A locomotive that strengthens the Jordan-Hölder theorem*

Gábor Czédli and E. Tamás Schmidt

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http://www.math.bme.hu/∼schmidt/

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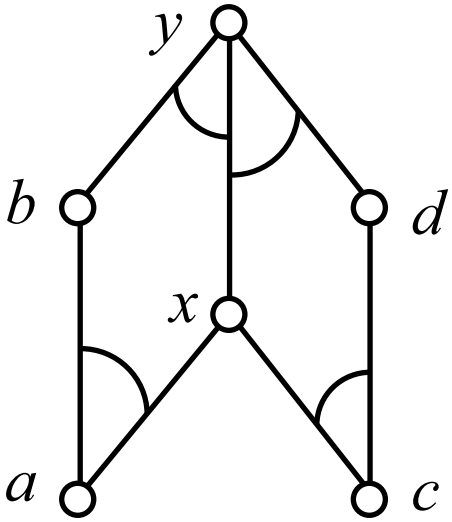
From now on, we are in a semimodular lattice L.

$[a,b] \mathrel{/\!\searrow} [c,d]$ iff $[a,b] \nearrow [x,y]$ and $[x,y] \searrow [c,d]$ for some interval $[x,y]$, that is,



1. Theorem (Main). *Assume that* L *is semimodular, and*

$$C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\} \text{ and}$$

$$D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}. \text{ Then}$$

- $n = m$, *and there is a permutation* π *of the set* $\{1, \ldots, n\}$ *such that the interval* $[c_{i-1}, c_i]$ *is* **up-and-down** *projective to the interval* $[d_{\pi(i)-1}, d_{\pi(i)}]$, *for all* i. *(Jordan-Hölder theorem for sm lattices +* **G**

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- *moreover, this permutation* π *is* uniquely determined, *and, for all* $i, j \in \{1, \ldots, n\}$, $[c_{i-1}, c_i] \not\searrow [d_{j-1}, d_j]$ *implies* $j \leq \pi(i)$.

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The last (red) part implies the uniqueness since if ϱ and π are permutations and $\varrho \leq \pi$, then $\varrho = \pi$.

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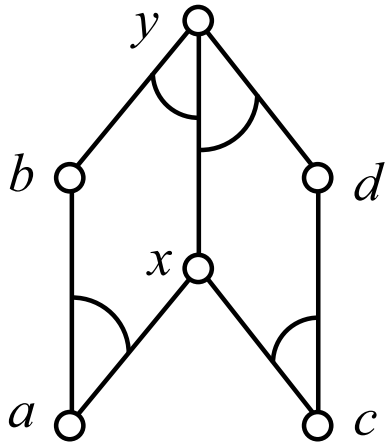
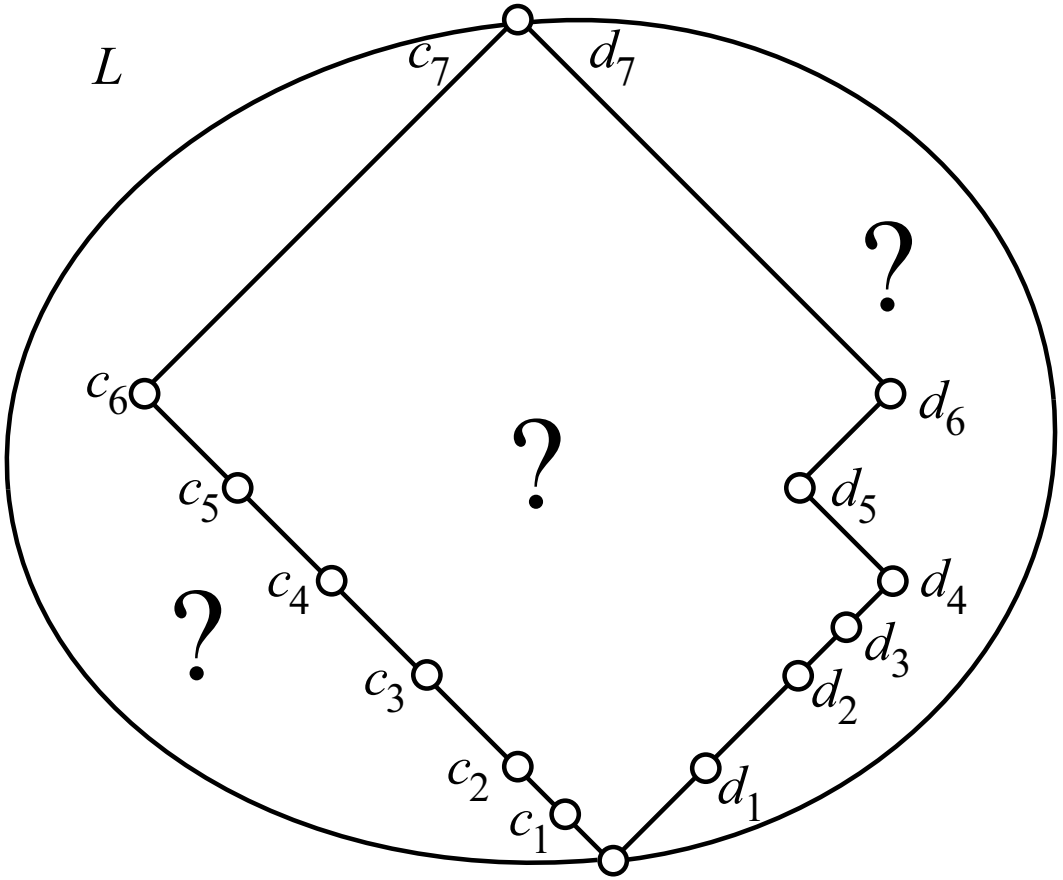
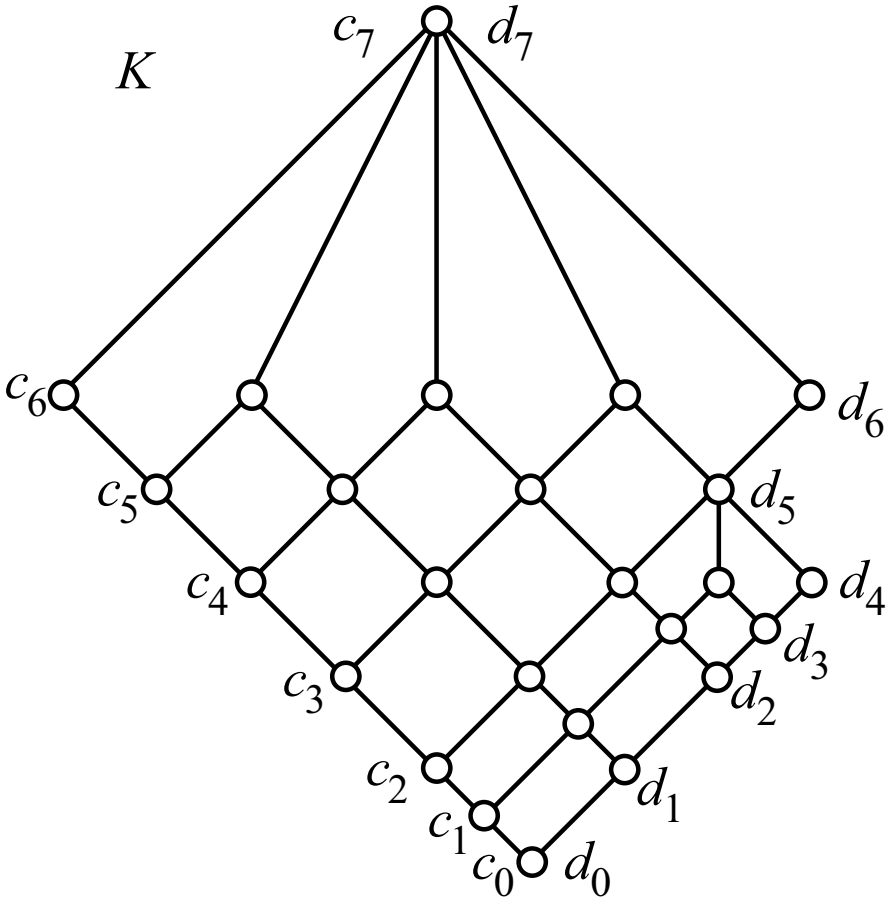
• Let $[a, b]$, $[x, y]$, and $[c, d]$ be **prime** intervals. Then the validity of $[a, b] \nearrow [x, y]$ and $[x, y] \searrow [c, d]$ depends only on $\vee$! E.g.,



