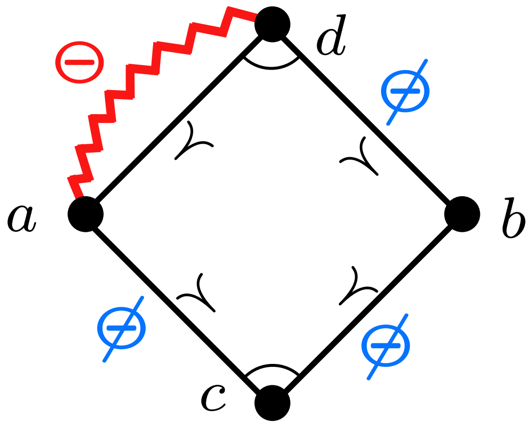
$$\varphi : C_1 \times \cdots \times C_k \to L, \qquad (x_1, \ldots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

Clearly, φ is a $\vee$-homomorphism. Since $J(L) \subseteq C_1 \cup \cdots \cup C_k$, φ is surjective.

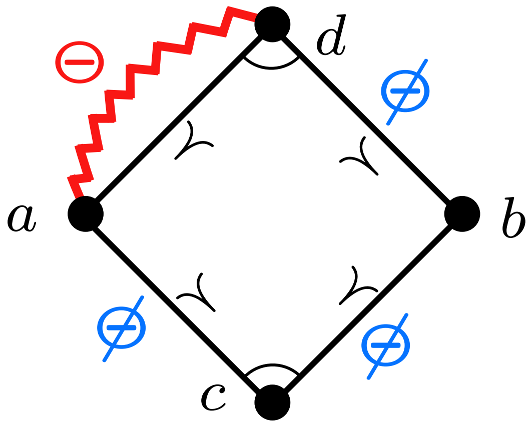
Next, assume that $\vec{u} \preceq \vec{v}$. Then, apart from indexing, $\vec{u} = (a, c_2, \ldots, c_k)$ and $\vec{v} = (b, c_2, \ldots, c_k)$, where $a \preceq b$ in L. This relation is preserved by $x \vee c_2 \vee \cdots \vee c_k$, so $\varphi(\vec{u}) \preceq \varphi(\vec{v})$. $\surd$

The original proof was longer, due to an important lemma of separate interest. Sm lattices are better uderstood as D/Θ, so we should understand what is Θ. Define *cover-preserving $\vee$-congruences* as those Θ for which the natural $L \to L/\Theta$, $x \mapsto [x]\Theta$ is a $\mathbf{H}^{\vee}_{\preceq}$ homomorphisms.

Lemma Let Θ be a $\vee$-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no such covering square:

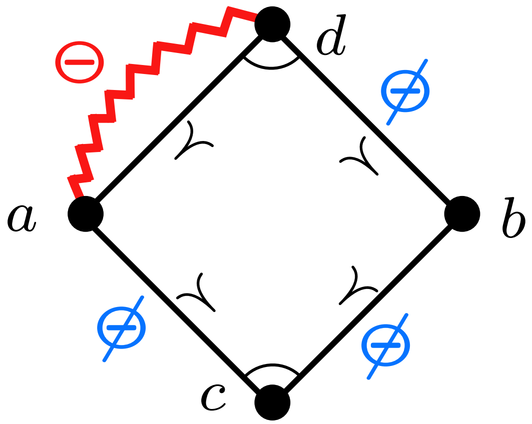


Lemma Let Θ be a ∨-congruence of a finite semimod. lattice L.
Then Θ is cover-preserving iff there is no such covering square:



Since dist $\Longrightarrow$ sm., we get

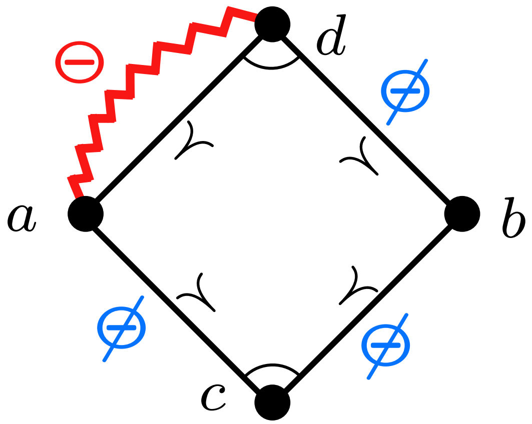
Lemma Let Θ be a ∨-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no such covering square:



Since dist $\Longrightarrow$ sm., we get

Corollary Finite semimodular lattices are characterized as D/Θ where D is a finite distr. lattice and Θ is as above.

Lemma Let Θ be a ∨-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no such covering square:



Since dist $\Longrightarrow$ sm., we get

Corollary Finite semimodular lattices are characterized as D/Θ where D is a finite distr. lattice and Θ is as above.

Before proving the lemma, we mention some applications of this corollary.

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \leq |L|/2$.

(V

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \leq |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

Let m

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \le |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

Let $m := |J(L)|$. Clearly, $|L|$

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \leq |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

Let $m := |J(L)|$. Clearly, $|L| \leq 2^m$.

Theorem (

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \le |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

Let $m := |J(L)|$. Clearly, $|L| \le 2^m$.

Theorem (Czédli–Maróti–Schmidt 2009) the *averaged Frankl's* for $|L| \ge \frac{2}{3} \cdot 2^m$ **plus** Frankl's for all L' with $|J(L')| \le m$.

Theorem

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \leq |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

Let $m := |J(L)|$. Clearly, $|L| \leq 2^m$.

Theorem (Czédli–Maróti–Schmidt 2009) the *averaged Frankl's* for $|L| \geq \frac{2}{3} \cdot 2^m$ **plus** Frankl's for all L' with $|J(L')| \leq m$.

Theorem (Czédli–Schmidt 2008, later!) Frankl's for L sm with $|L| \geq \frac{5}{8} \cdot 2^m$.

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \le |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

Let $m := |J(L)|$. Clearly, $|L| \le 2^m$.

Theorem (Czédli–Maróti–Schmidt 2009) the *averaged Frankl's* for $|L| \ge \frac{2}{3} \cdot 2^m$ **plus** Frankl's for all L' with $|J(L')| \le m$.

Theorem (Czédli–Schmidt 2008, later!) Frankl's for L sm with $|L| \ge \frac{5}{8} \cdot 2^m$. ($\frac{5}{8}$ is better than $\frac{2}{3}$, and no ugly red assumption!)

Theorem (

Frankl's conjecture: $\exists f \in J(L)$ high enough (i.e., $|\uparrow f| \leq |L|/2$. (Valdimir Božin, Novi Sad, 2009). Dozens of positive particular cases. E.g., Reinhold in 2000 for dually sm lattices. Open for sm lattices!)

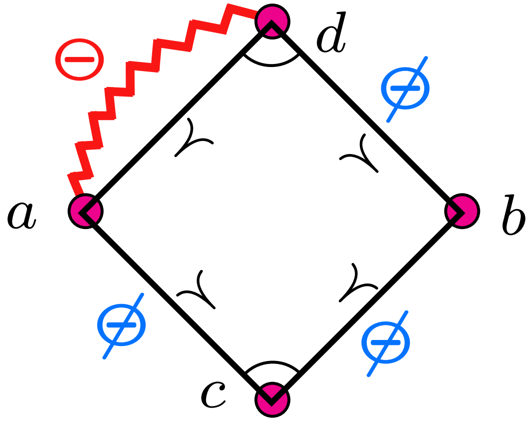
Let $m := |J(L)|$. Clearly, $|L| \leq 2^m$.

Theorem (Czédli–Maróti–Schmidt 2009) the *averaged Frankl's* for $|L| \geq \frac{2}{3} \cdot 2^m$ **plus** Frankl's for all L' with $|J(L')| \leq m$.

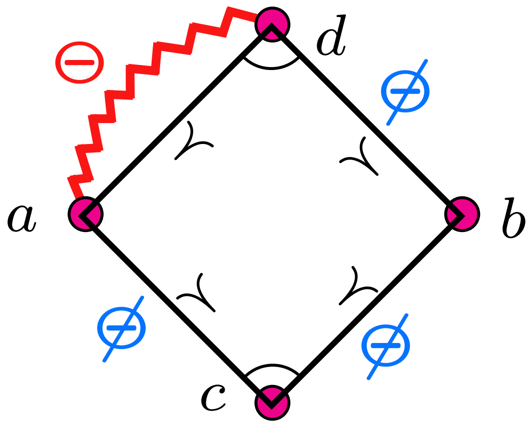
Theorem (Czédli–Schmidt 2008, later!) Frankl's for L sm with $|L| \geq \frac{5}{8} \cdot 2^m$. ($\frac{5}{8}$ is better than $\frac{2}{3}$, and no ugly red assumption!)

Theorem (Czédli–Schmidt 2008) Frankl's conjecture holds for finite planar sm lattices.

Lemma (Cz–Sch 2008)Let Θ be a ∨-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no:

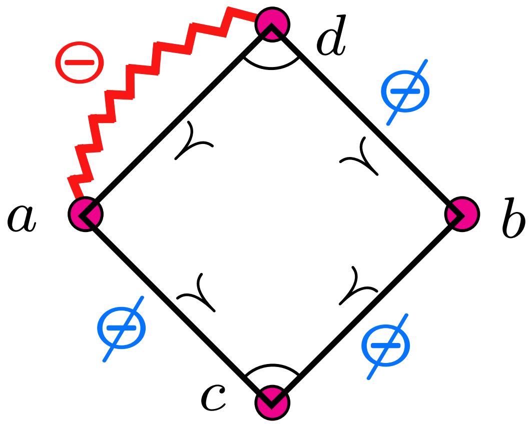


Lemma (Cz–Sch 2008)Let Θ be a ∨-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no:



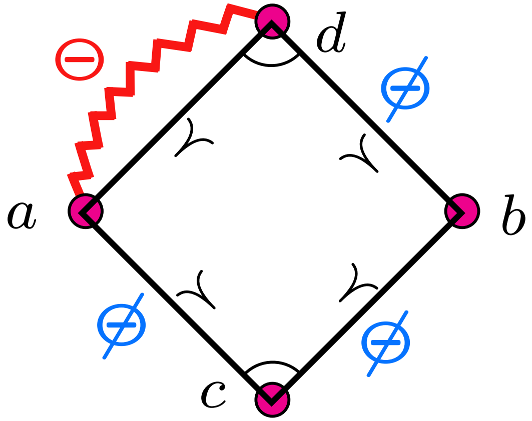
Proof (of the lemma). Suppose ∃ such a square. Then $c \prec a$, but

Lemma (Cz–Sch 2008)Let Θ be a $\vee$-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no:

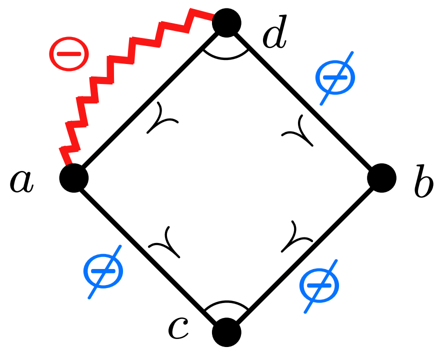


Proof (of the lemma). Suppose $\exists$ such a square. Then $c \prec a$, but $[c]\Theta < [b]\Theta < [d]\Theta = [a]\Theta$, so

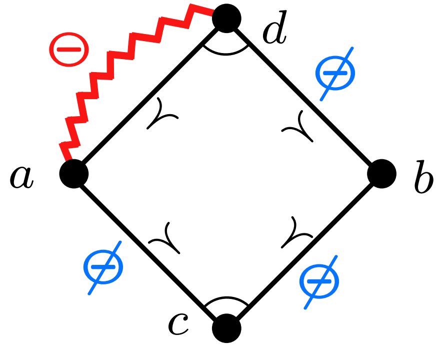
Lemma (Cz–Sch 2008)Let Θ be a $\vee$-congruence of a finite semimod. lattice L. Then Θ is cover-preserving iff there is no:



Proof (of the lemma). Suppose $\exists$ such a square. Then $c \prec a$, but $[c]\Theta < [b]\Theta < [d]\Theta = [a]\Theta$, so Θ is not cover-preserving.

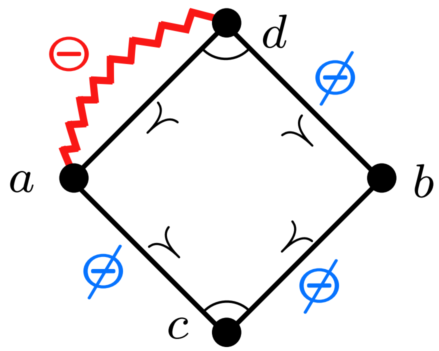


Conversely, by way of contradiction, assume that there is no such square but $\exists\ a \prec b \in L$ with $\underbrace{[a]\Theta}_{A} < \underbrace{[c]\Theta}_{C} < \underbrace{[b]\Theta}_{B}$.



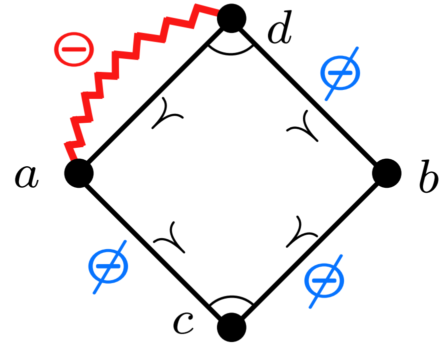
Conversely, by way of contradiction, assume that there is no such square but $\exists\ a \prec b \in L$ with $\underbrace{[a]\Theta}_{A} < \underbrace{[c]\Theta}_{C} < \underbrace{[b]\Theta}_{B}$.

We may assume that $a \le c$,



Conversely, by way of contradiction, assume that there is no such square but $\exists\ a \prec b \in L$ with $\underbrace{[a]\Theta}_{A} < \underbrace{[c]\Theta}_{C} < \underbrace{[b]\Theta}_{B}$.

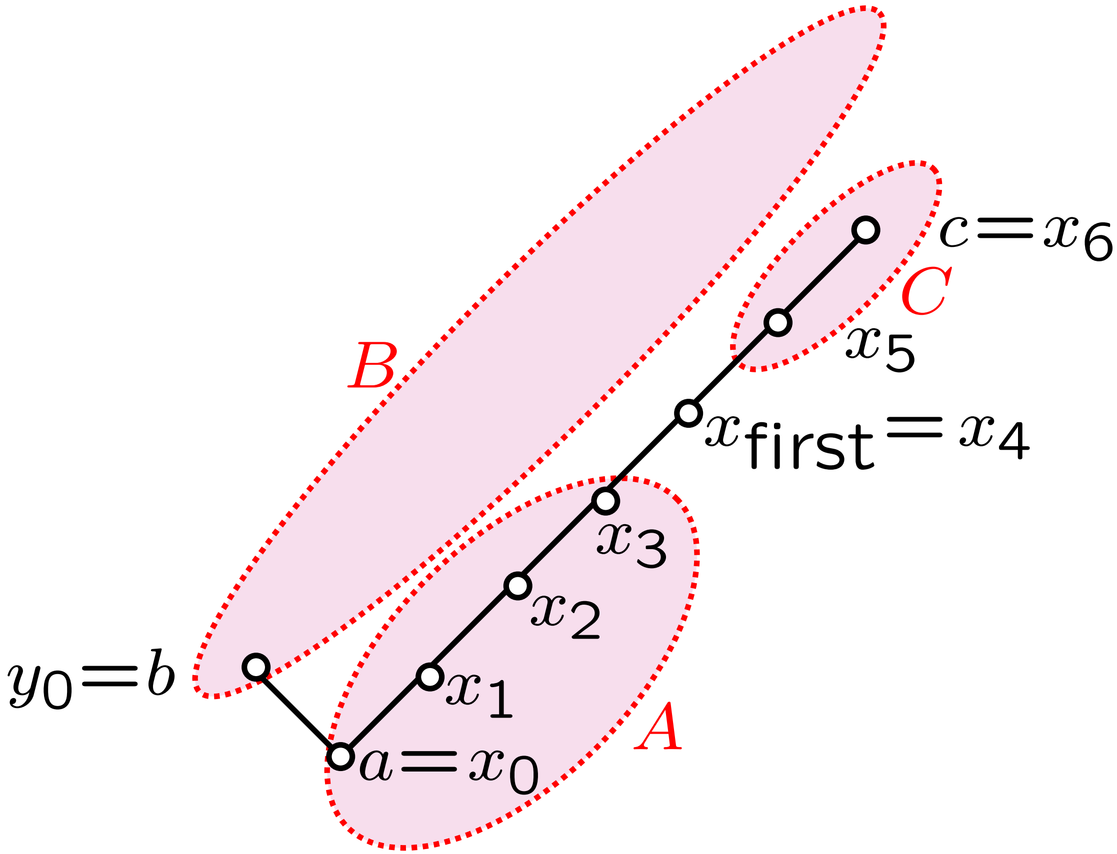
We may assume that $a \le c$, for otherwise $c := a \vee c$.



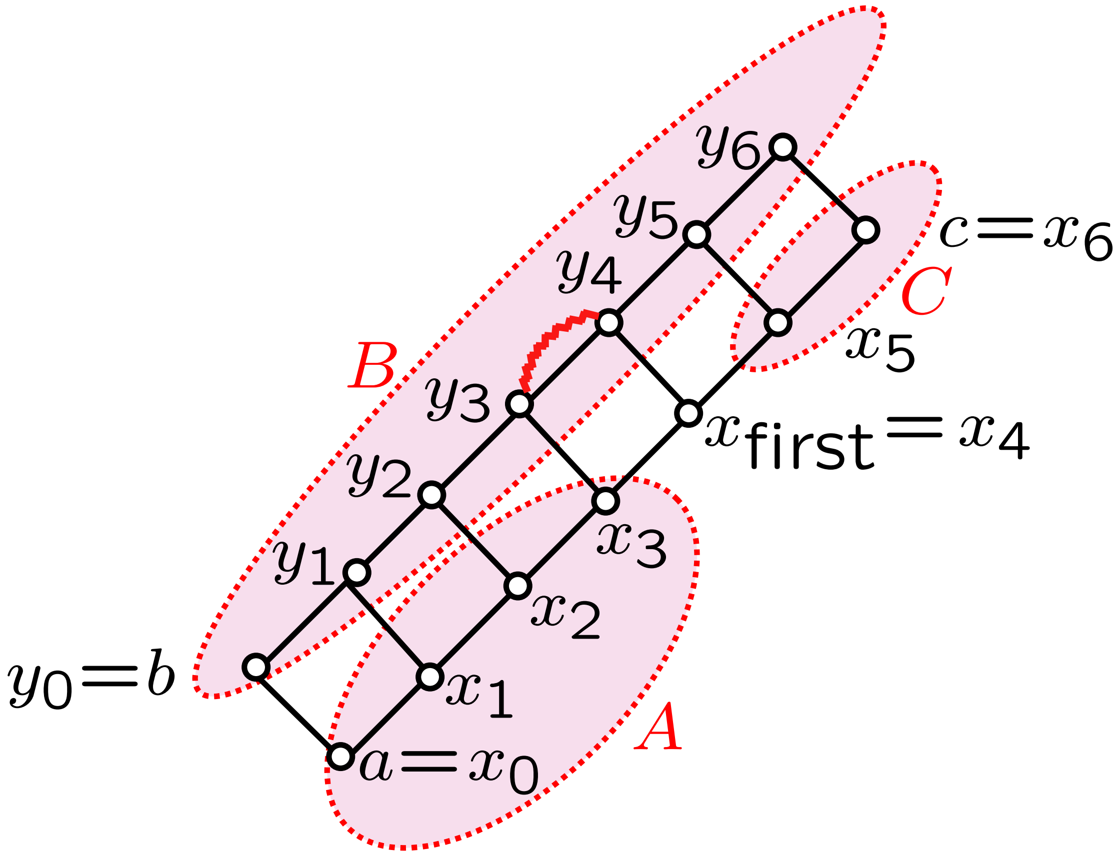
Conversely, by way of contradiction, assume that there is no such square but $\exists\ a \prec b \in L$ with $\underbrace{[a]\Theta}_{A} < \underbrace{[c]\Theta}_{C} < \underbrace{[b]\Theta}_{B}$.