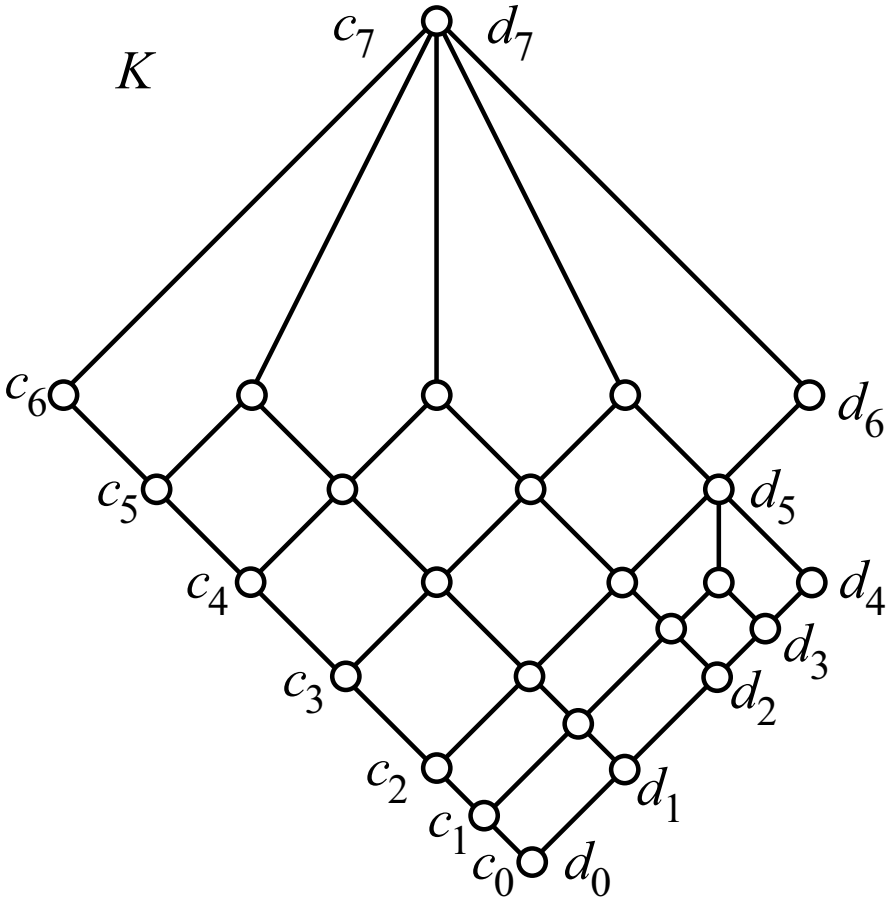
$[a, b] \nearrow [x, y]$ iff $b \vee x = y$ (trivial exercise).

- So, take the **join**-subsemilattice K generated by $C \cup D$!





We get a **slim** (terminology from G. Grätzer and E. Knapp)



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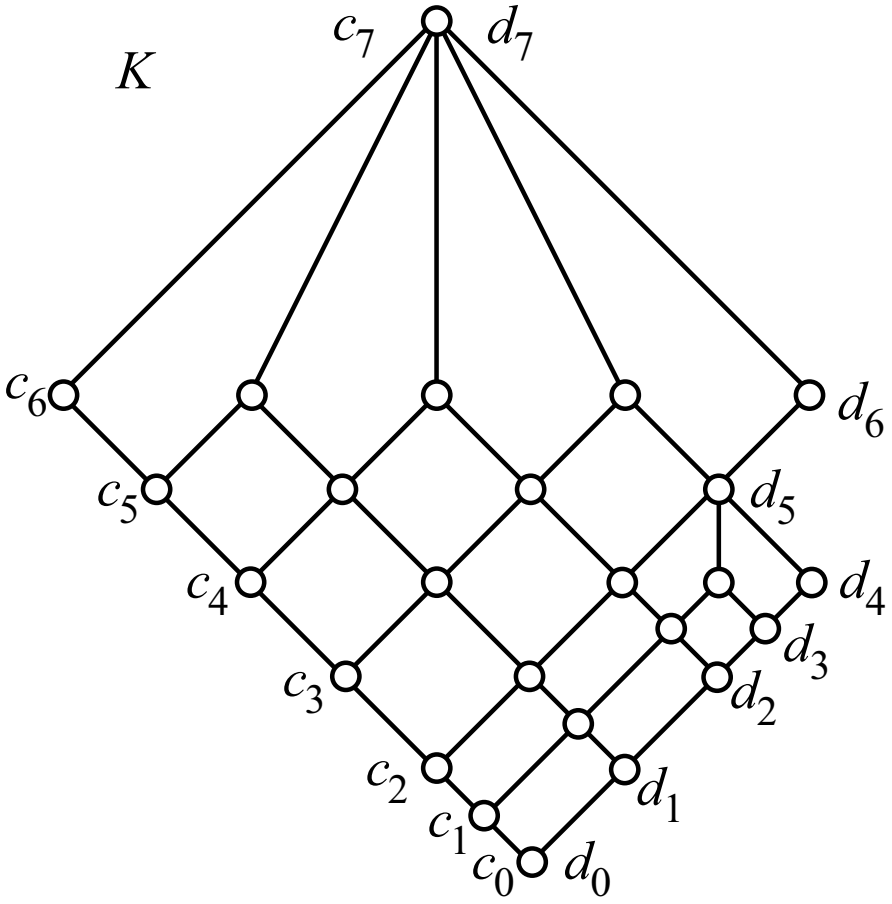

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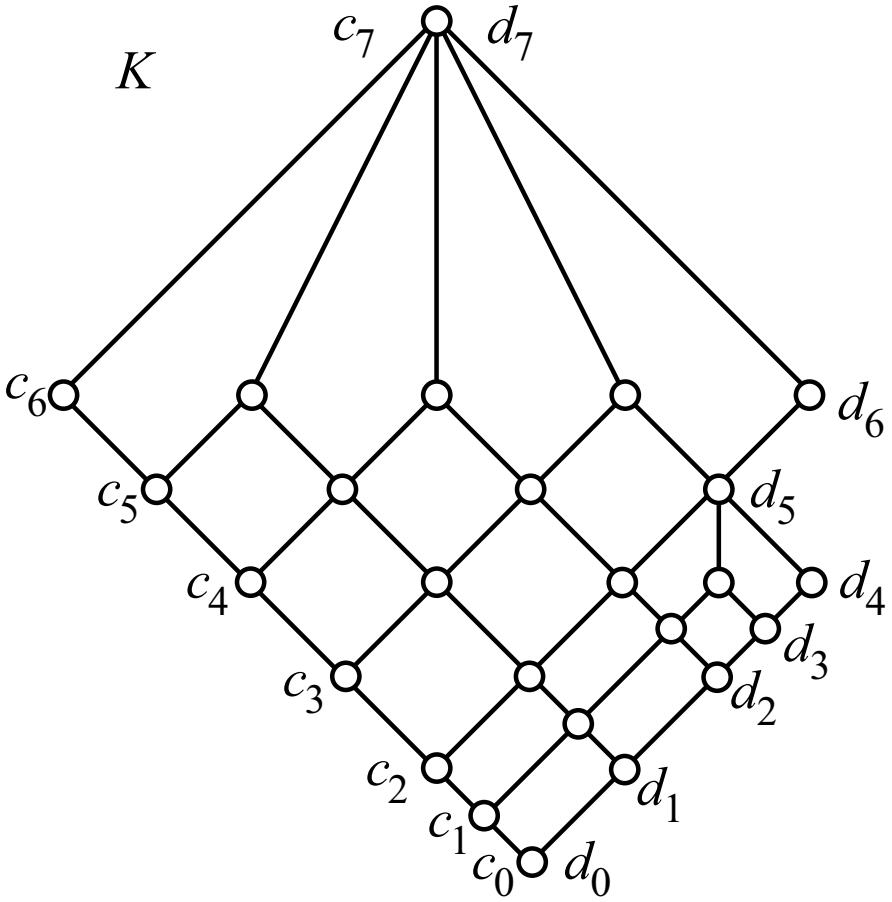

We get a **slim** (terminology from G. Grätzer and E. Knapp) **planar** semimodular lattice K, with left boundary chain C and right boundary chain D. (Straightforward.)