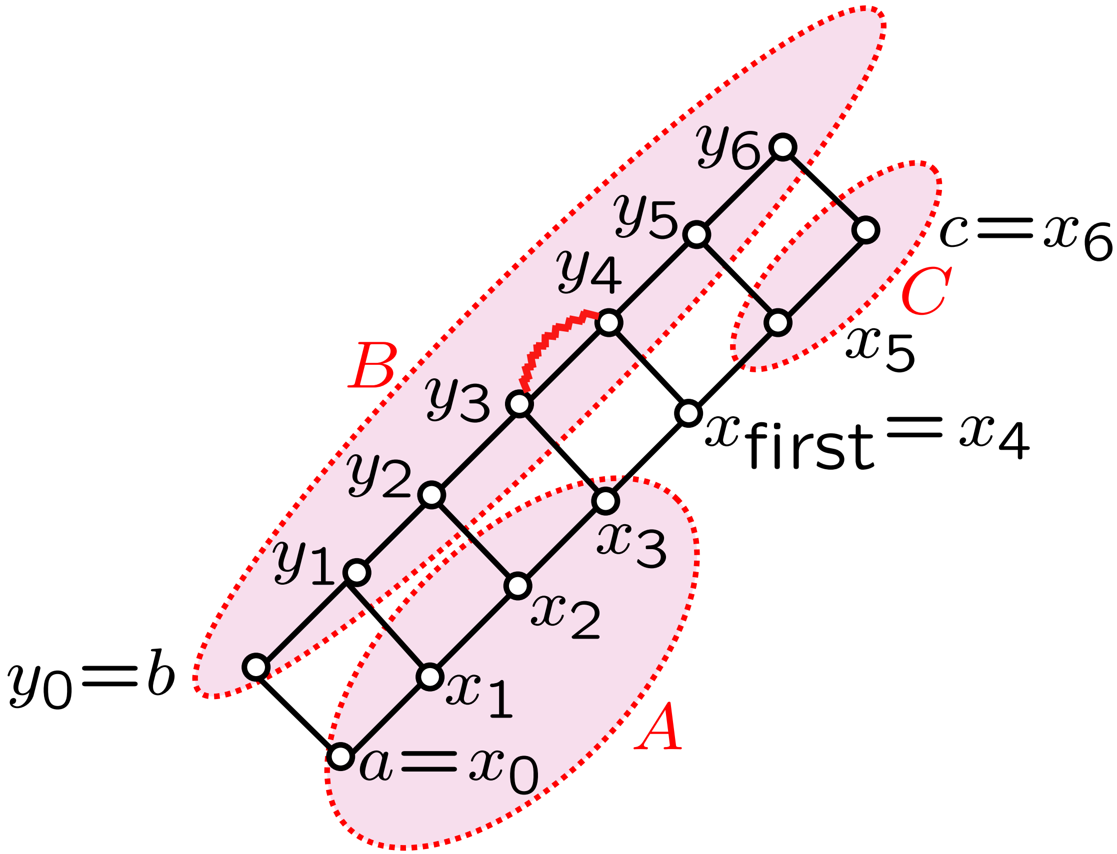
We may assume that $a \le c$, for otherwise $c := a \vee c$.

Take a maximal chain in $[a, c]$:



$c = x_6$
C
x_5
B
$x_{\text{first}} = x_4$
x_3
x_2
$y_0 = b$
x_1
A
$a = x_0$


y_6
y_5
$c = x_6$
y_4
C
B
x_5
y_3
$x_{\text{first}} = x_4$
y_2
x_3
y_1
x_2
$y_0 = b$
x_1
A
$a = x_0$



and the $\{x_3, x_4, y_3, y_4\}$ square gives a contradiction. $\surd$

Theorem (Czédli–Schmidt 2008) Let L be finite semimodular. Then ∃! finite distributive D such that L is a $\mathbf{H}^{\vee}_{\prec}$ image of D, $J(D)$ equals $J(L)$, and the $\vee$-homomorphism acts identically on $J(D)$.

Theorem (Czédli–Schmidt 2008) Let L be finite semimodular. Then $\exists!$ finite distributive D such that L is a $\mathbf{H}^{\vee}_{\prec}$ image of D, $J(D)$ equals $J(L)$, and the $\vee$-homomorphism acts identically on $J(D)$.

A semimodular lattice of finite length is called is called a **finite dimensional geometric lattice** (sometimes called matroid), if it is atomistic.

Theorem (Czédli–Schmidt 2008) Let L be finite semimodular. Then $\exists!$ finite distributive D such that L is a $\mathbf{H}^{\vee}_{\prec}$ image of D, $J(D)$ equals $J(L)$, and the $\vee$-homomorphism acts identically on $J(D)$.

A semimodular lattice of finite length is called is called a **finite dimensional geometric lattice** (sometimes called matroid), if it is atomistic. In this case, $J(L)$ is an antichain, and we obtain

Corollary Finite geometric lattices are characterized as $\mathbf{H}^{\vee}_{\prec}$-images of finite boolean lattices.

Like in case of $\mathbf{H}$, sublattices of a semimodular lattice need not be semimodular.

Like in case of $\mathbf{H}$, sublattices of a semimodular lattice need not be semimodular. The right operator is $\mathbf{S}_{\prec}$, forming **cover-preserving sublattices**. (The reverse procedure is called **cover-preserving embedding**.)

Like in case of $\mathbf{H}$, sublattices of a semimodular lattice need not be semimodular. The right operator is $\mathbf{S}_{\prec}$, forming **cover-preserving sublattices**. (The reverse procedure is called **cover-preserving embedding**.)

Grätzer and E.W. Kiss (1986): each finite sm lattice has a $\mathbf{S}_{\prec}$-embedding into a *finite* geometric lattice.

W

Like in case of $\mathbf{H}$, sublattices of a semimodular lattice need not be semimodular. The right operator is $\mathbf{S}_{\prec}$, forming **cover-preserving sublattices**. (The reverse procedure is called **cover-preserving embedding**.)

Grätzer and E.W. Kiss (1986): each finite sm lattice has a $\mathbf{S}_{\prec}$-embedding into a *finite* geometric lattice.

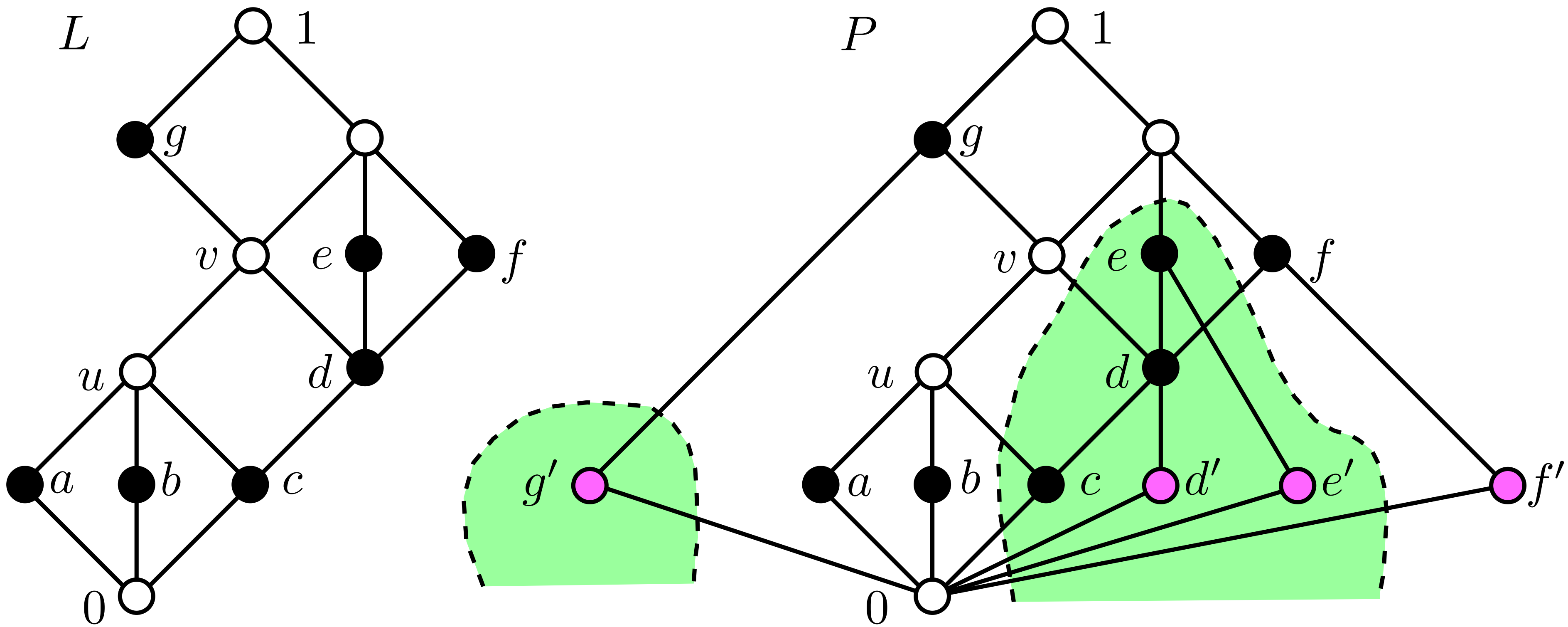
We strengthened this result by constructing a lattice $G(L)$ such that

Theorem (Czédli–Schmidt, 2008) Let L be a semimodular lattice of **finite length**. Then $G = G(L)$ is a geometric lattice such that L is a cover-preserving sublattice of G, $|J(L)| = |A(G)|$, and length(L) = length(G).

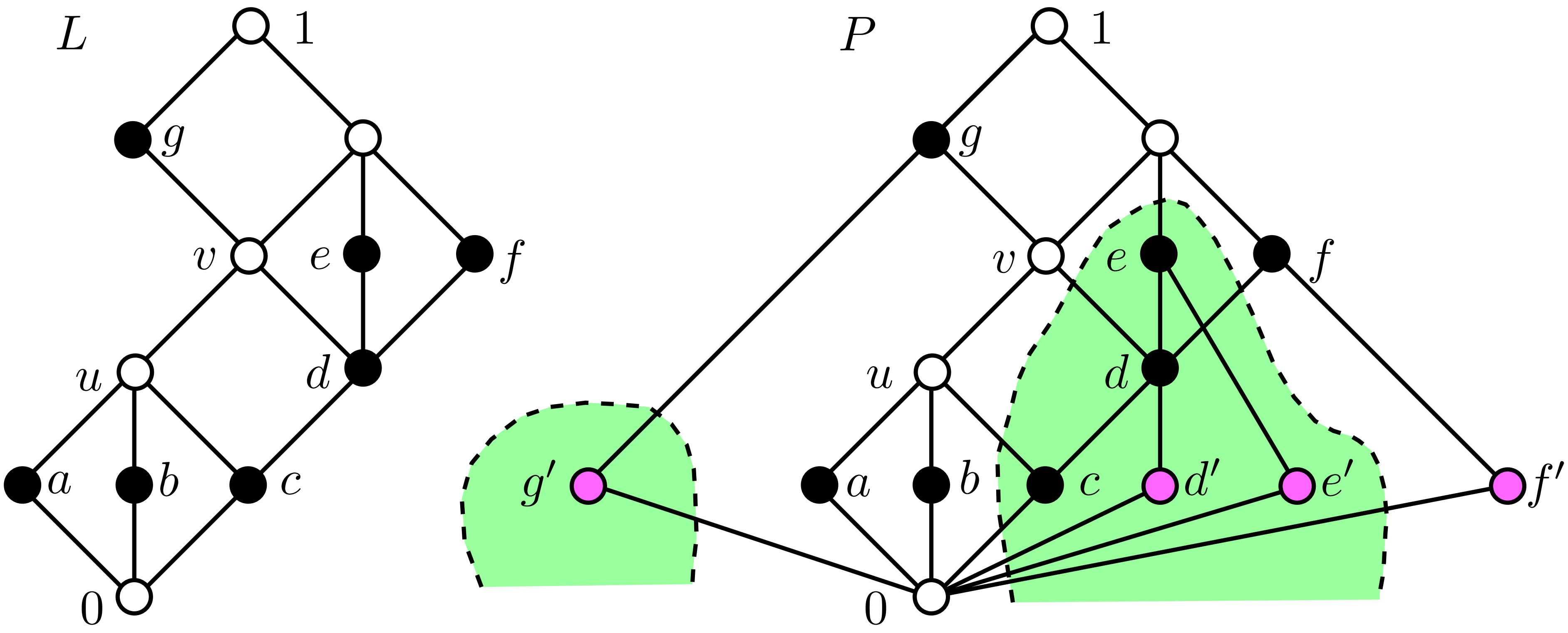
The construction of $G(L)$ (7 page proof):

$H(L) := J(L) \setminus A(L)$ („high” join-irreducibles”).

For each $x \in H(L)$, we insert a new x' such that $0 \prec x' \prec x$. This way we obtain $P = (P; \leq)$:

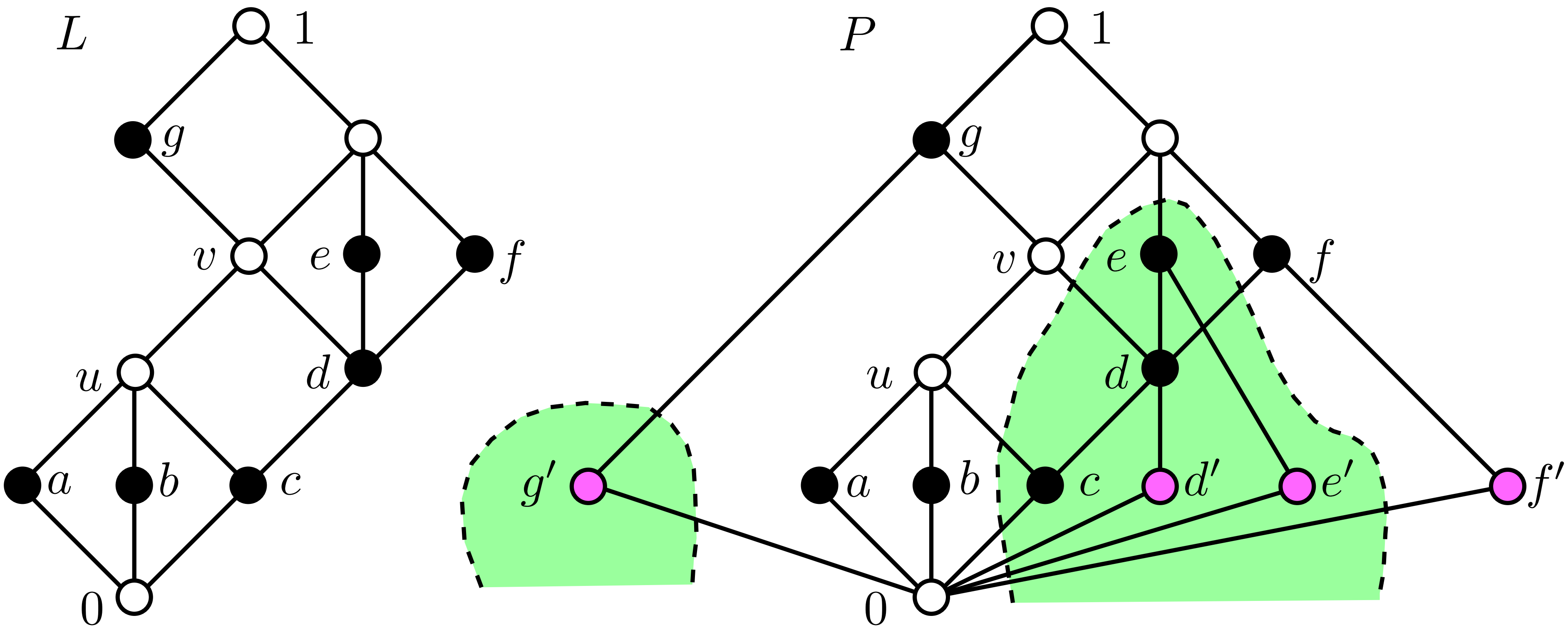


For each $x \in H(L)$, we insert a new x' such that $0 \prec x' \prec x$. This way we obtain $P = (P; \leq)$:



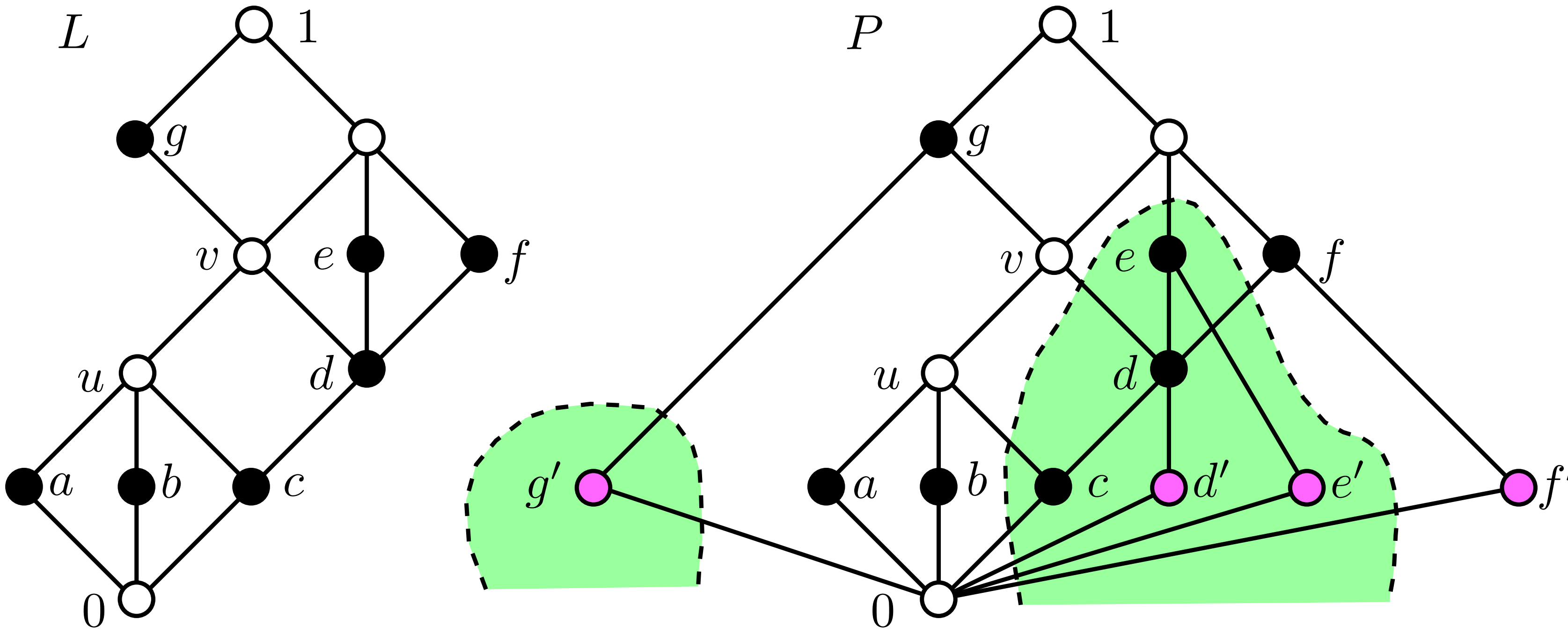
Note:

For each $x \in H(L)$, we insert a new x' such that $0 \prec x' \prec x$. This way we obtain $P = (P; \leq)$:

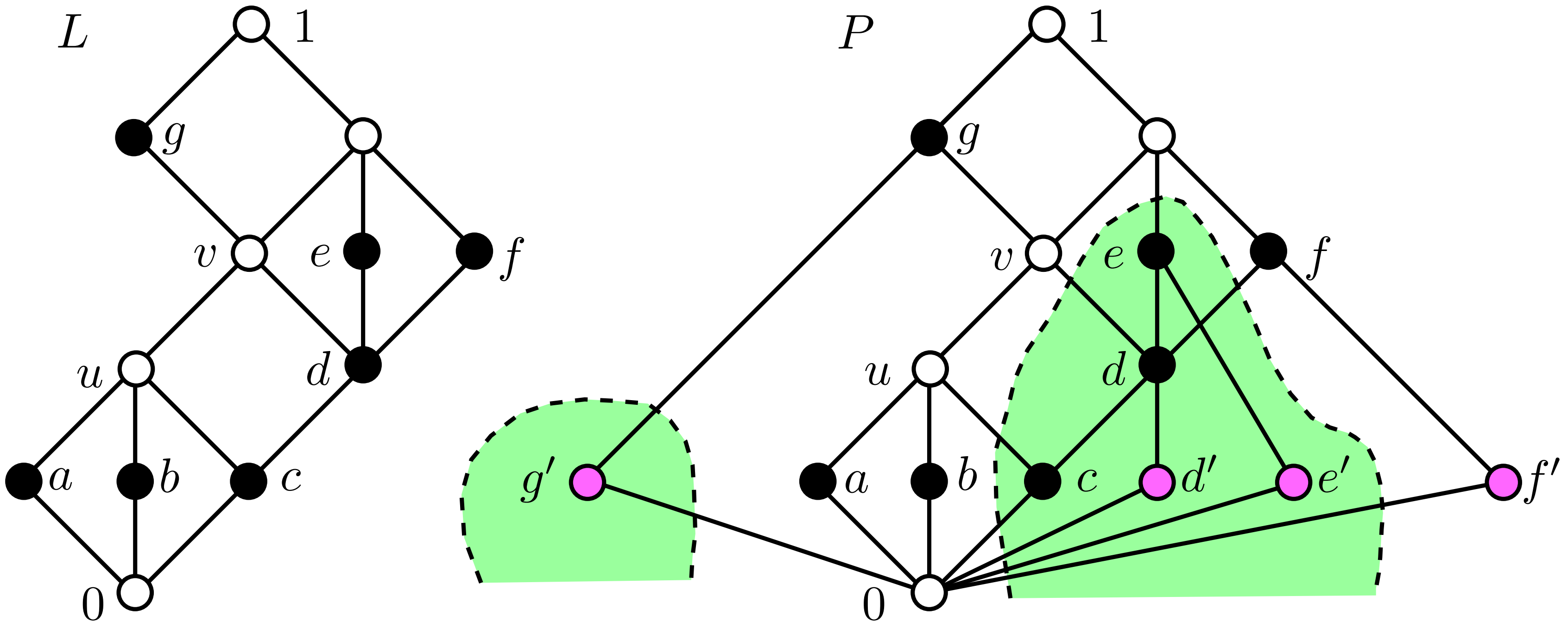


Note: $0 \prec g' \prec g \prec 1$ and $0 \prec c \prec d \prec e \prec e \vee f \prec 1$; P not sm !

Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**

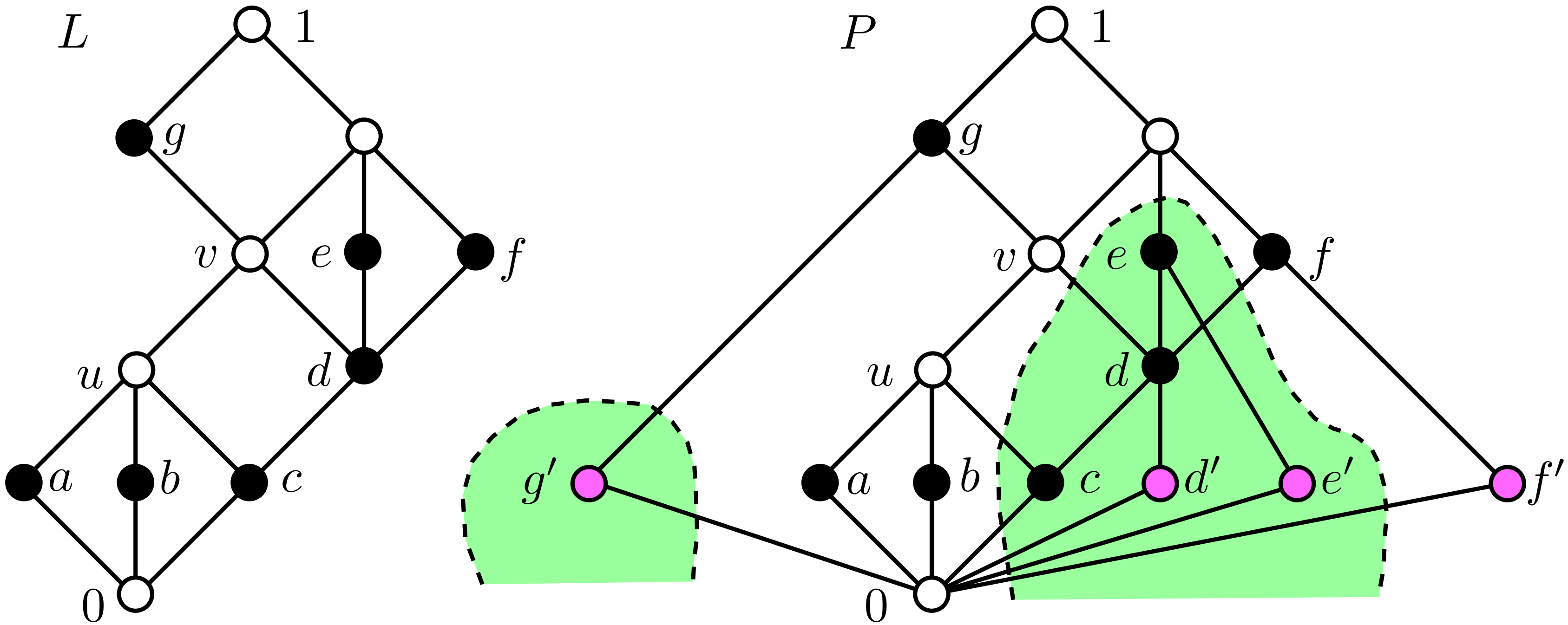


Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



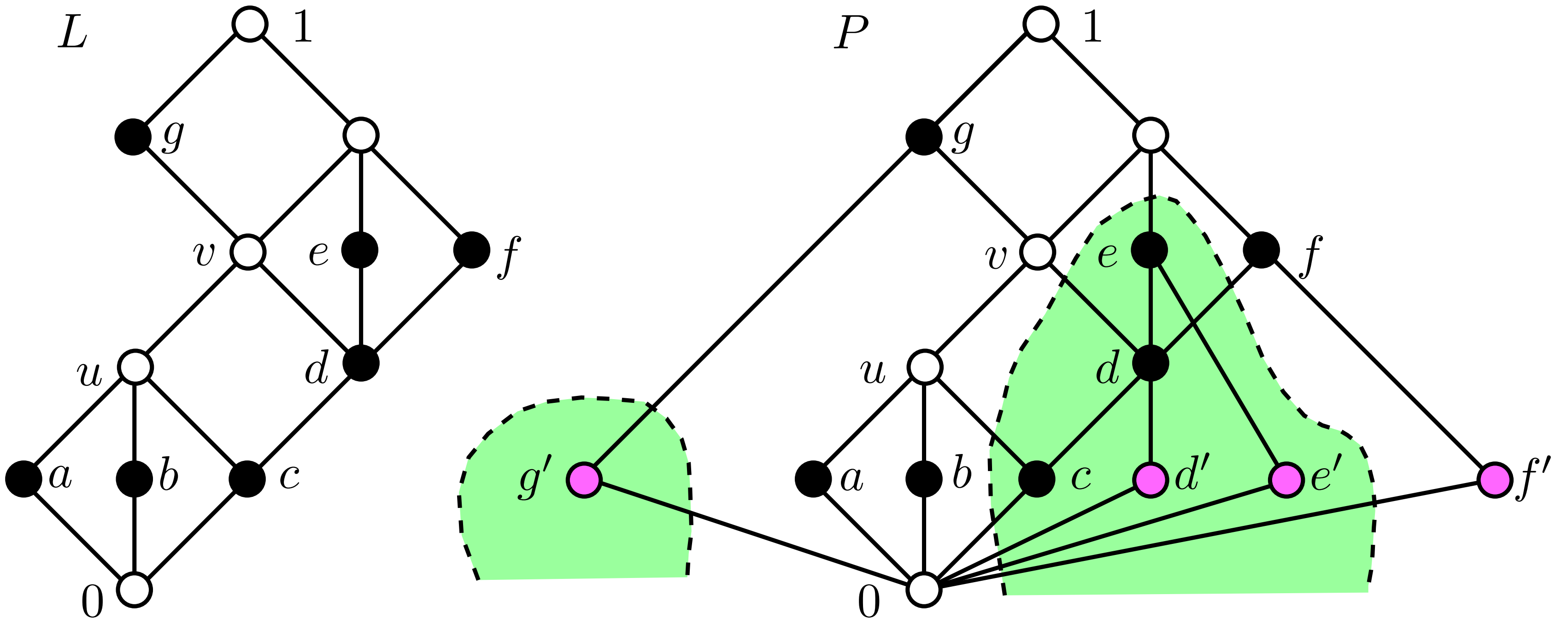
for comparables,

Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



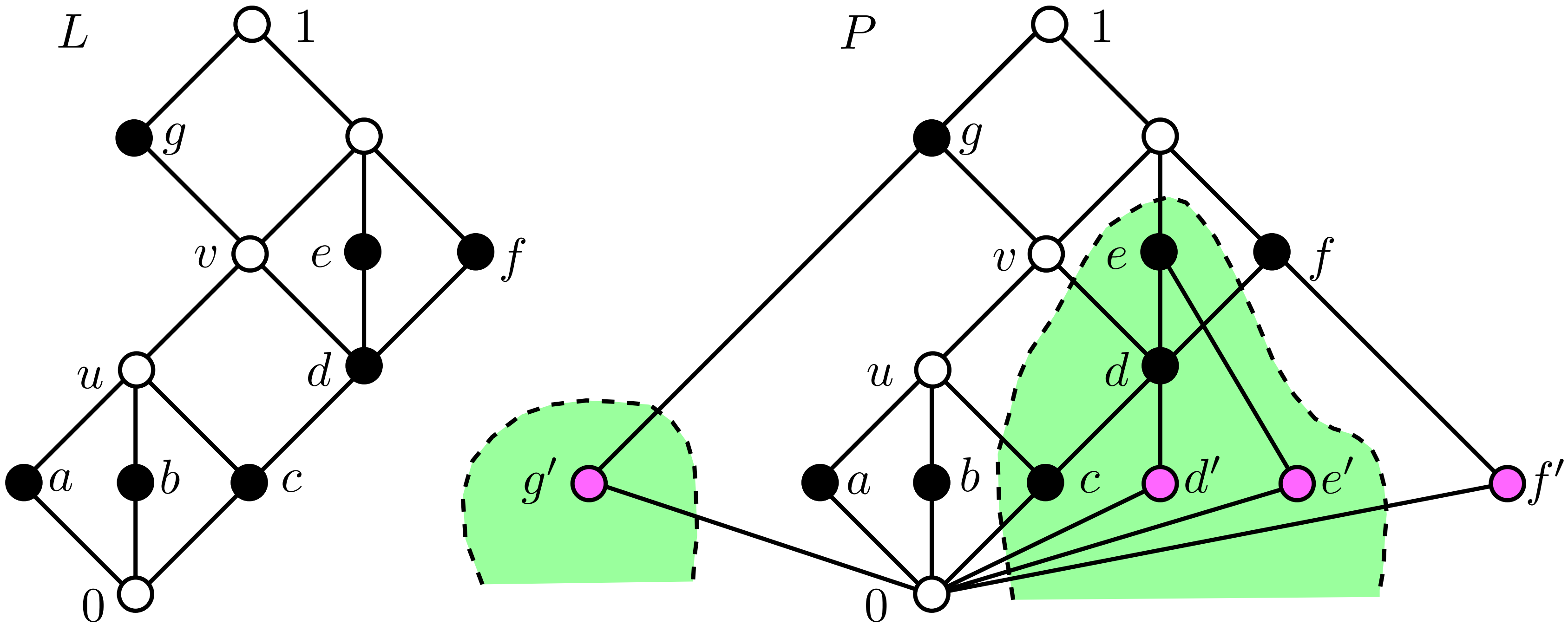
for comparables, for old elements,

Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



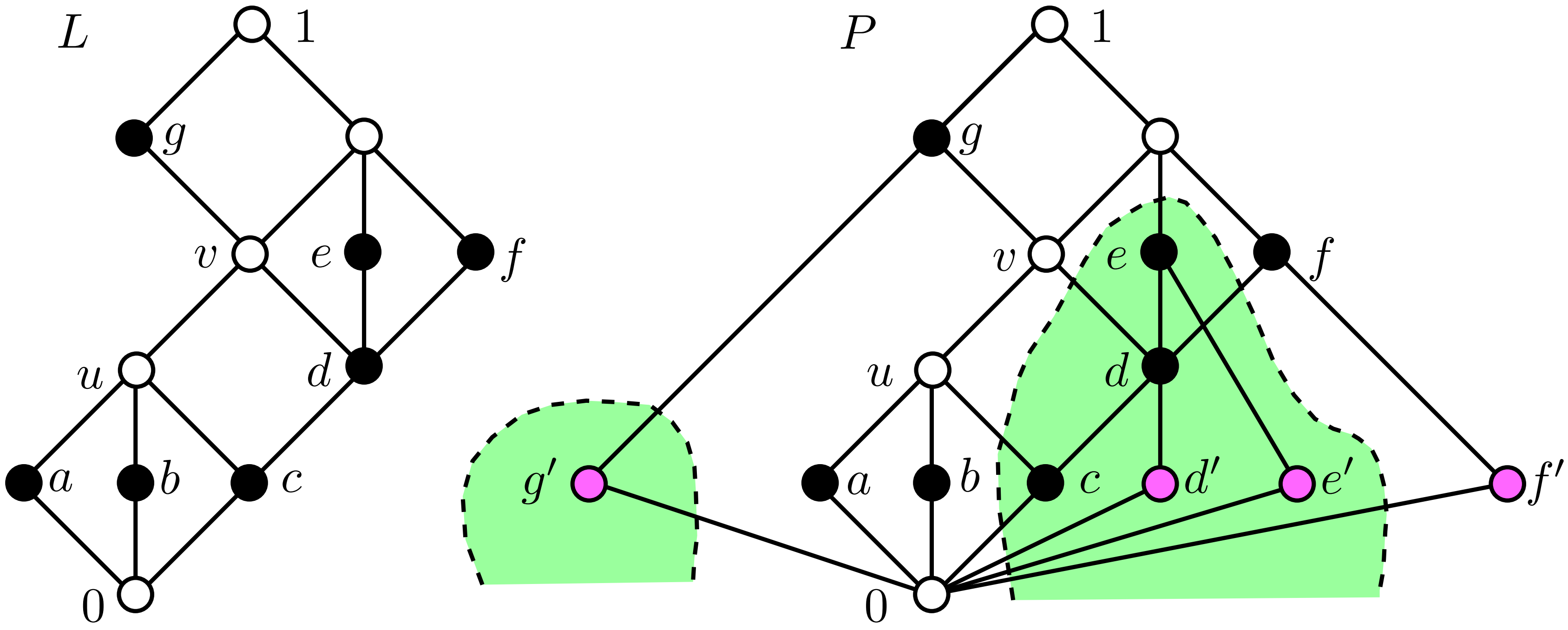
for comparables, for old elements, $x \vee_P y'$ is defined if $x \vee_L y \succ_L x$ (then $x \vee_P y' := x \vee_L y$),

Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



for comparables, for old elements, $x \vee_P y'$ is defined if $x \vee_L y \succ_L x$ (then $x \vee_P y' := x \vee_L y$), and the same for $x' \vee y$.

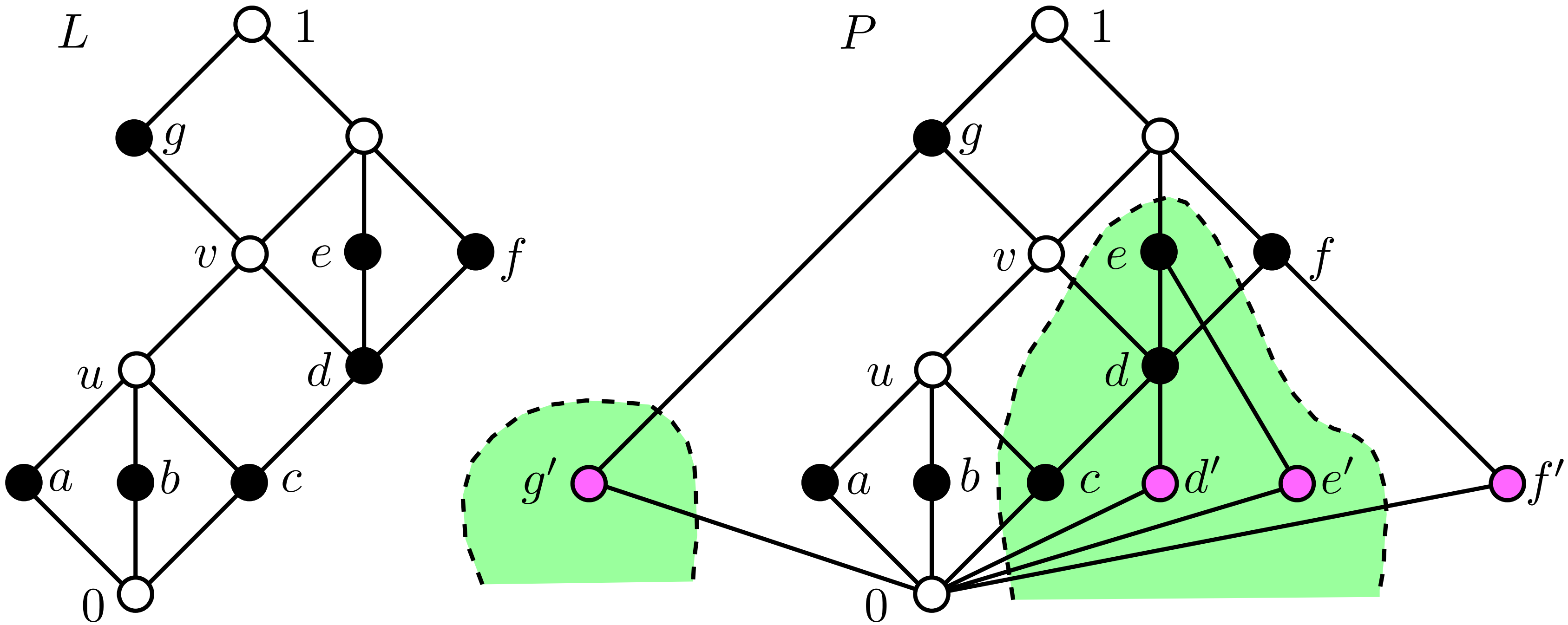
Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



for comparables, for old elements, $x \vee_P y'$ is defined if $x \vee_L y \succ_L x$ (then $x \vee_P y' := x \vee_L y$), and the same for $x' \vee y$.

E.g., $u \vee_P d' =$

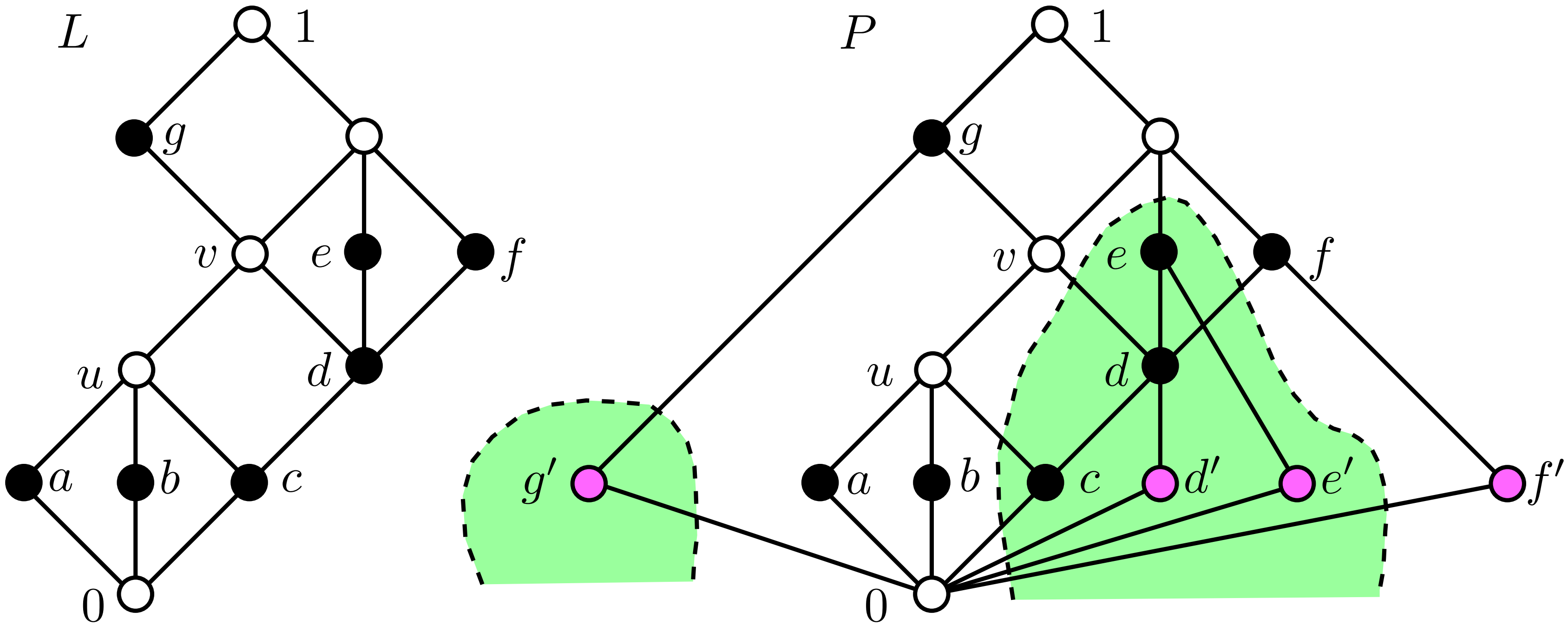
Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



for comparables, for old elements, $x \vee_P y'$ is defined if $x \vee_L y \succ_L x$ (then $x \vee_P y' := x \vee_L y$), and the same for $x' \vee y$.