• **Locomotive Lemma:**(Up-and-down) projectivity between **prime**

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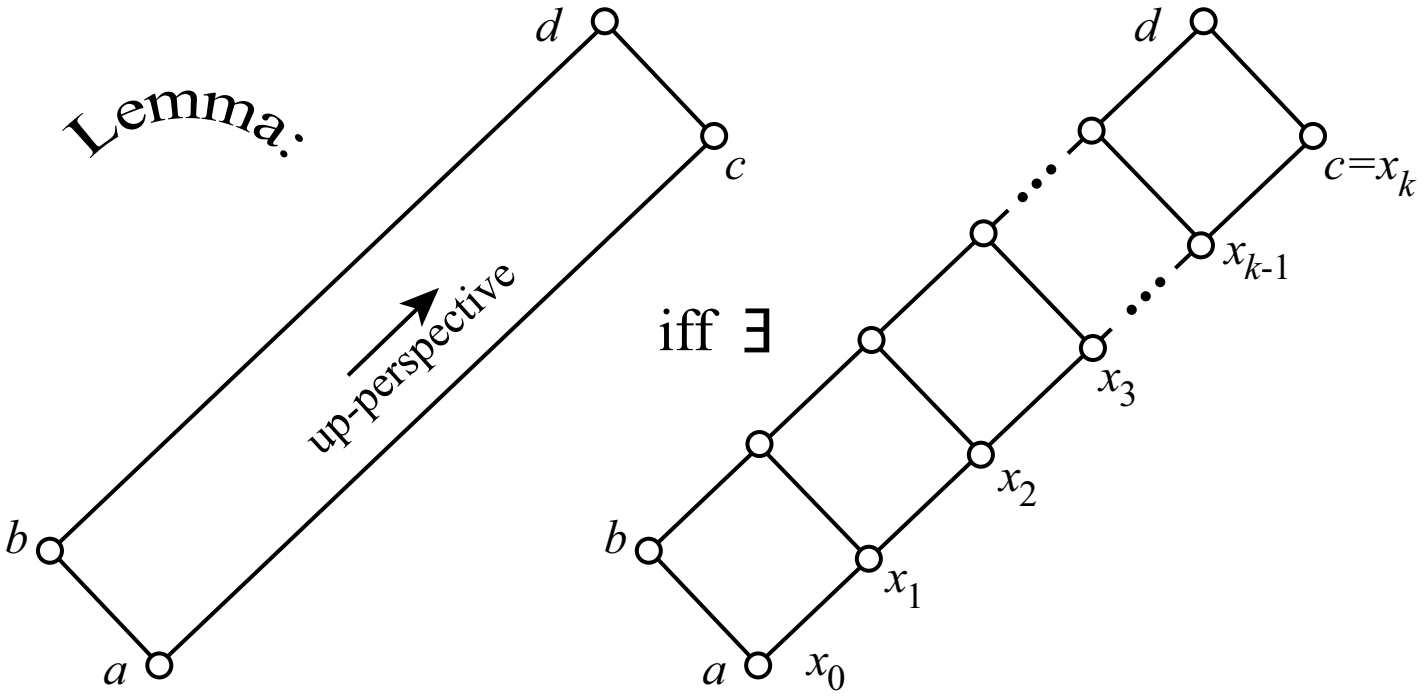