E.g., $u \vee_P d' = v$,

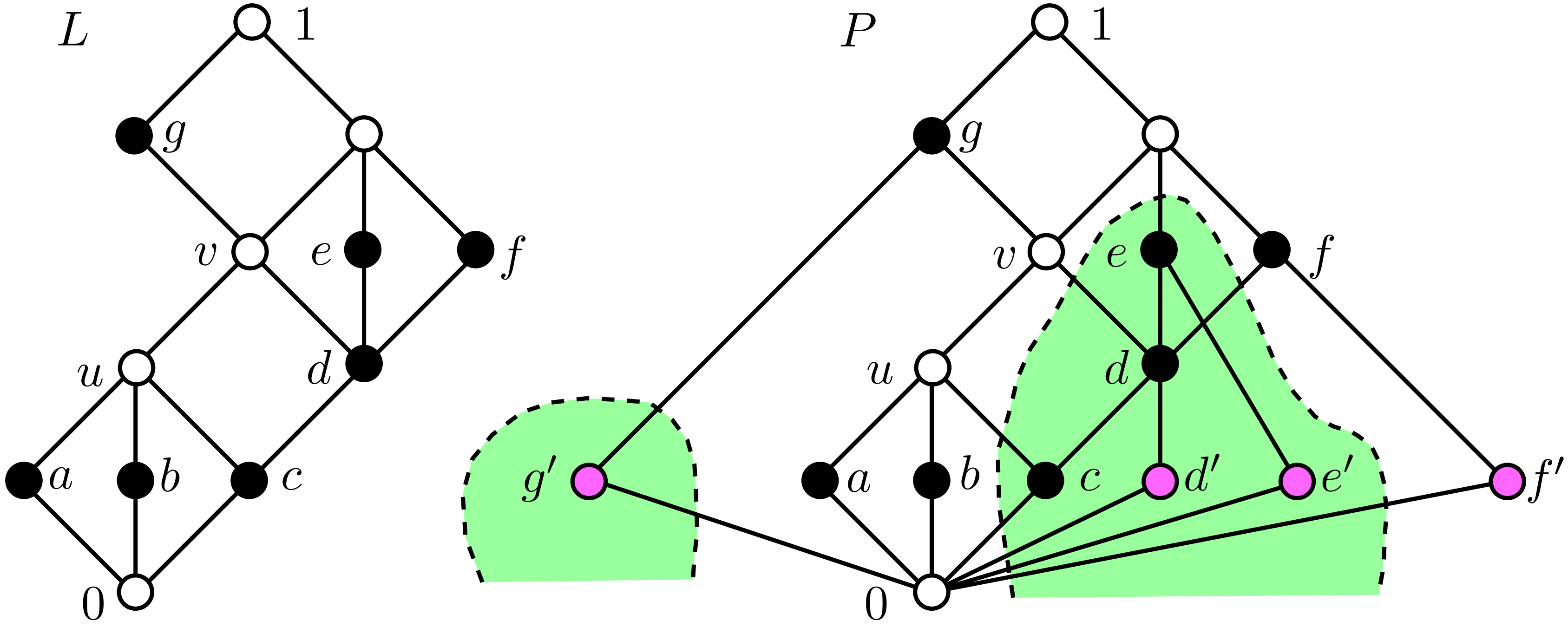
Consider P a partial $\vee$-semilattice! $\vee_P$ of two elements is **defined**



for comparables, for old elements, $x \vee_P y'$ is defined if $x \vee_L y \succ_L x$ (then $x \vee_P y' := x \vee_L y$), and the same for $x' \vee y$.

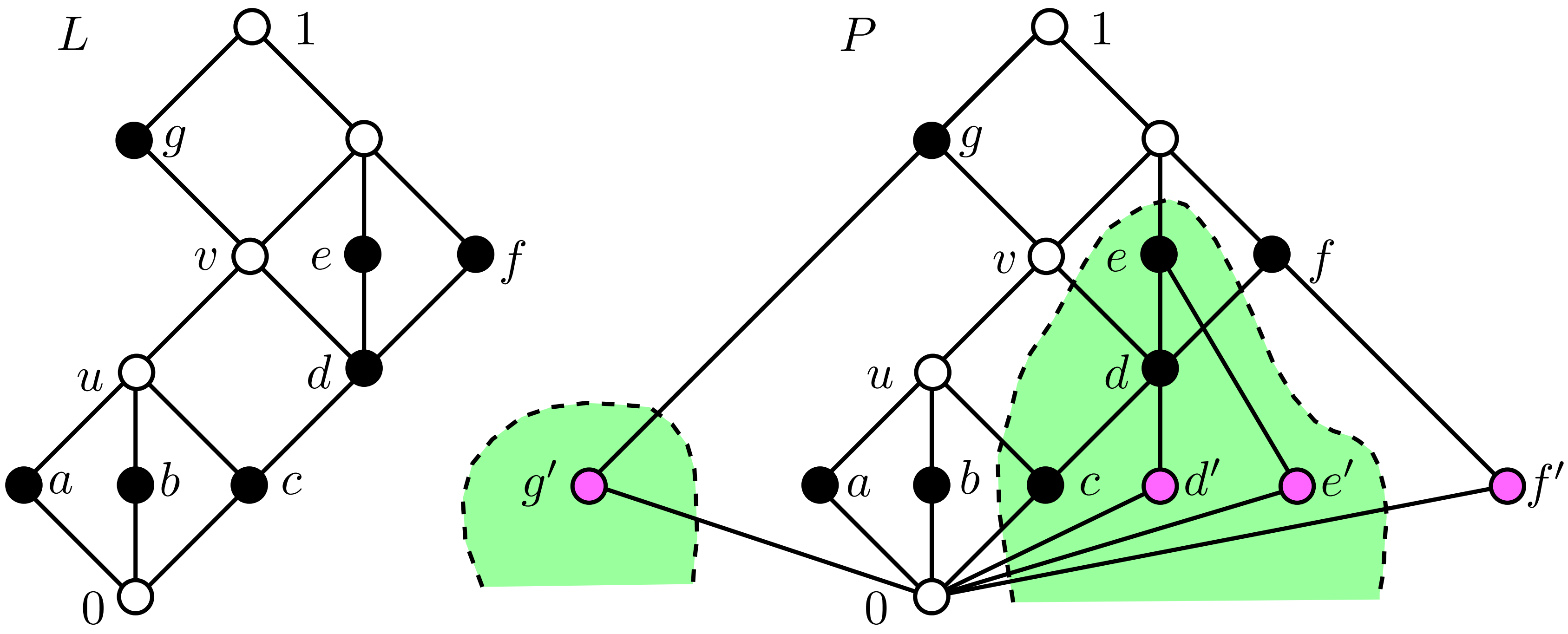
E.g., $u \vee_P d' = v$, $g \vee_P f' = 1$.

$\vee_P$ of two elements is **undefined**



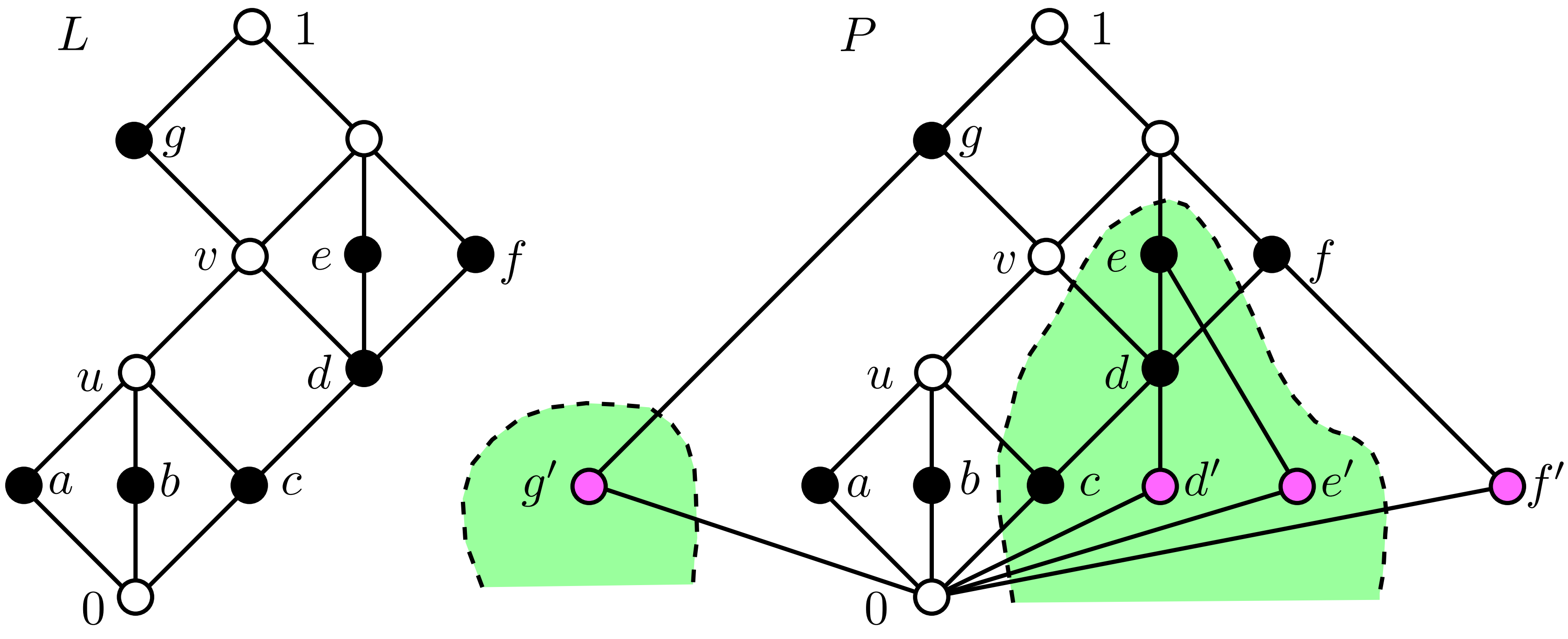
for two distinct new elements,

$\vee_P$ of two elements is **undefined**



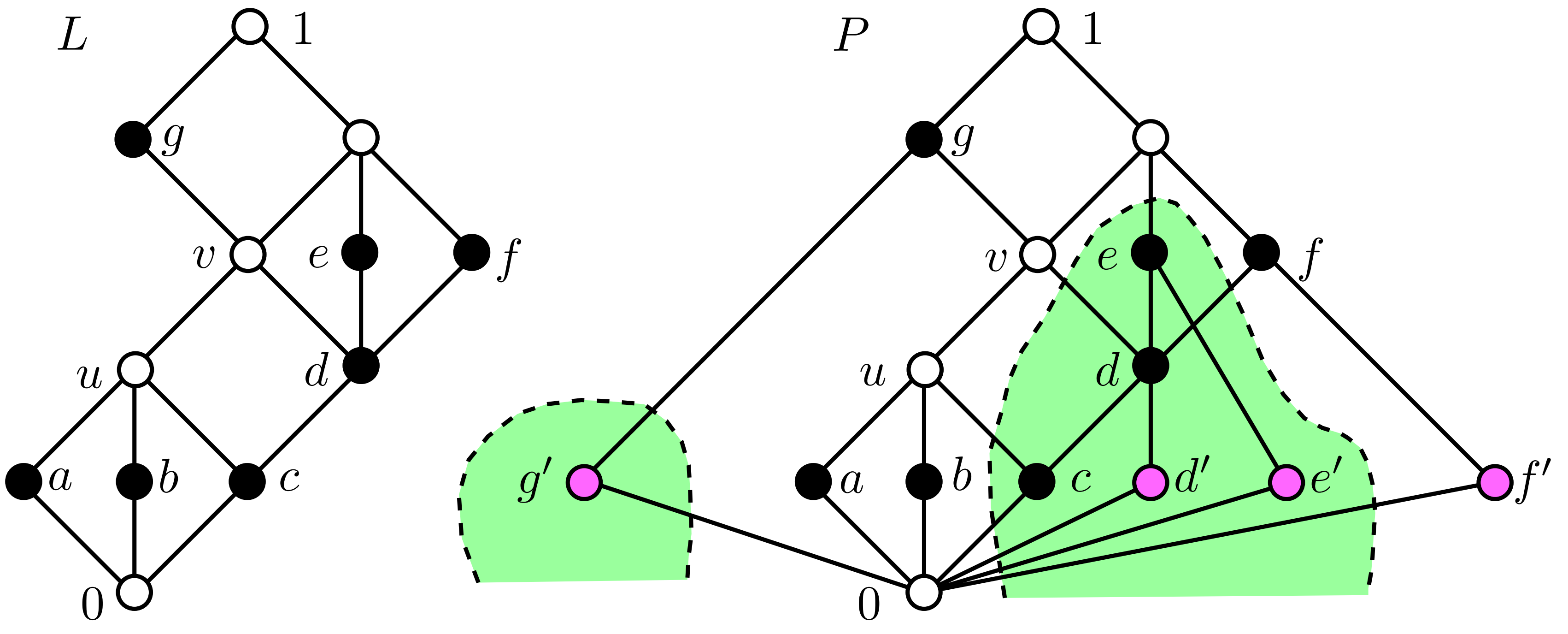
for two distinct new elements, for x and y' when $x \vee_L y \not\succ_L x$, and similarly (commutatively) for x' and y.

$\vee_P$ of two elements is **undefined**

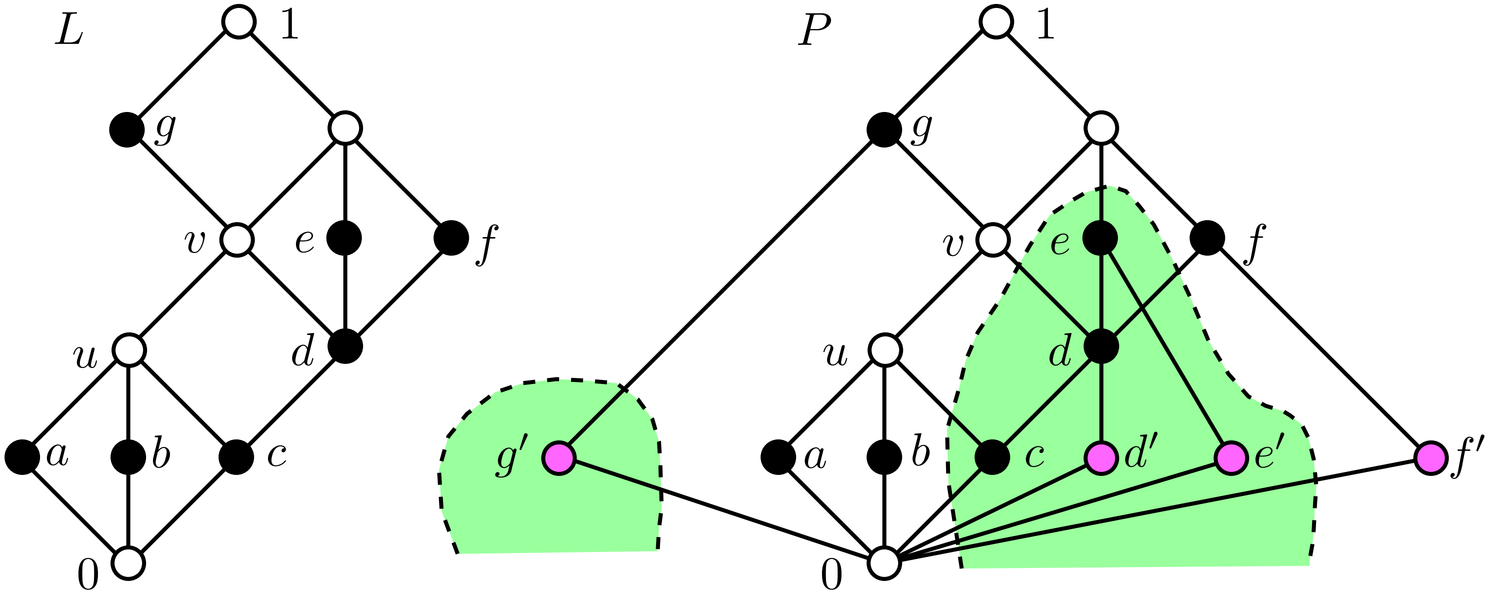


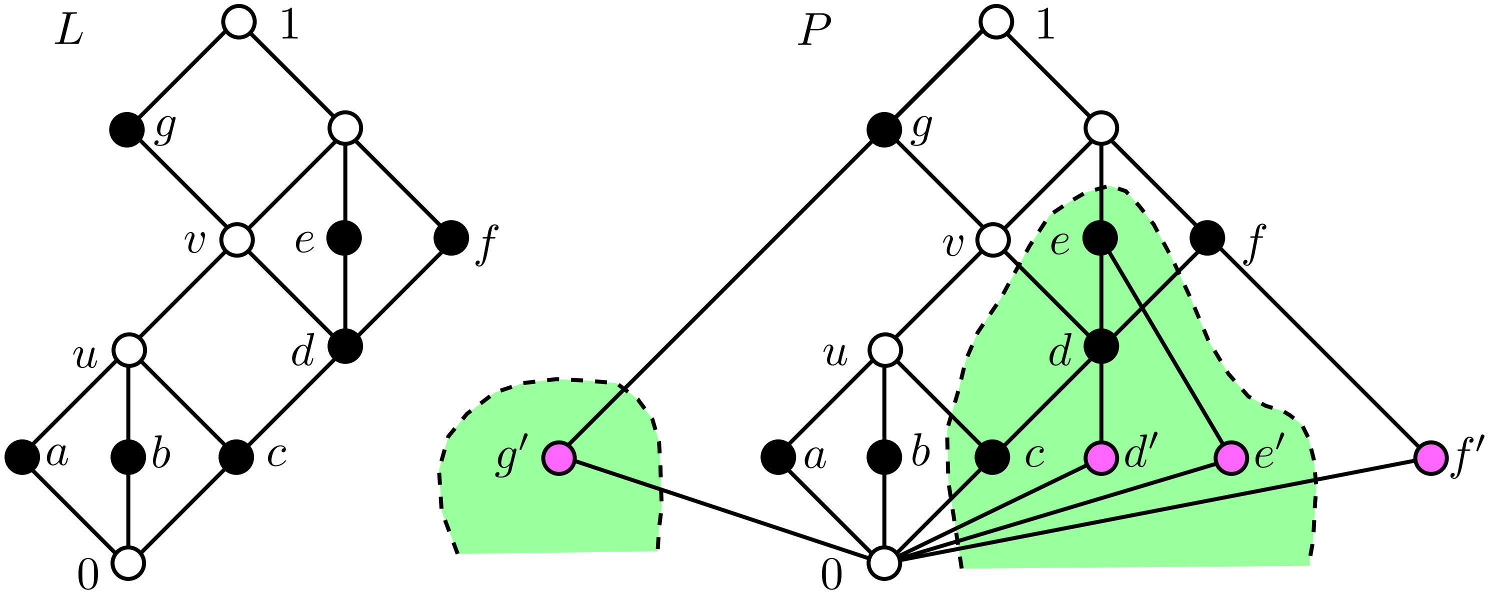
for two distinct new elements, for x and y' when $x \vee_L y \not\succ_L x$, and similarly (commutatively) for x' and y. E.g., $c \vee_P g'$ and

$\vee_P$ of two elements is **undefined**

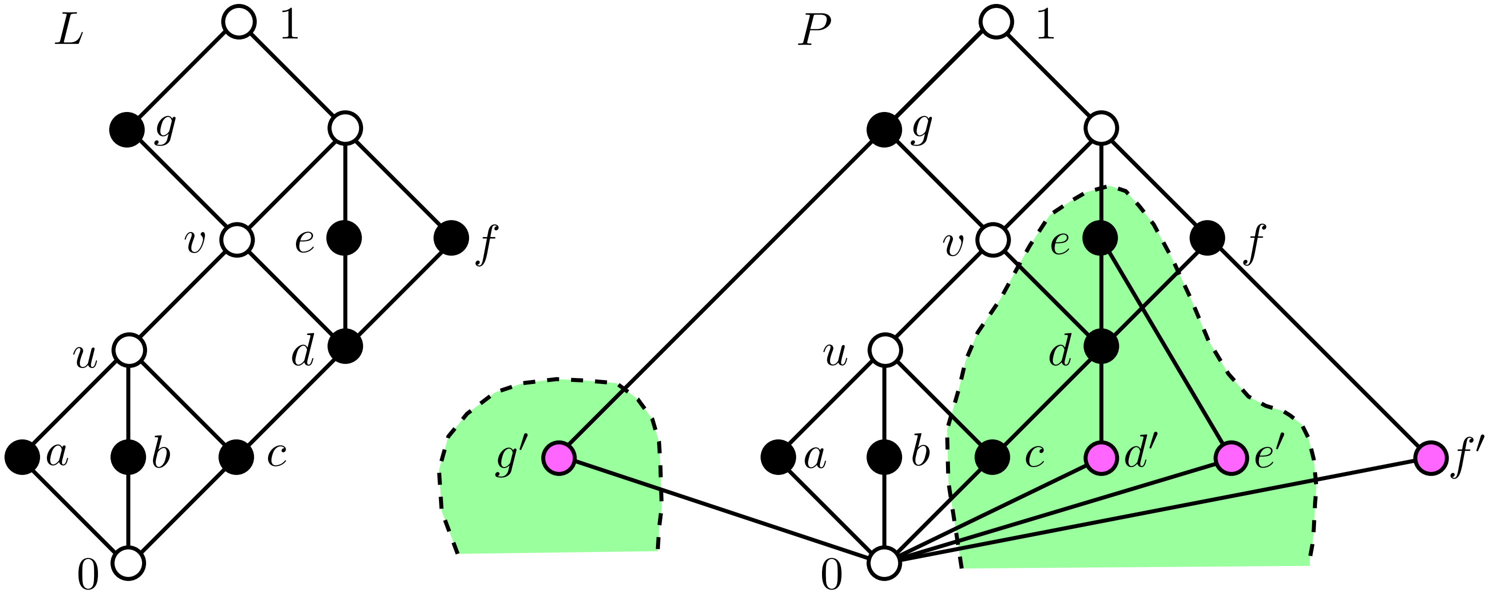


for two distinct new elements, for x and y' when $x \vee_L y \not\succ_L x$, and similarly (commutatively) for x' and y. E.g., $c \vee_P g'$ and $g' \vee_P f$, and $d' \vee_P e'$ are undefined.

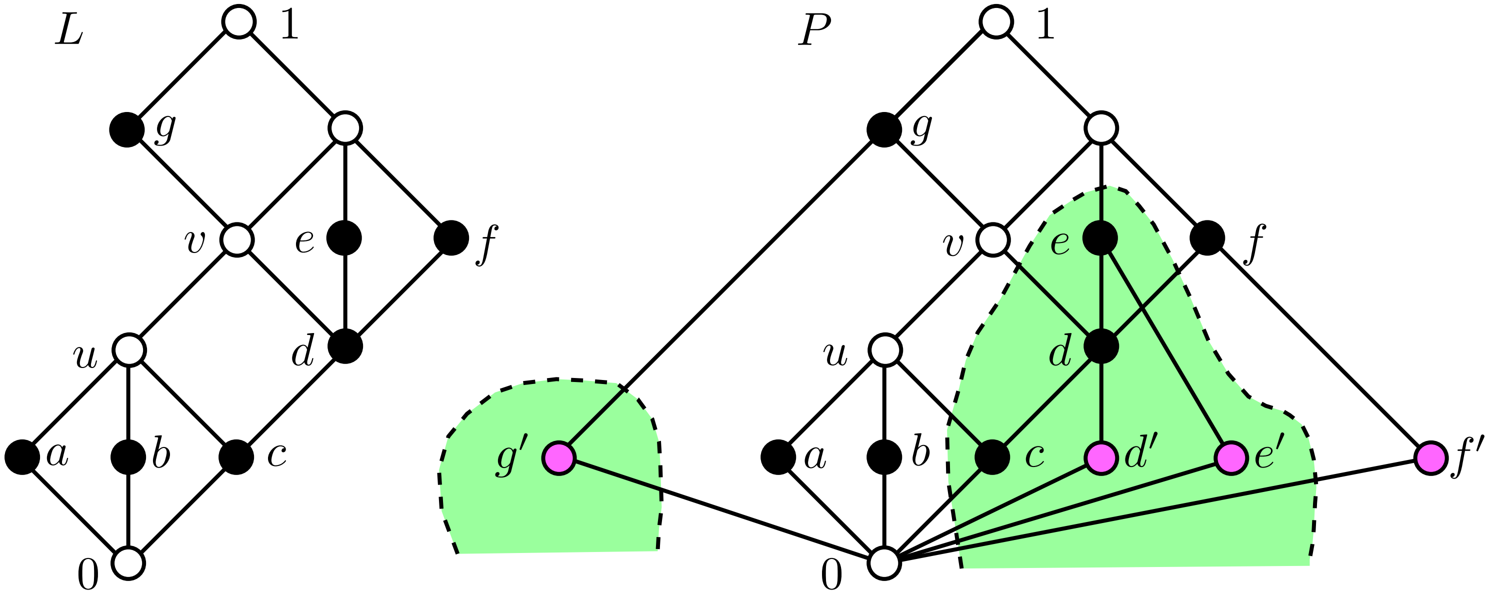

L
1
g
v
e
f
u
d
a
b
c
0
P
1
g
v
e
f
u
d
g′
a
b
c
d′
e′
f′
0



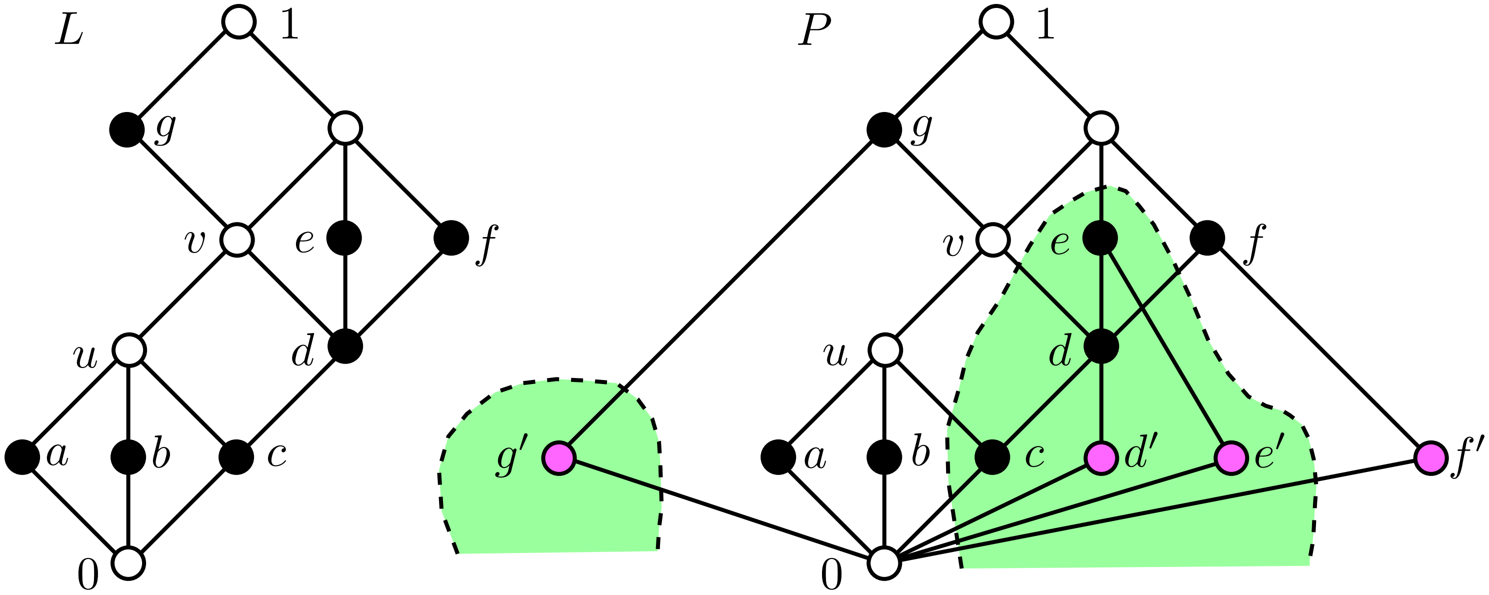
Ideal of P: down-set closed w.r. existing joins.



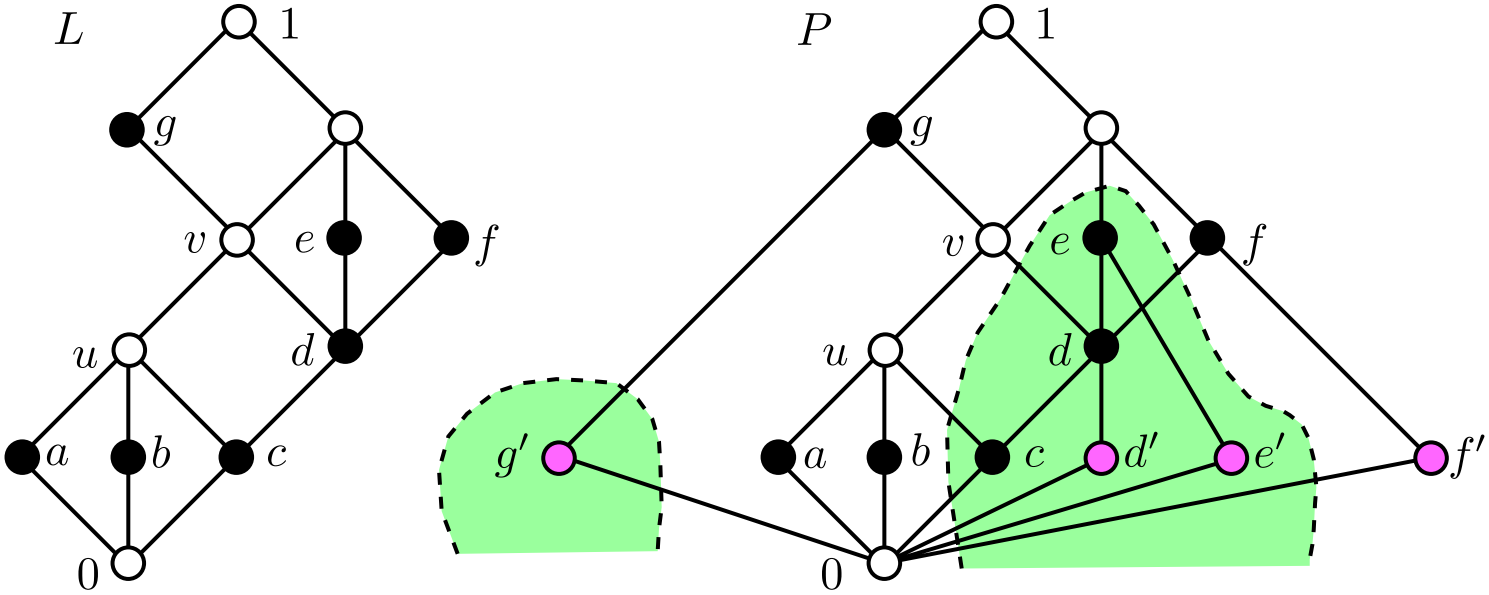
Ideal of P: down-set closed w.r. existing joins. $\mathcal{I}(P) = (\mathcal{I}(P), \subseteq)$.



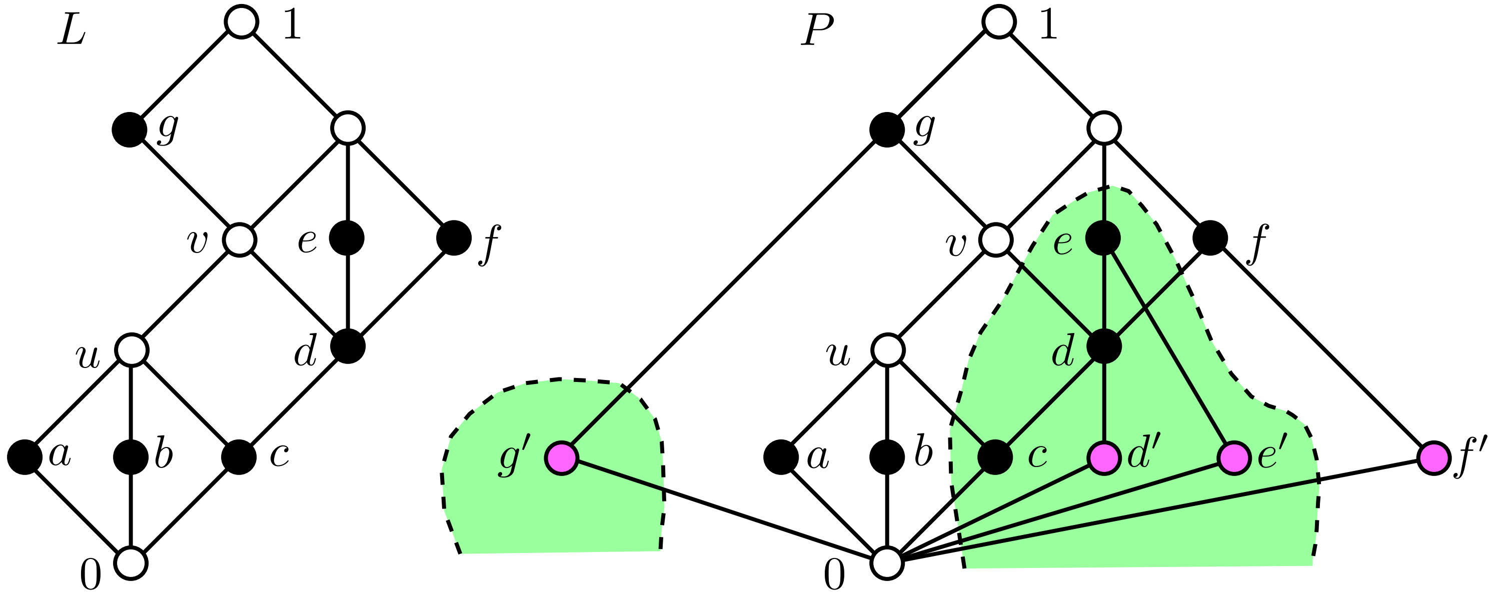
Ideal of P: down-set closed w.r. existing joins. $\mathcal{I}(P) = (\mathcal{I}(P), \subseteq)$. For $I \in \mathcal{I}(P)$,



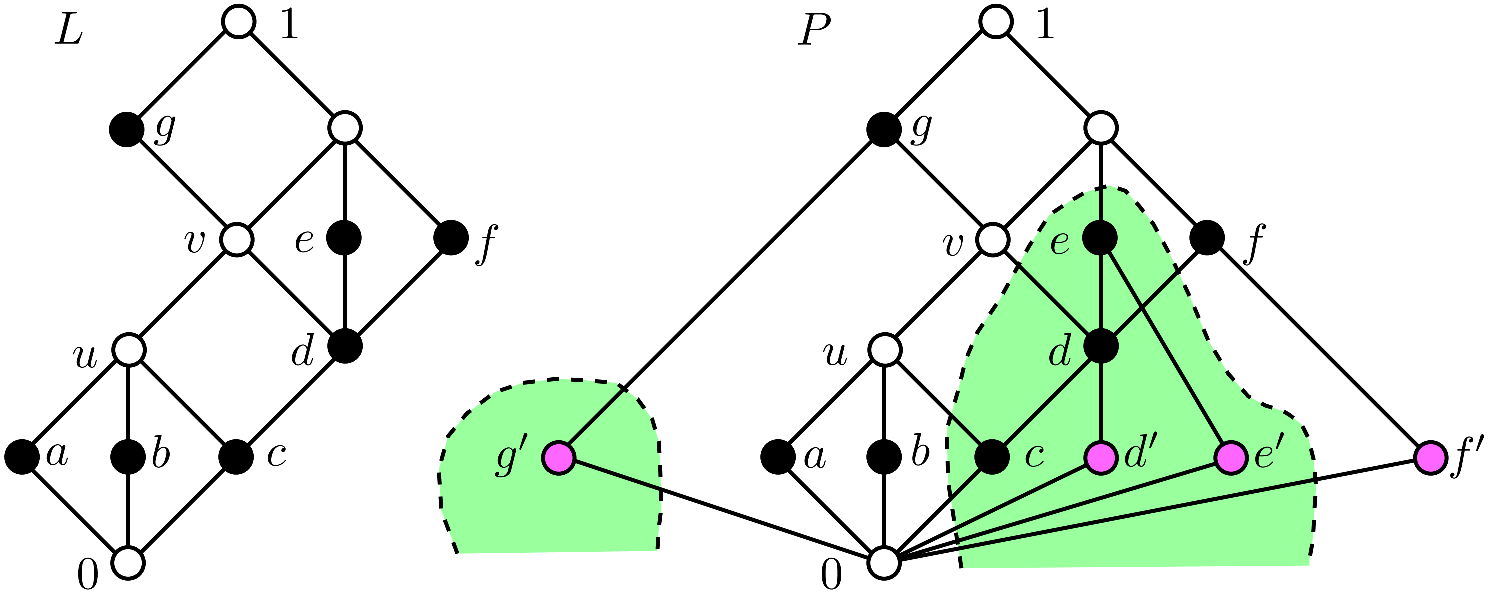
trunk(I) denotes the largest element of $I \cap L$.



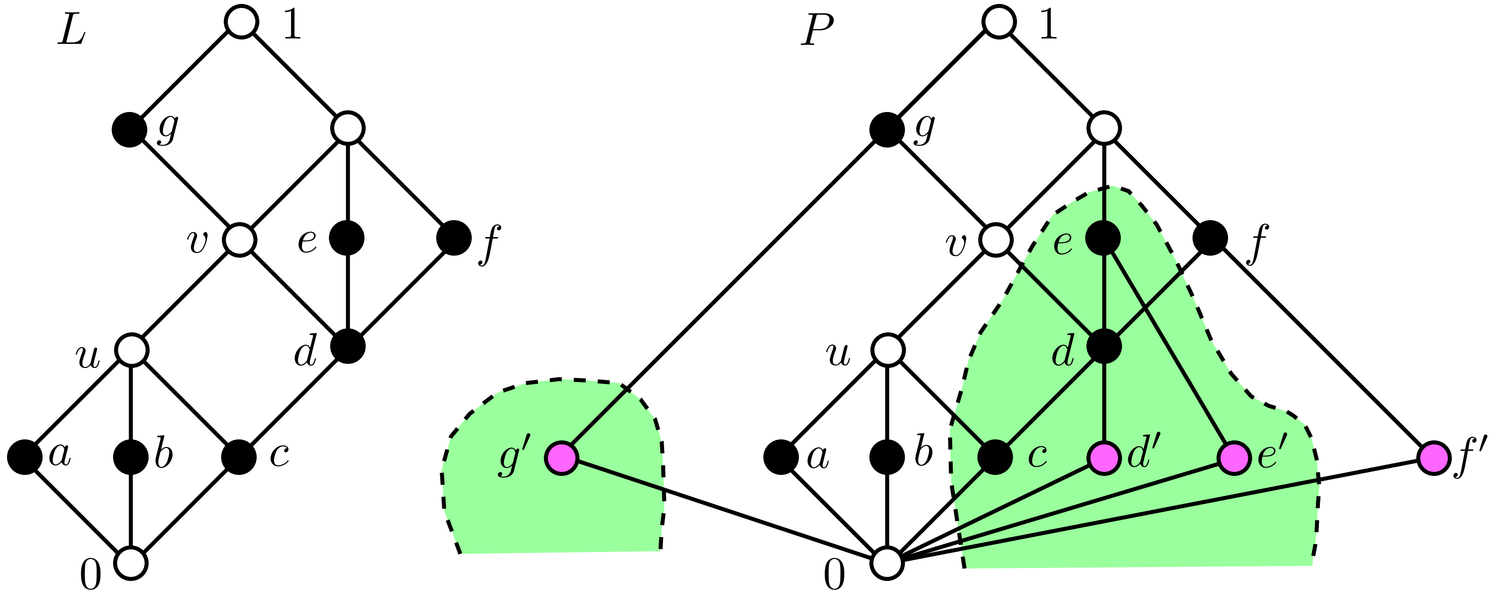
trunk(I) denotes the largest element of $I \cap L$. I.e., the largest old element of I. For example, trunk($\{0, c, d, e, d', e', g'\}$) $= e$.



trunk(I) denotes the largest element of $I \cap L$. I.e., the largest old element of I. For example, trunk($\{0, c, d, e, d', e', g'\}$) $= e$. Define branch(I) $:= I \setminus \downarrow$trunk($I$) $=$ {new elements not below the trunk}.

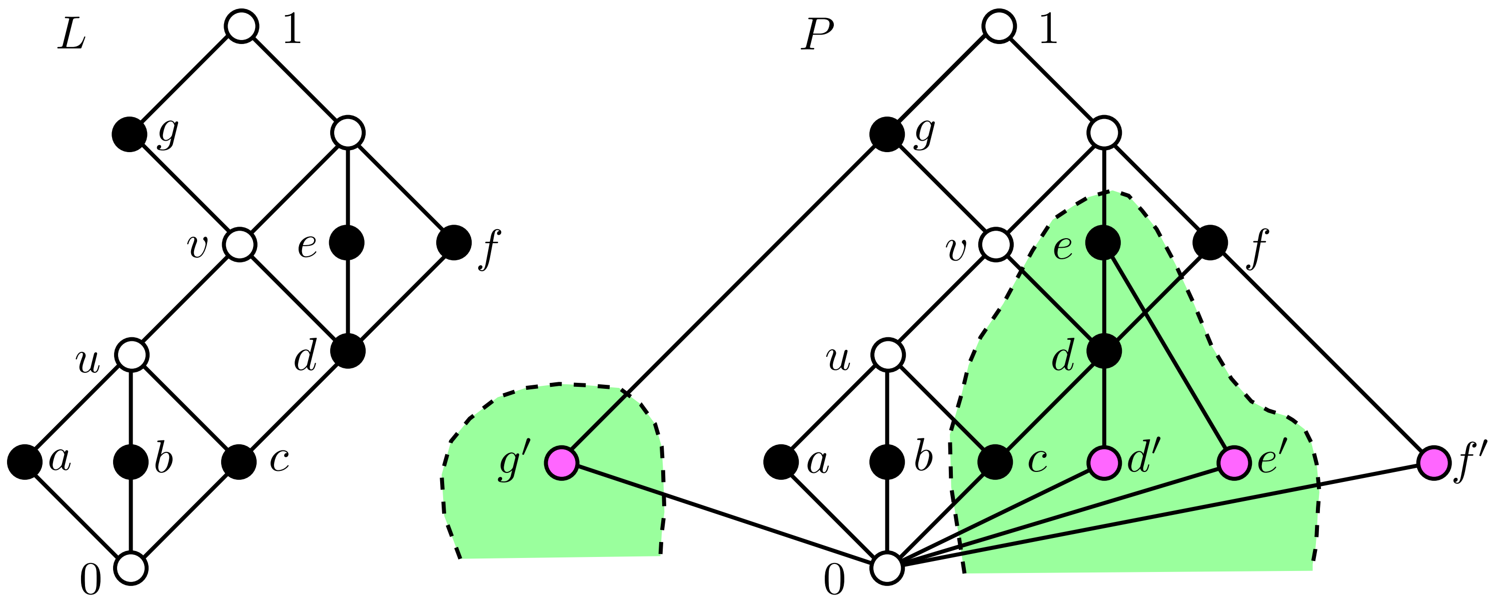


trunk(I) denotes the largest element of $I \cap L$. I.e., the largest old element of I. For example, trunk($\{0, c, d, e, d', e', g'\}$) $= e$. Define branch(I) $:= I \setminus \downarrow$trunk($I$) $=$ {new elements not below the trunk}. E.g., branch($\{0, c, d, e, d', e', g'\}$) $= \{g'\}$.