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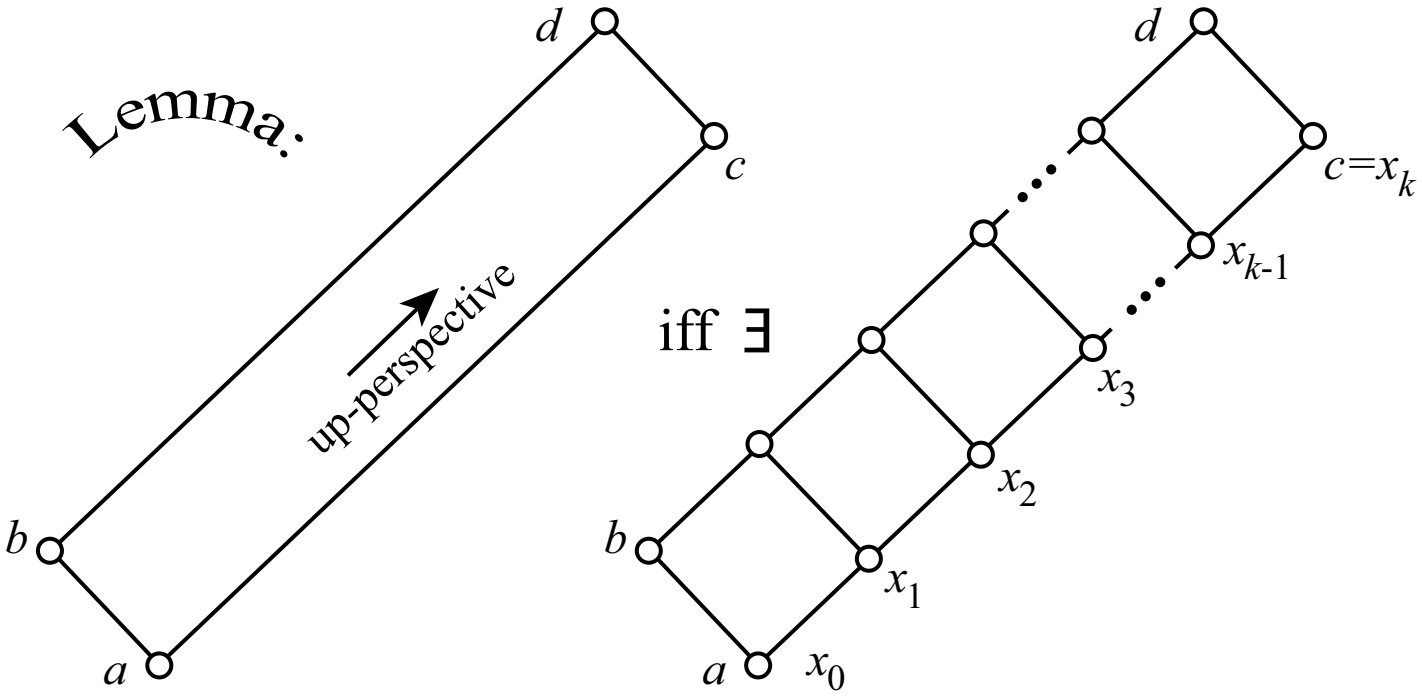
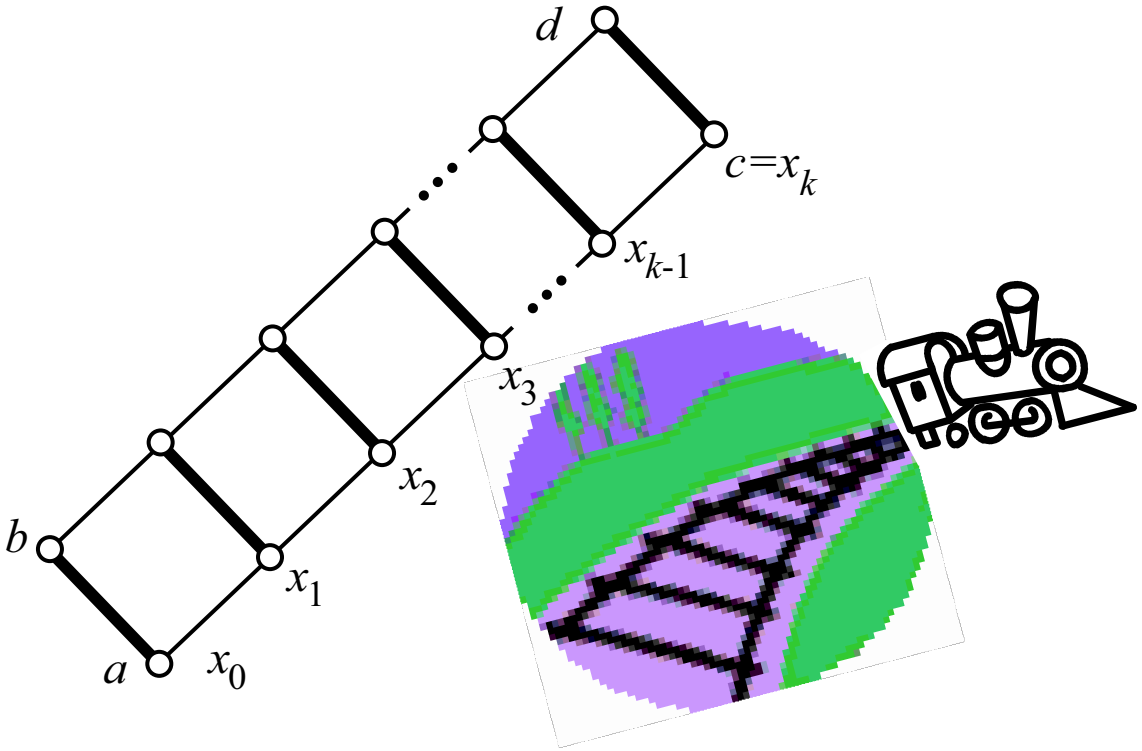
• **Locomotive Lemma:**(Up-and-down) projectivity between **prime** intervals is captured by **covering squares**.

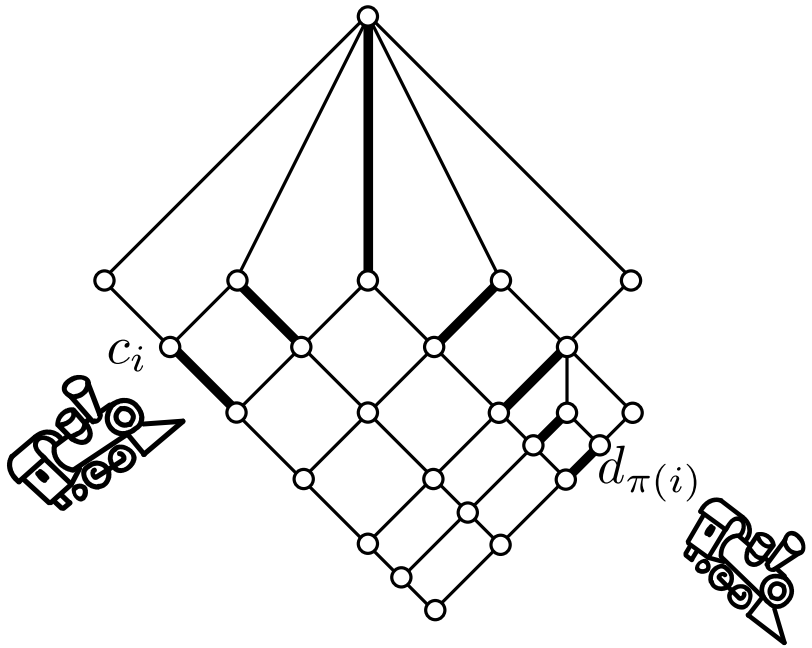


(Straightforward; any maximal chain in $[a, c]$ will do.)