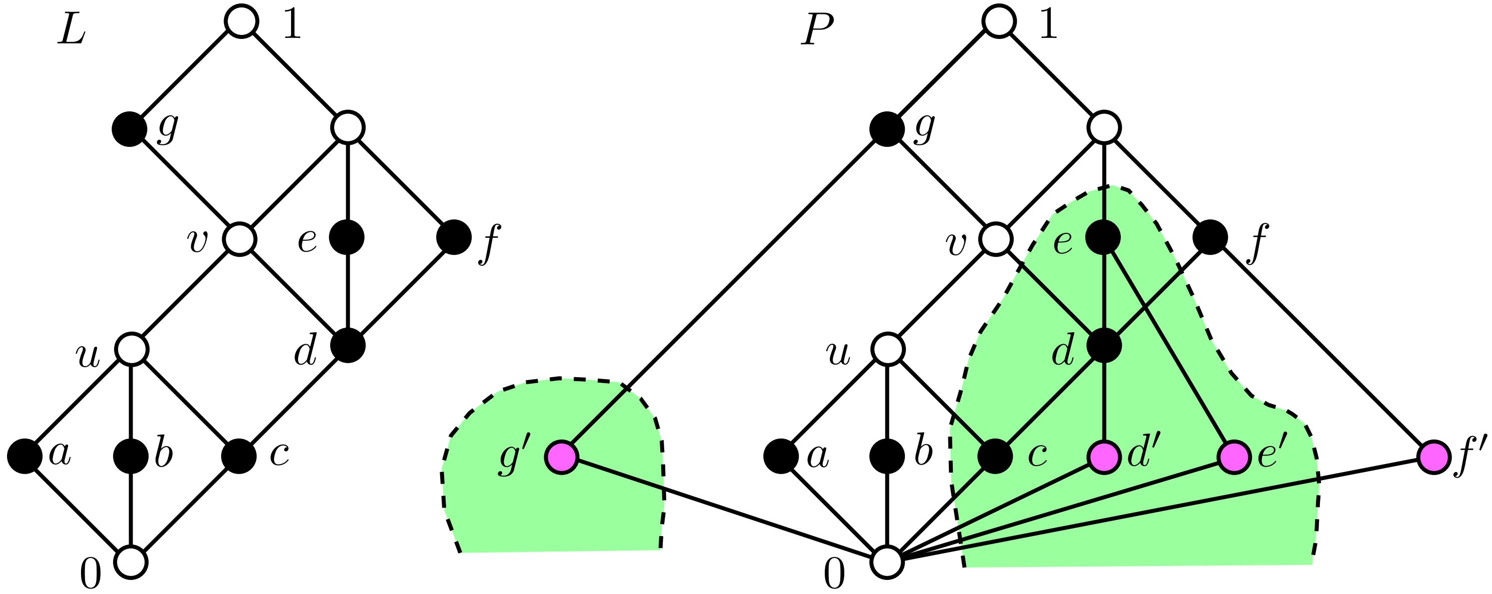


trunk(I) denotes the largest element of $I \cap L$. I.e., the largest old element of I. For example, trunk($\{0, c, d, e, d', e', g'\}$) $= e$. Define branch(I) $:= I \setminus \downarrow$trunk($I$) $=$ {new elements not below the trunk}. E.g., branch($\{0, c, d, e, d', e', g'\}$) $= \{g'\}$.

Trunk and branch together determine the ideal; notation: $t; B$

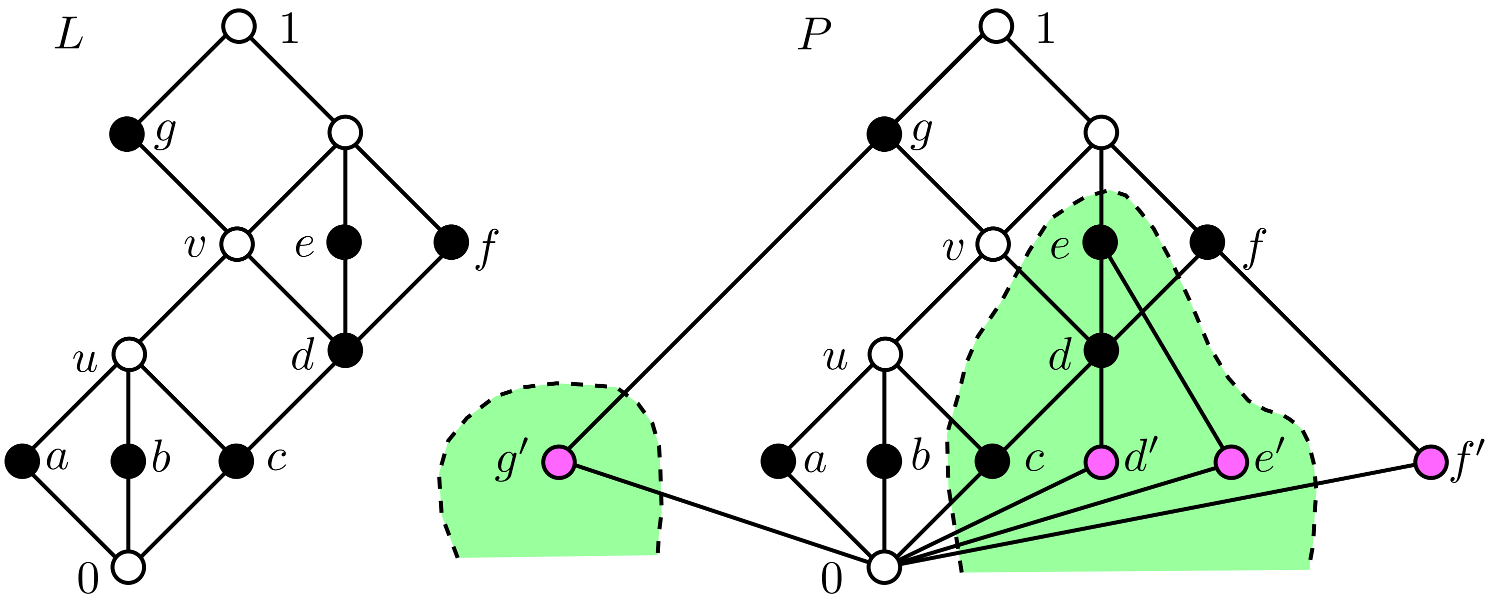


The trunk and branch cannot be chosen independently. The „green choice" $e; g'$ was possible but, say, $e; f'$ is forbidden.



The trunk and branch cannot be chosen independently. The „green choice" $e; g'$ was possible but, say, $e; f'$ is forbidden.

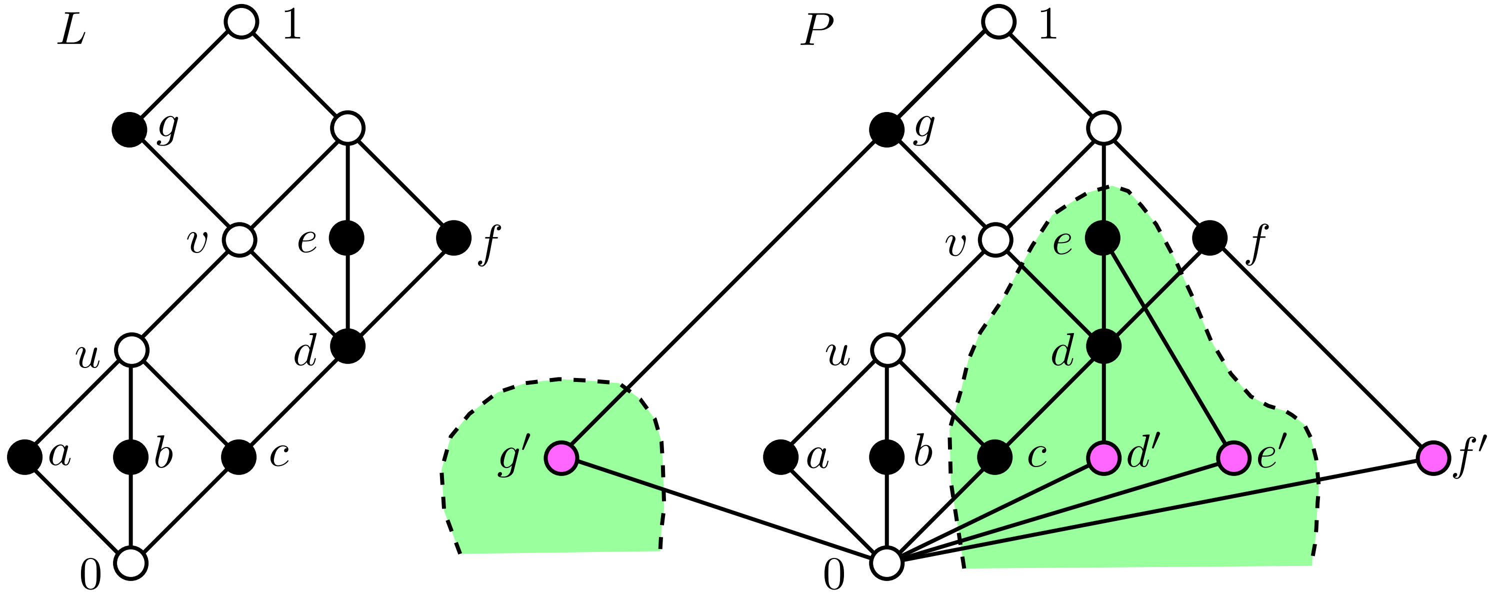
For $I \in \mathcal{I}(P)$, define $r(I) := h(\text{trunk}(I)) + |\text{branch}(I)|$, the **rank** of I. (Possibly infinite.)



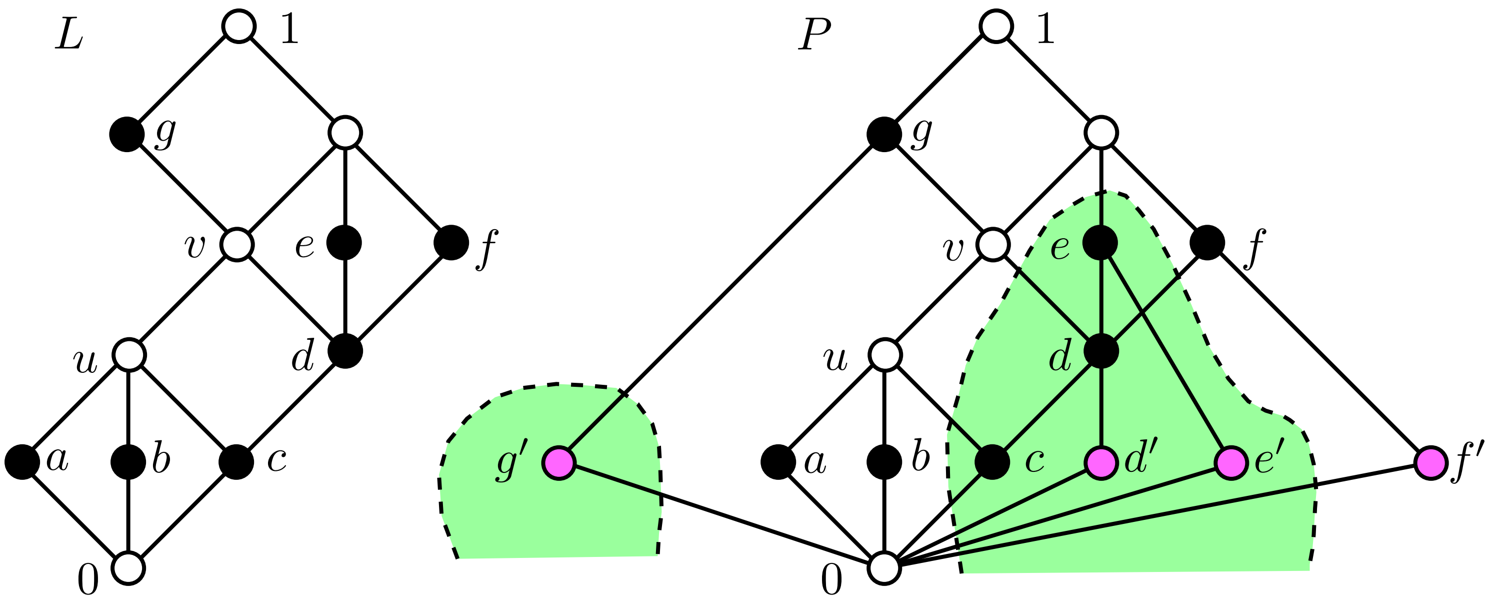
The trunk and branch cannot be chosen independently. The „green choice" $e; g'$ was possible but, say, $e; f'$ is forbidden.

For $I \in \mathcal{I}(P)$, define $r(I) := h(\text{trunk}(I)) + |\text{branch}(I)|$, the **rank** of I. (Possibly infinite.)

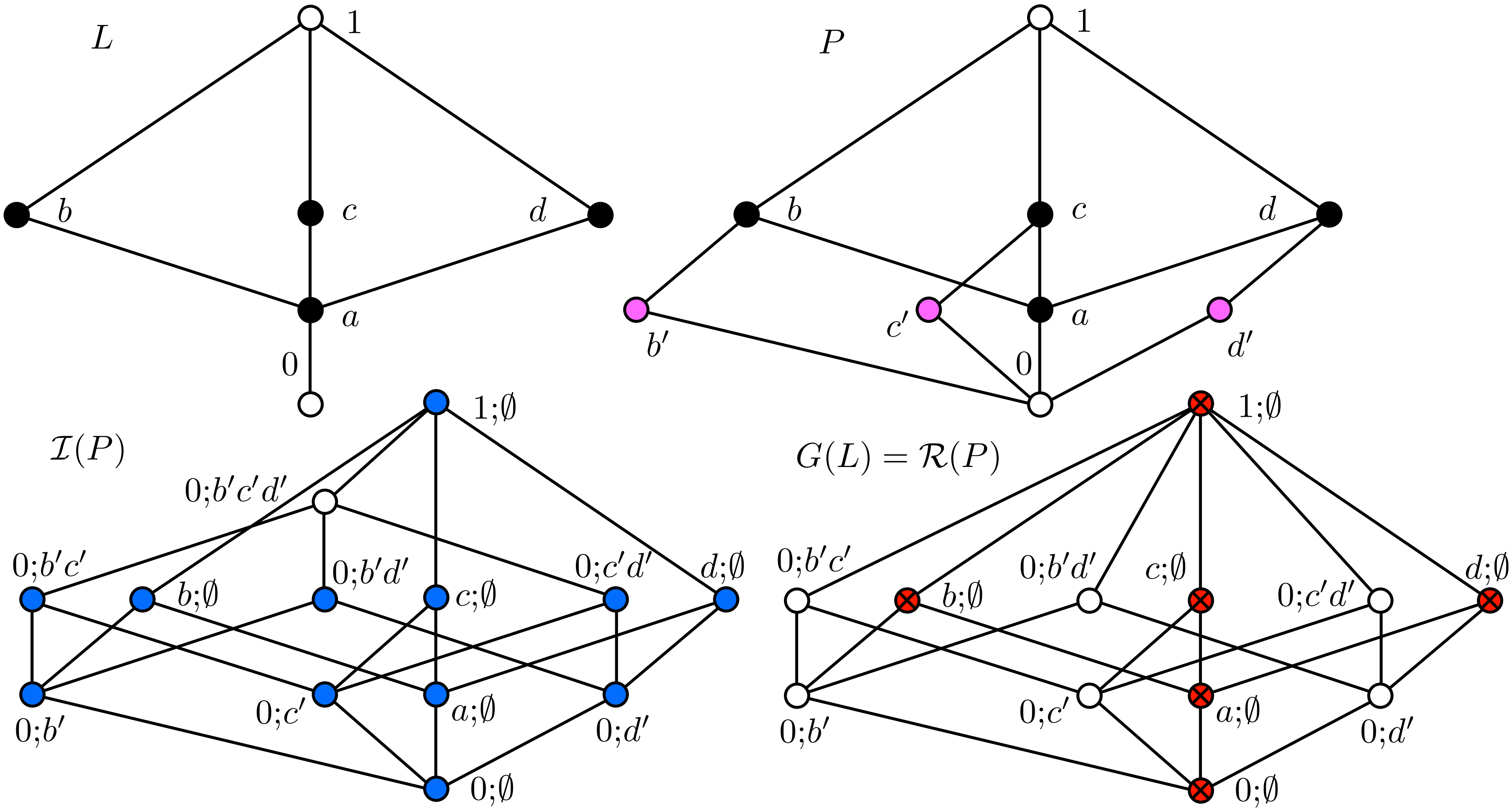
For example, branch$(P) = \emptyset$, so $r(P) = \text{h}(1)$ is finite.



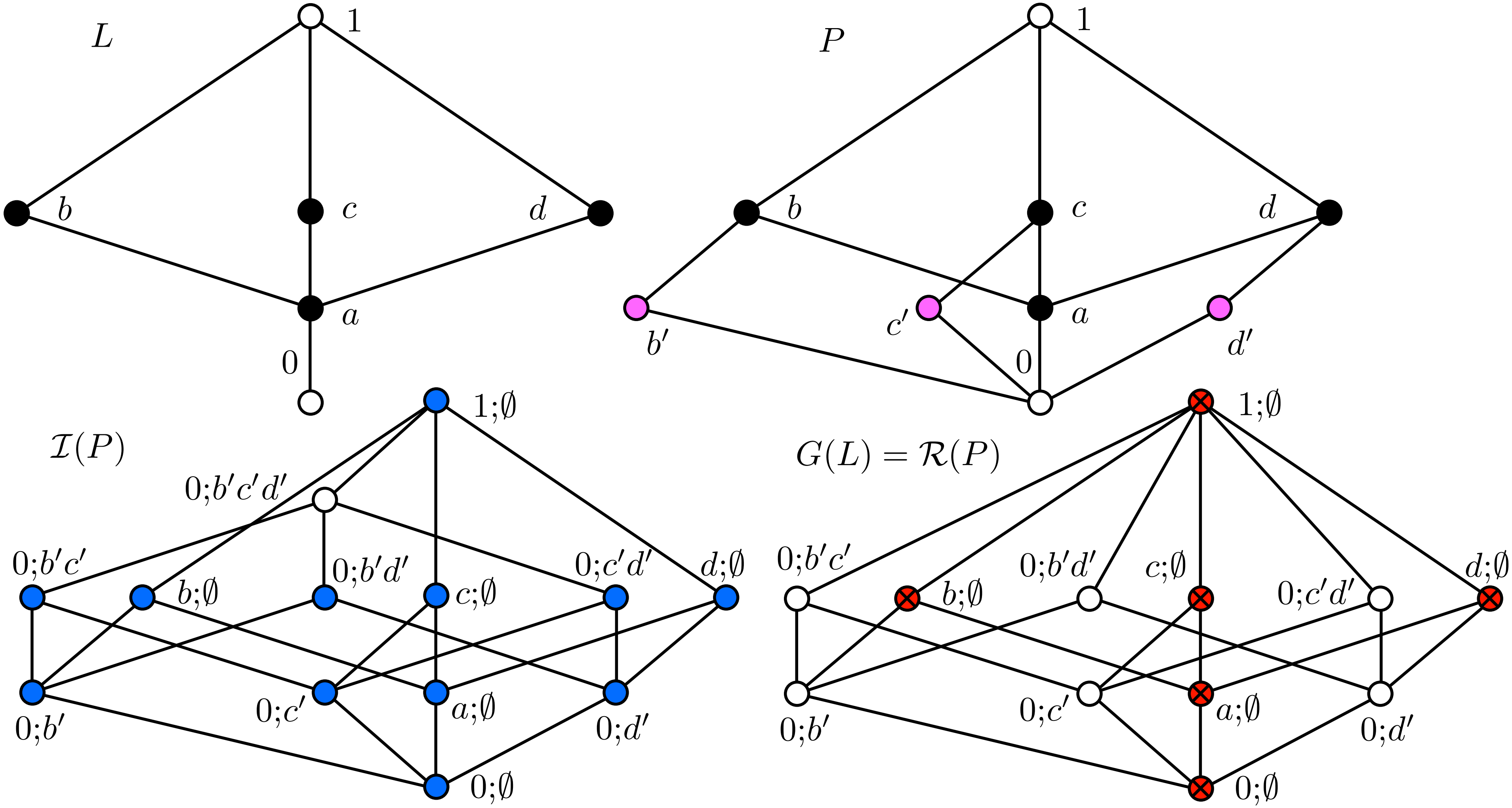
An ideal I is called a **rank-critical ideal**, if $\forall J \in \mathcal{I}(P)$, $I \subset J \implies r(I) < r(J)$.



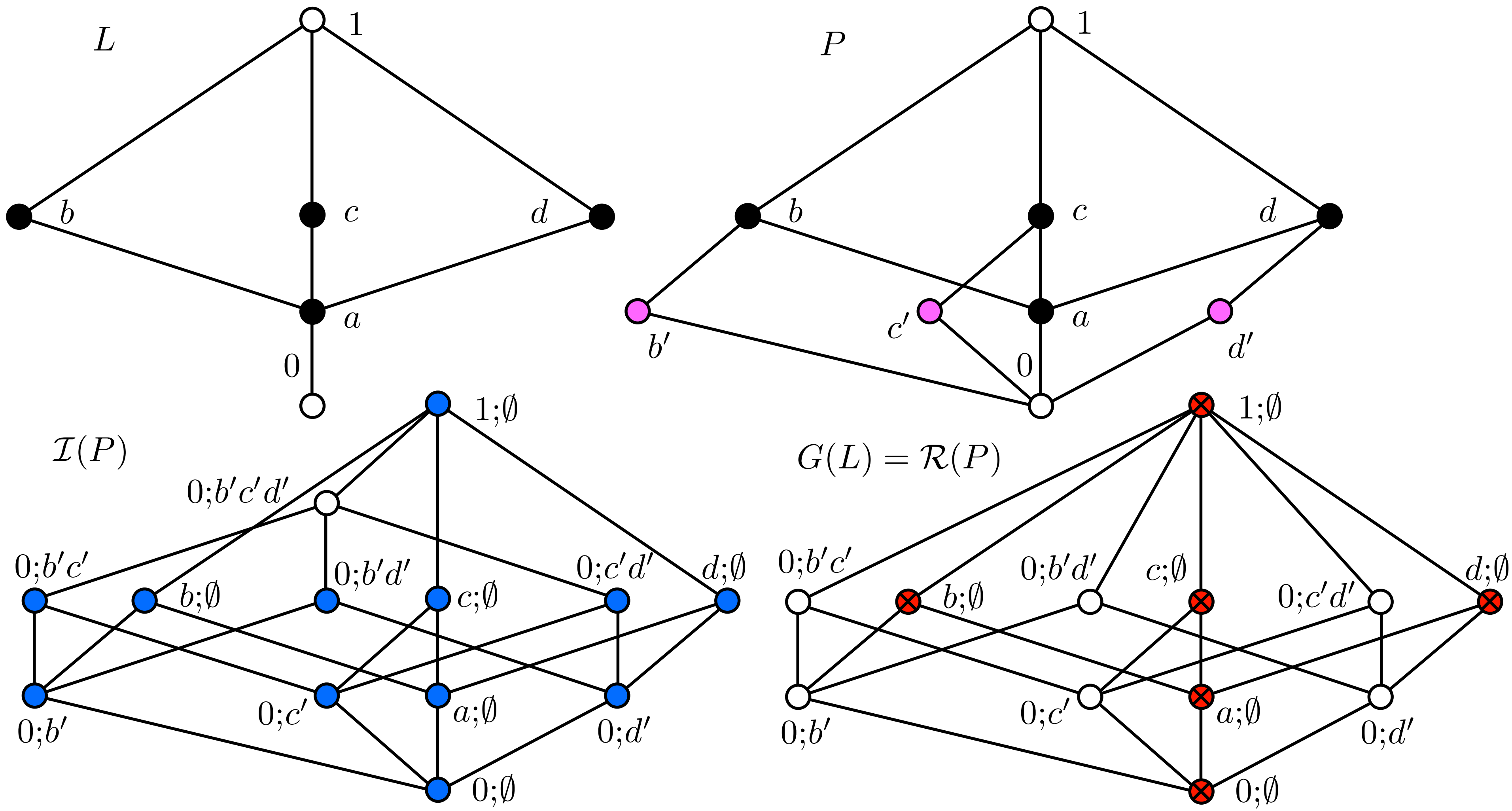
An ideal I is called a **rank-critical ideal**, if $\forall J \in \mathcal{I}(P)$, $I \subset J \implies r(I) < r(J)$. Notice that rank-critical ideals are necessarily of finite length!



The rank-critical ideals (the blue ones) form $\mathcal{R}(P) = (\mathcal{R}(P), \subseteq)$.

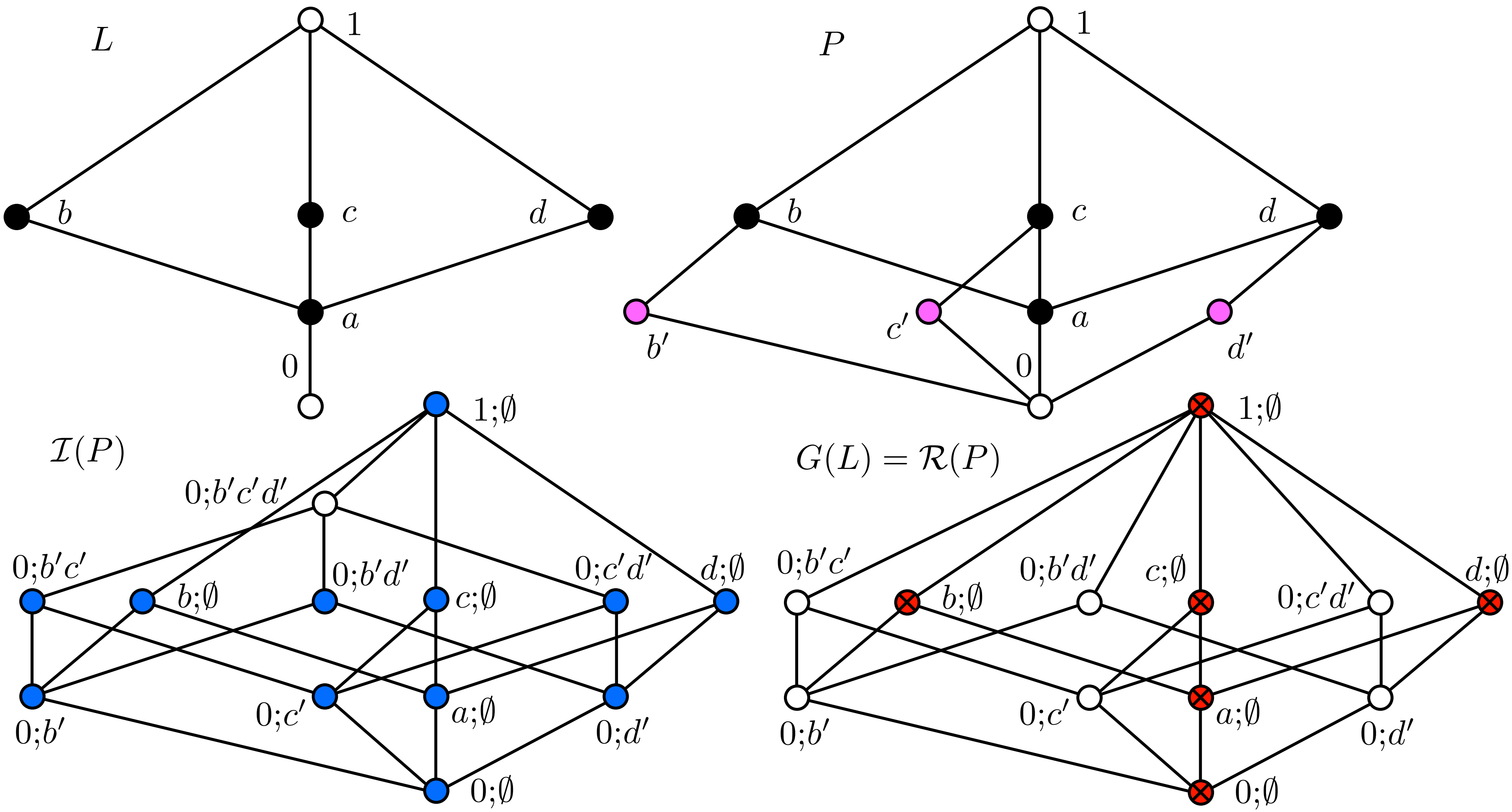


Why $0;b'c'd'$ is not rank-critical?

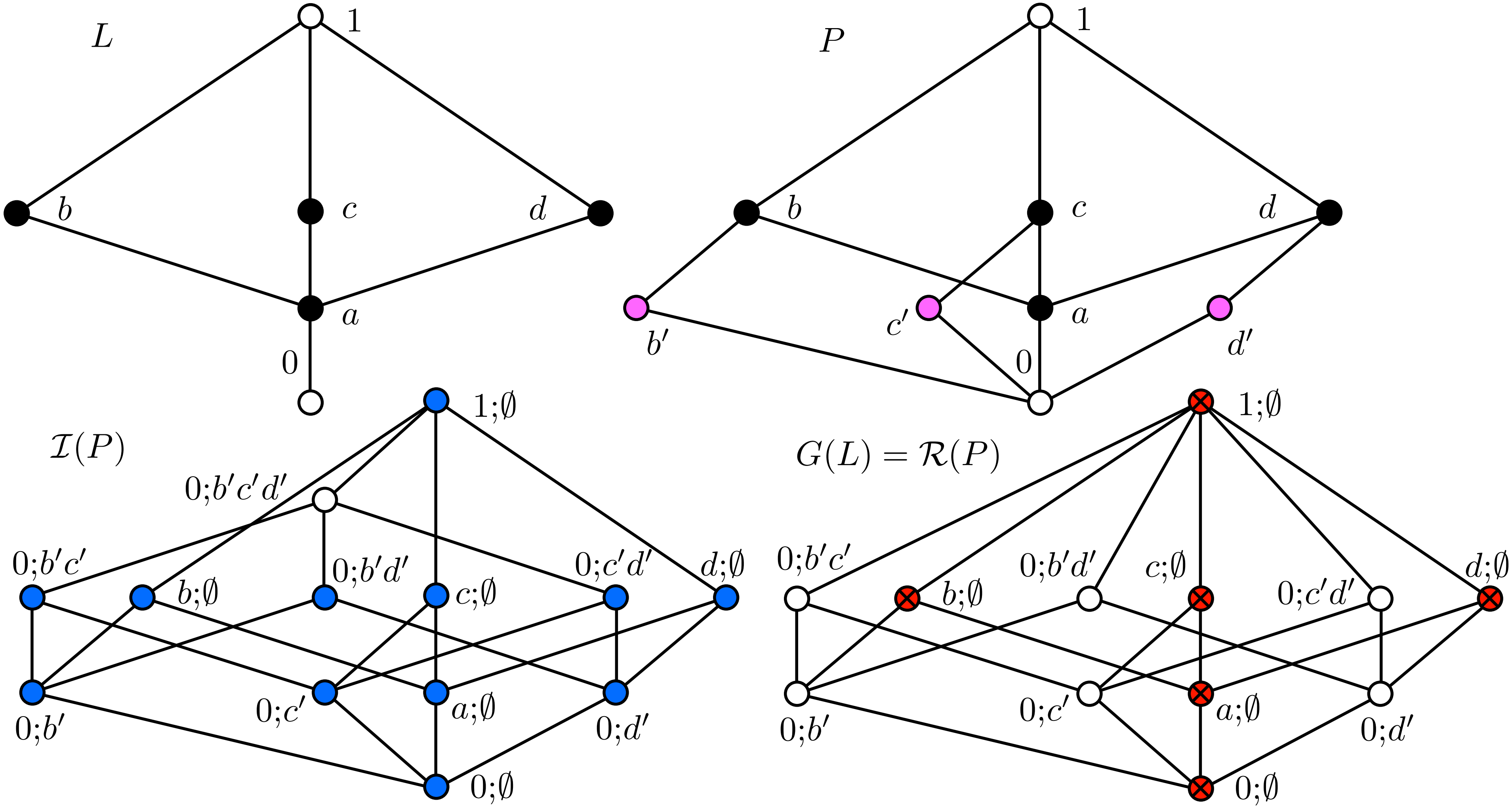


Why $0; b'c'd'$ is not rank-critical?

$\mathbf{S}_{\prec}$ construction: rank-critical ideals

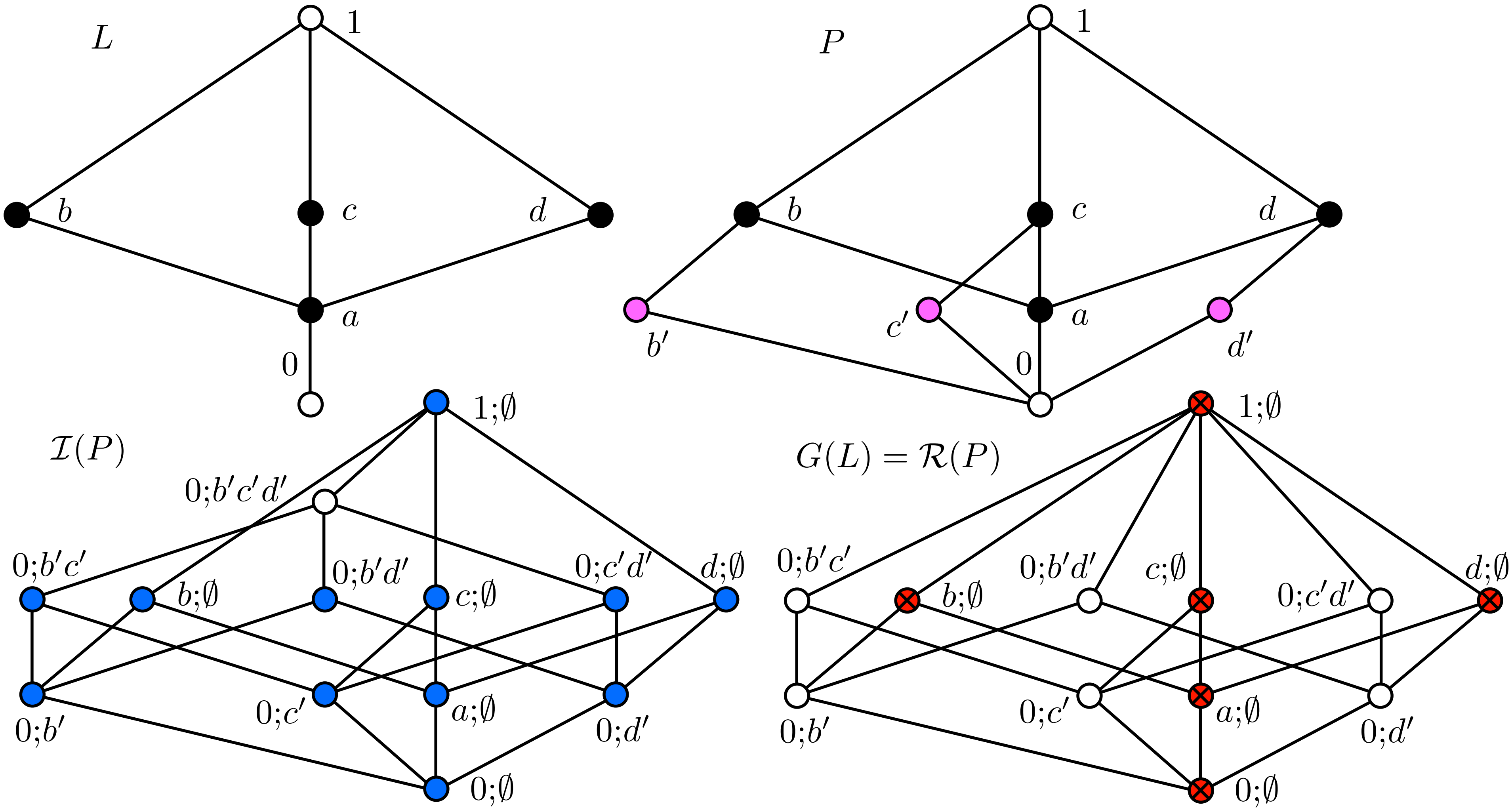


Why 0; $b'c'd'$ is not rank-critical? Since P

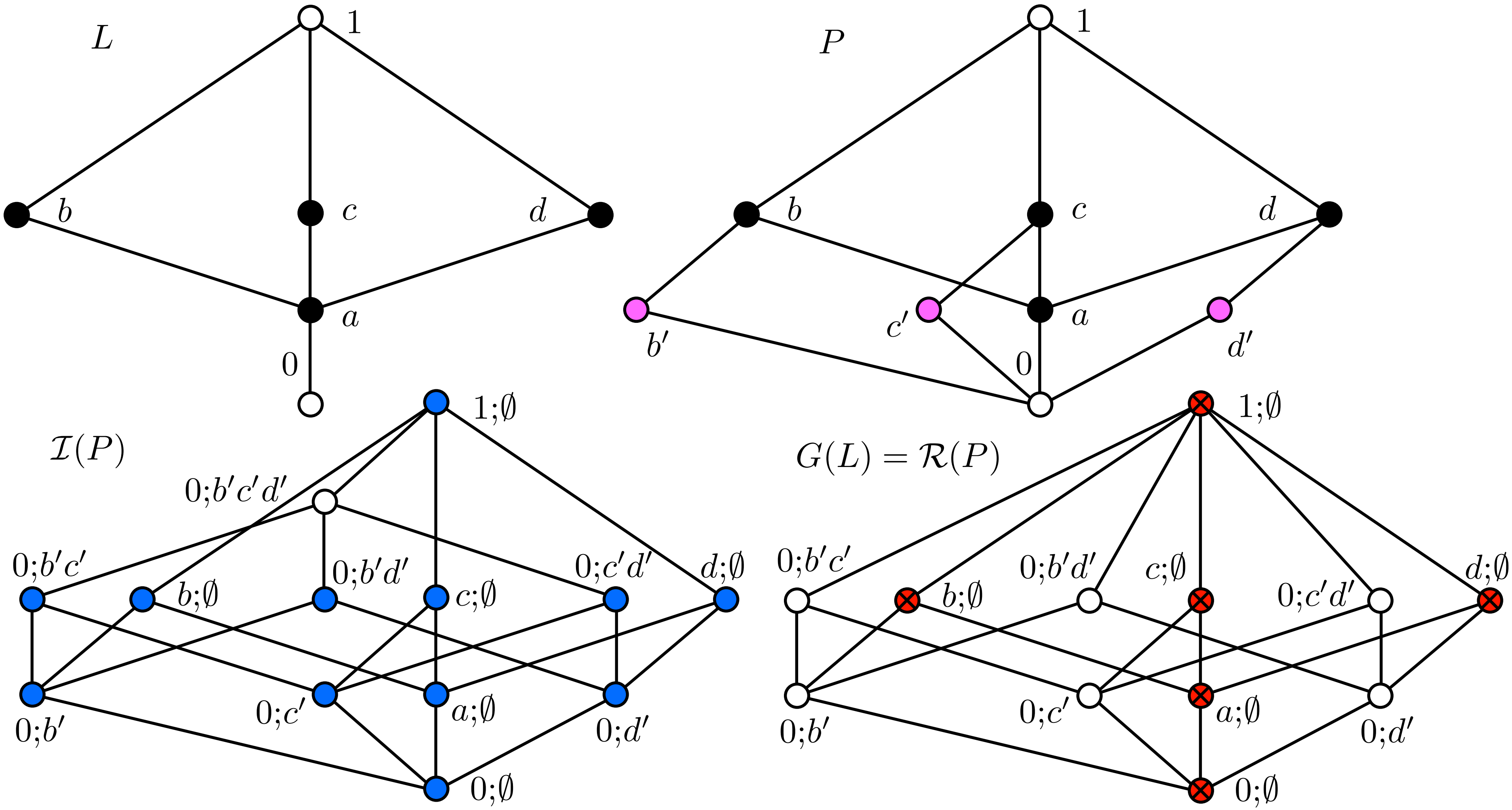


Why 0; $b'c'd'$ is not rank-critical? Since P has the same rank.