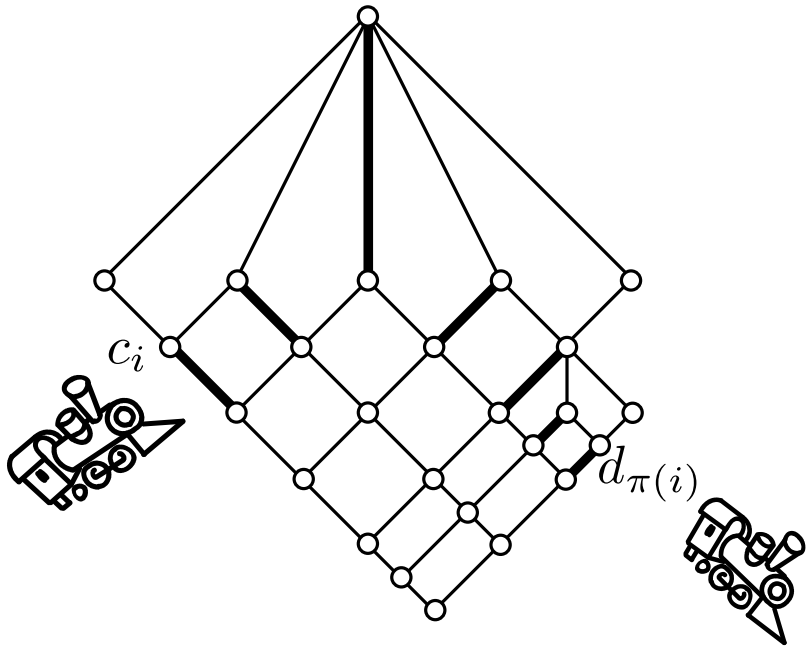
• Illustration: **thick** edges on the left ≈ beams holding the rail.



The consecutive prime intervals (= *thick* edges) form a **trajectory**. **Trajectory** = possible path of the **locomotive**. More precisely: trajectory = class of the „prime projective" equivalence relattion described by the Locomotive Lemma. Trajectory = railroad of a locomotive.

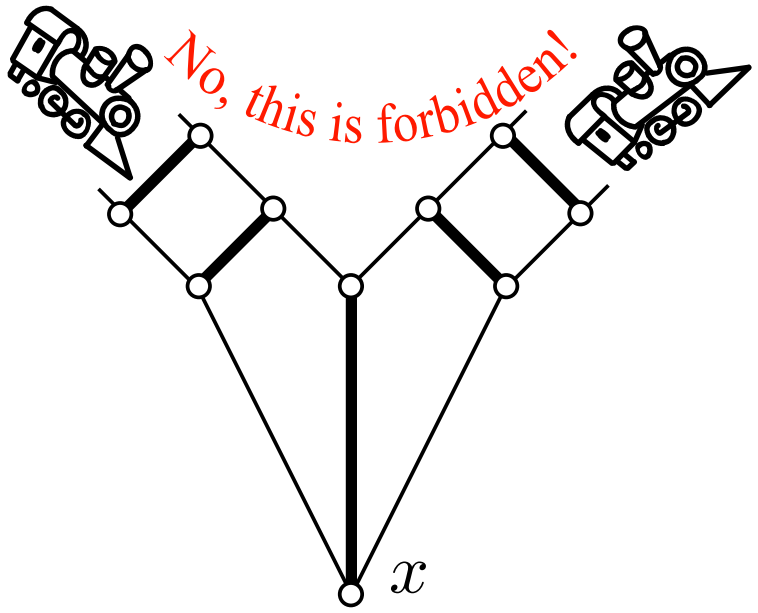


• Since opposite sides of covering squares (= cells) are uniquely determined, e



• Since opposite sides of covering squares (= cells) are uniquely determined, each prime interval of K belongs to a **unique** trajectory. In a trajectory, there is no fork, trajectories **never ramify**.

Usually, trajectories (locomotives) go from left to right.



In each moment, the locomotive (a trajectory) goes either to the northeast, or to the southeast.

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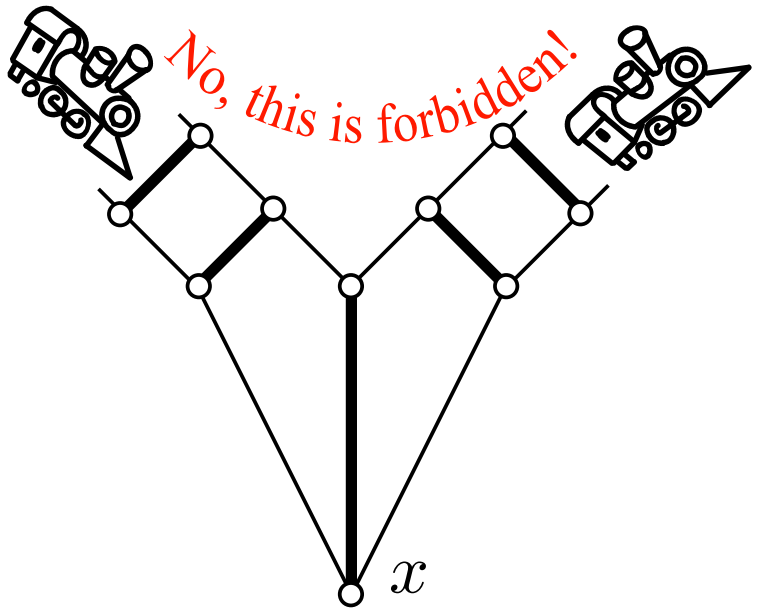

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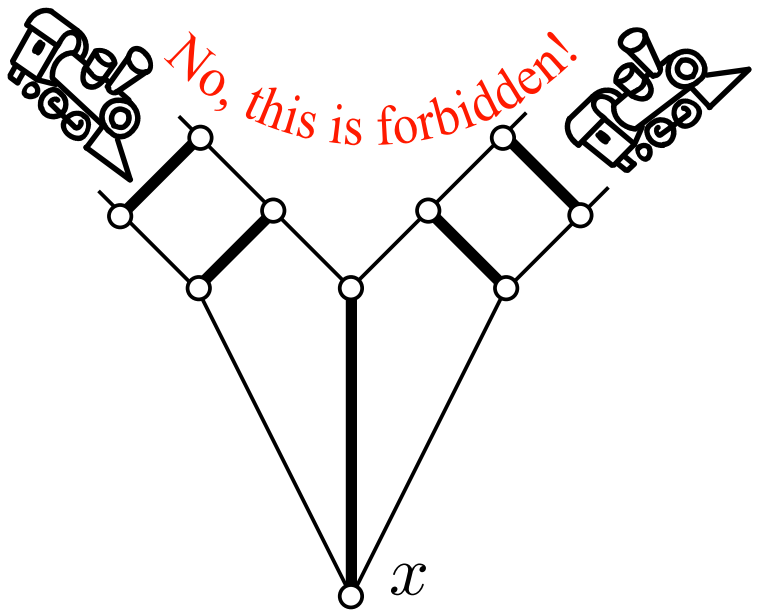

In each moment, the locomotive (a trajectory) goes either to the northeast, or to the southeast.

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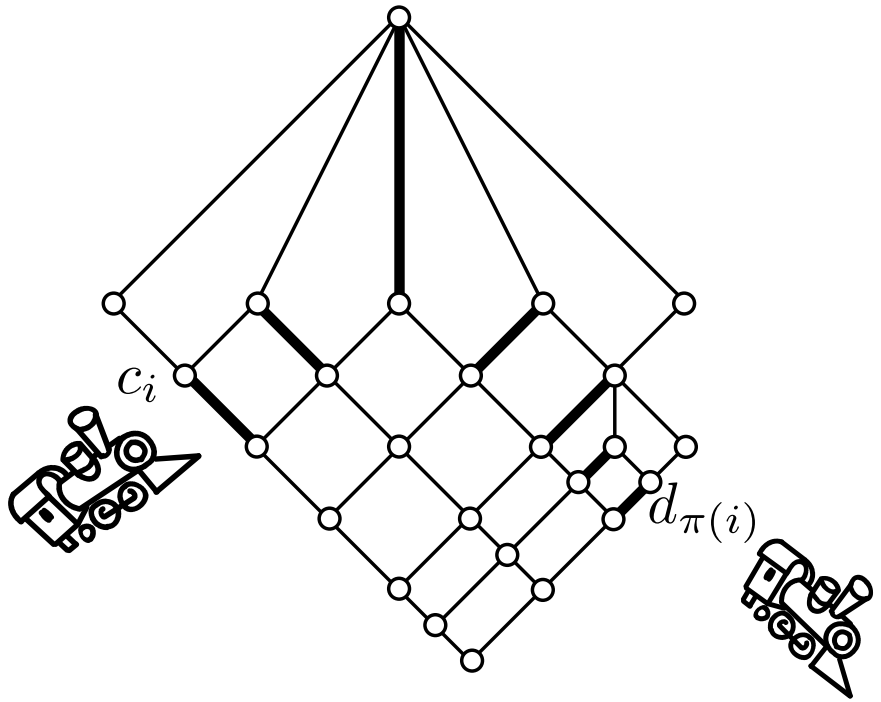
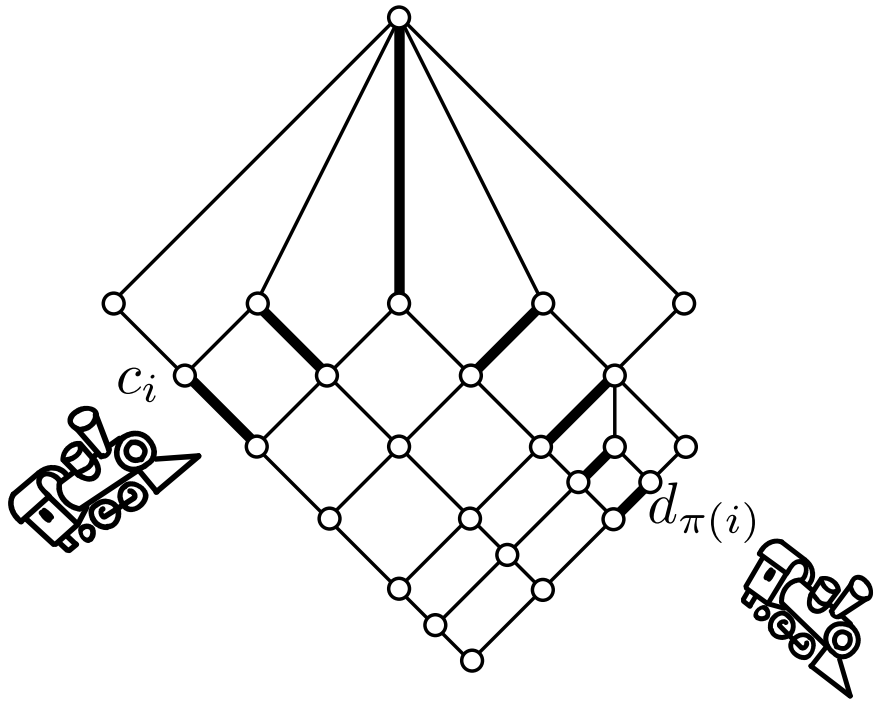
Indeed, otherwise x would have three upper covers, and slimness would easily lead to a contradiction (easy exercise).