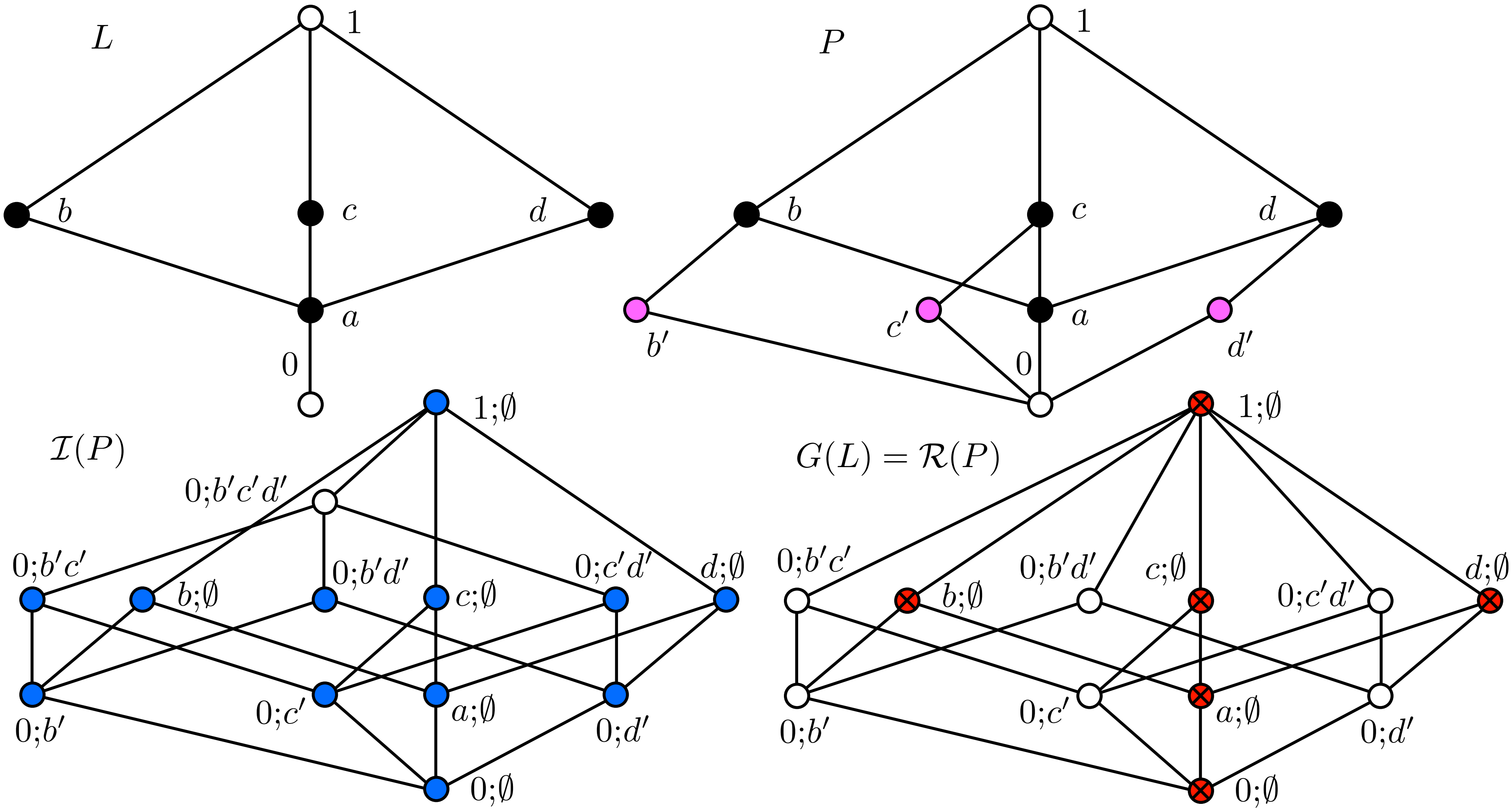
$\mathbf{S}_\prec$ construction: rank-critical ideals



Desired $G(L) := \mathcal{R}(P)$. ⊗ shows the $\mathbf{S}_\prec$ embedding. Q



Desired $G(L) := \mathcal{R}(P)$. $\otimes$ shows the $\mathbf{S}_{\prec}$ embedding. Q.e

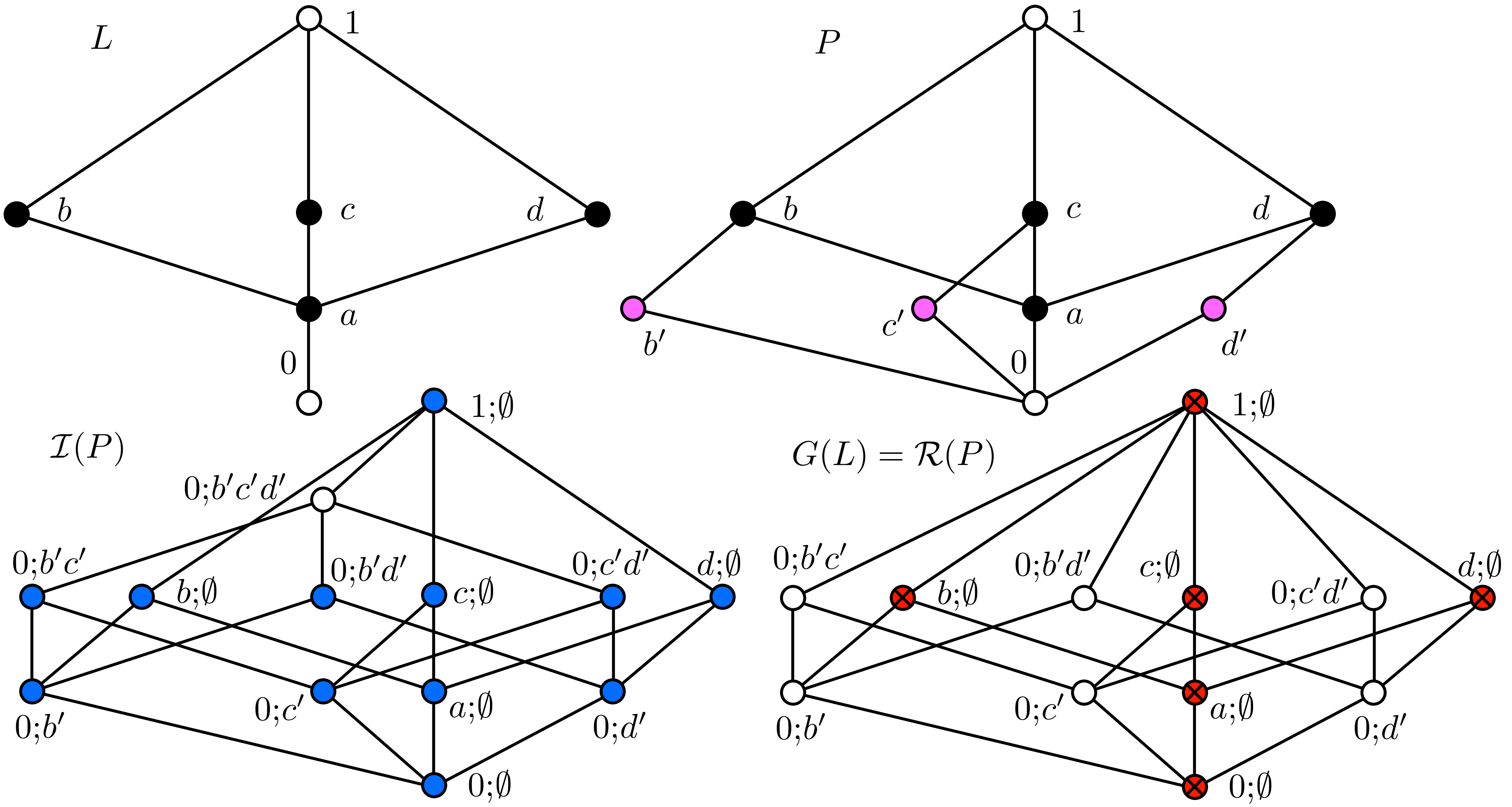


Desired $G(L) := \mathcal{R}(P)$. $\otimes$ shows the $\mathbf{S}_{\prec}$ embedding. Q.e.d.

If L is a finite distributive lattice, then so is our $G(L)$, If L is a finite distributive lattice, then so is our $G(L)$, while Grätzer and Kiss embeds the three-element chain (acting as L) into M_3.

If L is a finite distributive lattice, then so is our $G(L)$, while Grätzer and Kiss embeds the three-element chain (acting as L) into M_3.

No known construction minimizes $|G(L)|$. For example, L in the next slide embeds into $\mathbf{2} \times M_3$, which is ten-element, while our construction gives an eleven-element $G(L)$:


L
1
b
c
d
a
0
P
1
b
c
d
b′
c′
a
d′
0
I(P)
1;∅
0;b′c′d′
0;b′c′
b;∅
0;b′d′
c;∅
0;c′d′
d;∅
0;b′
0;c′
a;∅
0;d′
0;∅
G(L) = R(P)
1;∅
0;b′c′
b;∅
0;b′d′
c;∅
0;c′d′
d;∅
0;b′
0;c′
a;∅
0;d′
0;∅

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Can we stipulate that L be a finite geometric lattice?

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Can we stipulate that L be a finite geometric lattice?

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Can we stipulate that L be a finite geometric lattice?

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Can we stipulate that L be a finite geometric lattice?

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Can we stipulate that L be a finite geometric lattice? NO!

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

Can we stipulate that L be a finite geometric lattice? NO!

What is a finite geometric lattice: a finite

Dilworth theorem: every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

$\exists$ great many further results that stipulate some particular properties of L. The first one from Grätzer and Schmidt (1957): L can always be sectionally complemented.

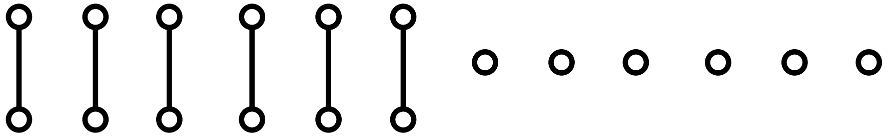
Can we stipulate that L be a finite geometric lattice? NO!

What is a finite geometric lattice: a finite sm lattice L such that $J(L)$ is an antichain, that is, $J(L)$ is the cardinal sum of one-element chains. This motivates the next definition:

Finite almost-geometric lattice = finite sm lattice such that $J(L)$ is the cardinal sum of at most two-element chains.

$J(L)$, if L is geometric:

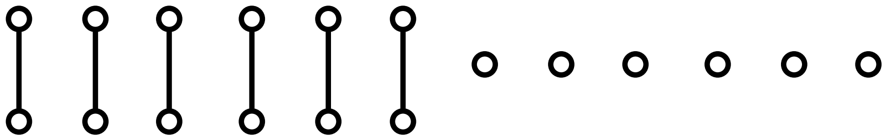
$J(L)$, if L is almost-geometric:



Finite almost-geometric lattice = finite sm lattice such that $J(L)$ is the cardinal sum of at most two-element chains.

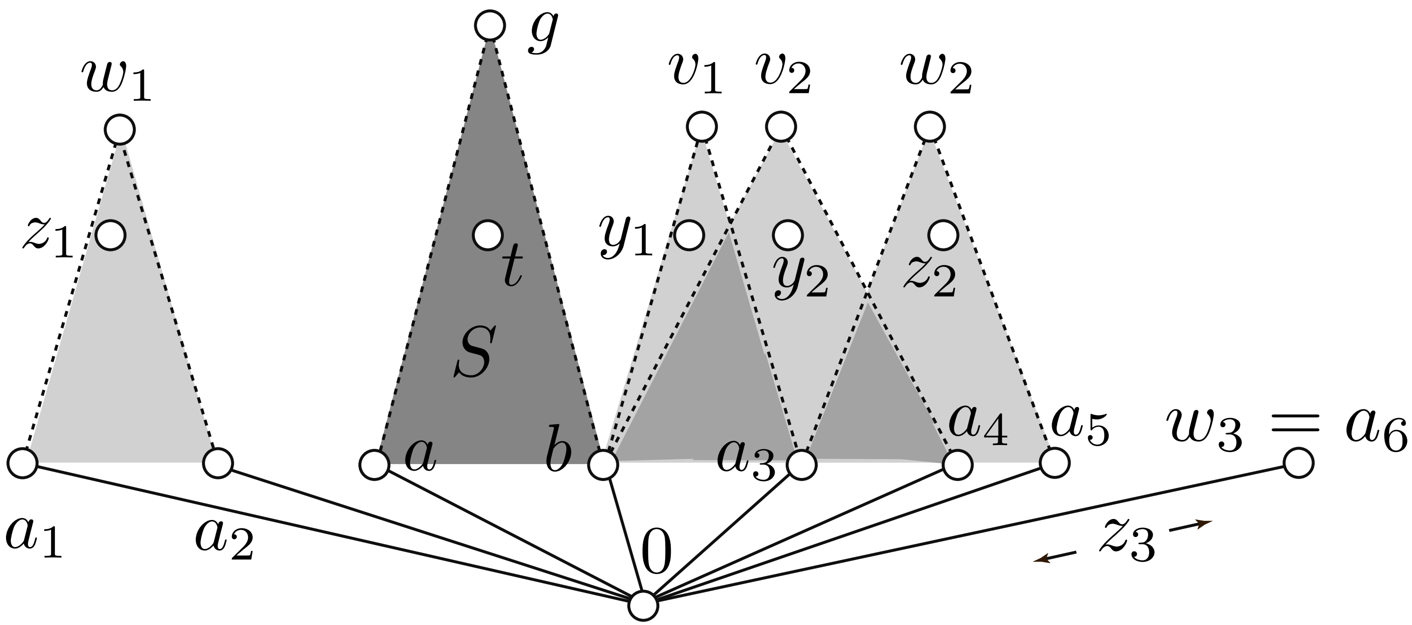
$J(L)$, if L is geometric:

$J(L)$, if L is almost-geometric:



Theorem (Czédli–Schmidt, 2008) Every finite distributive lattice D is isomorphic to the congruence lattice of a finite almost-geometric lattice G.

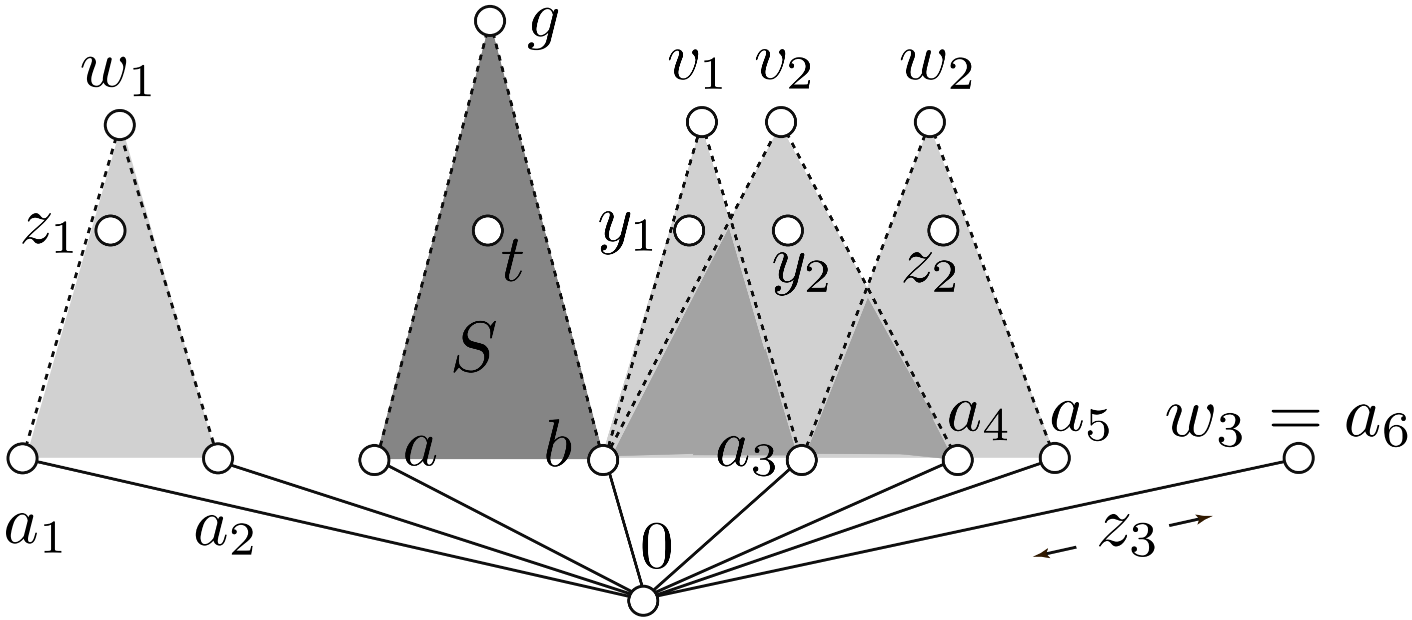
Proof



The chopped lattice $C = C_{m+1}$ in Case 2

See **www.math.u-szeged.hu/∼czedli/**

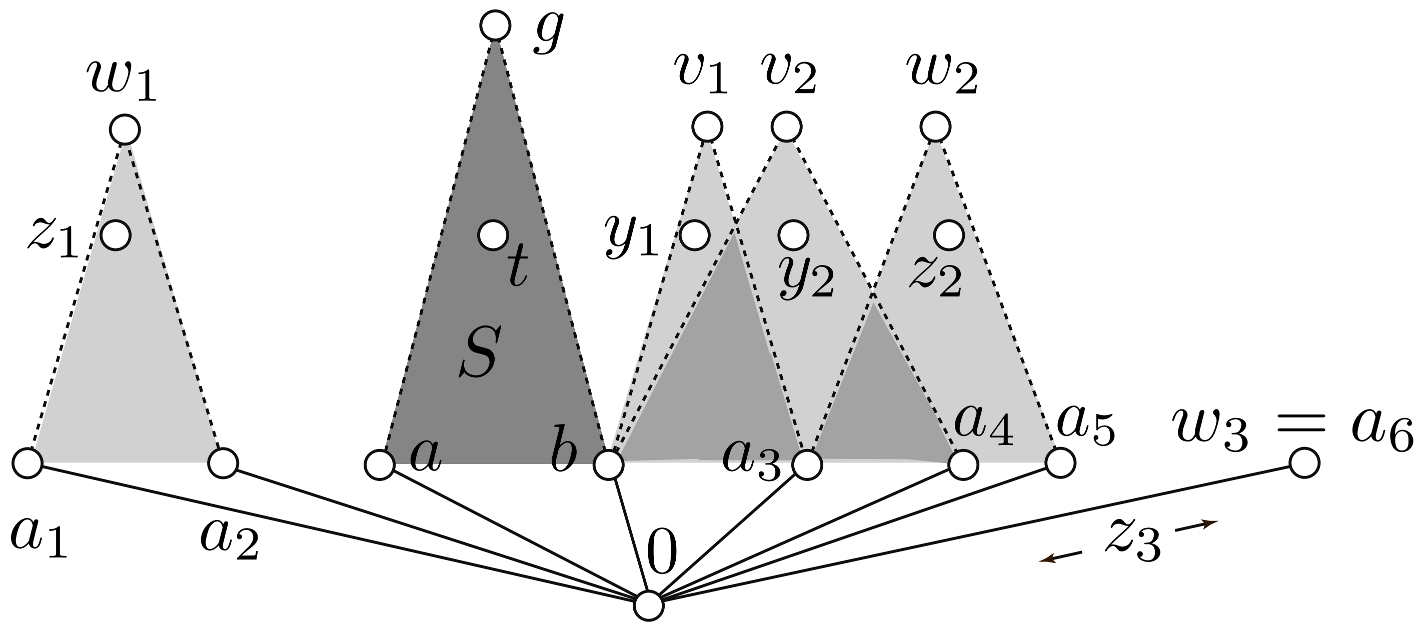
Proof



The chopped lattice $C = C_{m+1}$ in Case 2

See **www.math.u-szeged.hu/∼czedli/** (18 pages

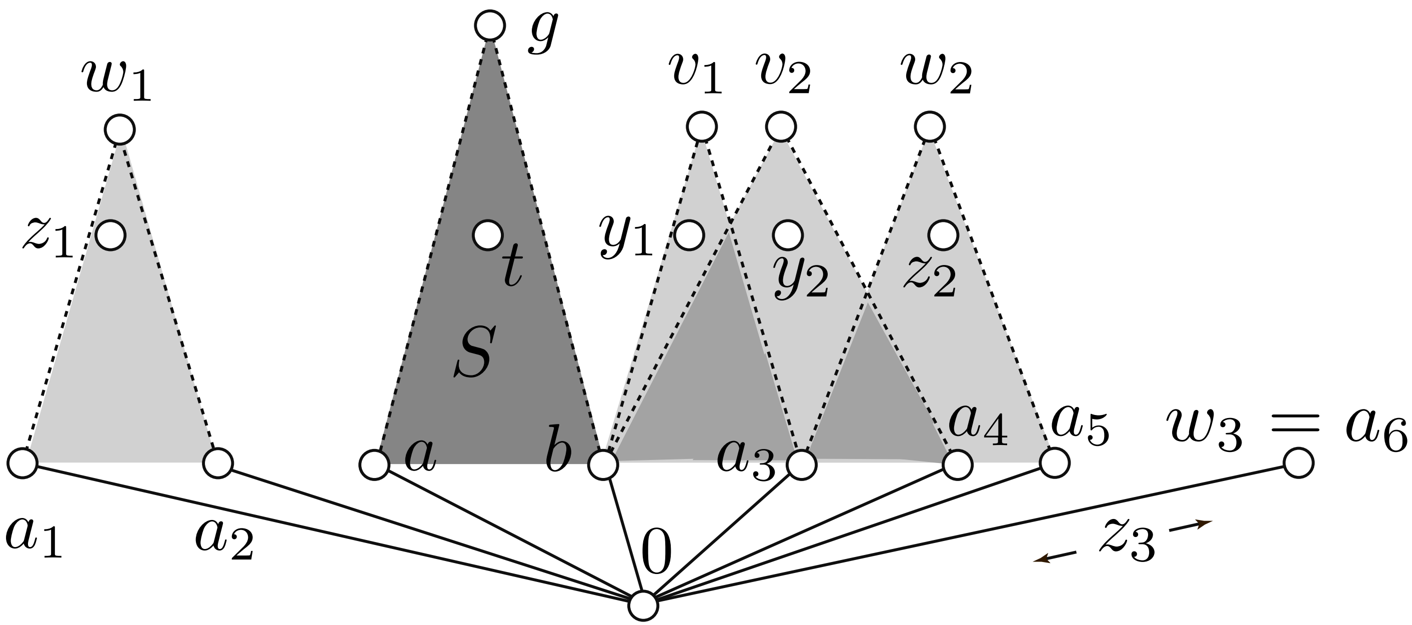
Proof



The chopped lattice $C = C_{m+1}$ in Case 2

See **www.math.u-szeged.hu/∼czedli/** (18 pages , which uses the „chopped lattice toolkit", see Grätzer: The congruences of a finite lattice — A Proof by Picture Approach).

Proof



The chopped lattice $C = C_{m+1}$ in Case 2

See **www.math.u-szeged.hu/∼czedli/** (18 pages , which uses the „chopped lattice toolkit", see Grätzer: The congruences of a finite lattice — A Proof by Picture Approach). (The present slides are also available there.)