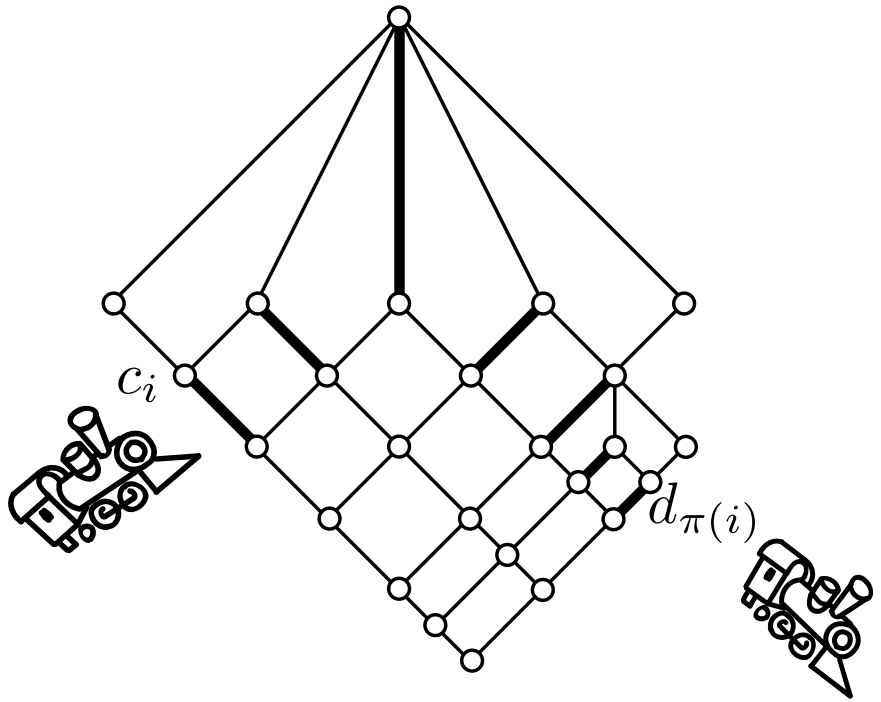
The "planar matching" by trajectories Czédli-Schmidt, 2010 10′/10’



• Prime intervals of the boundary chains C and D are described by their top elements c_i and d_j, and vice versa. Hence it suffices to deal with prime intervals instead of elements.



„Never ramify" property $\Rightarrow$ trajectories establish a bijection π between the prime intervals on the left boundary C, „departure side", and the prime intervals on the right boundary, „arrival side". For K, whence for L. (



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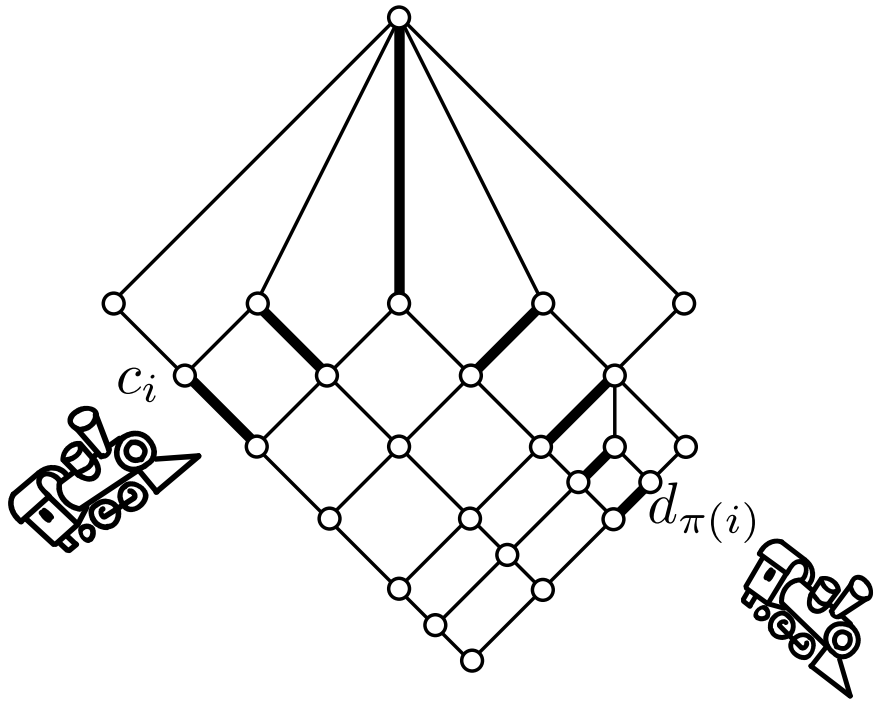

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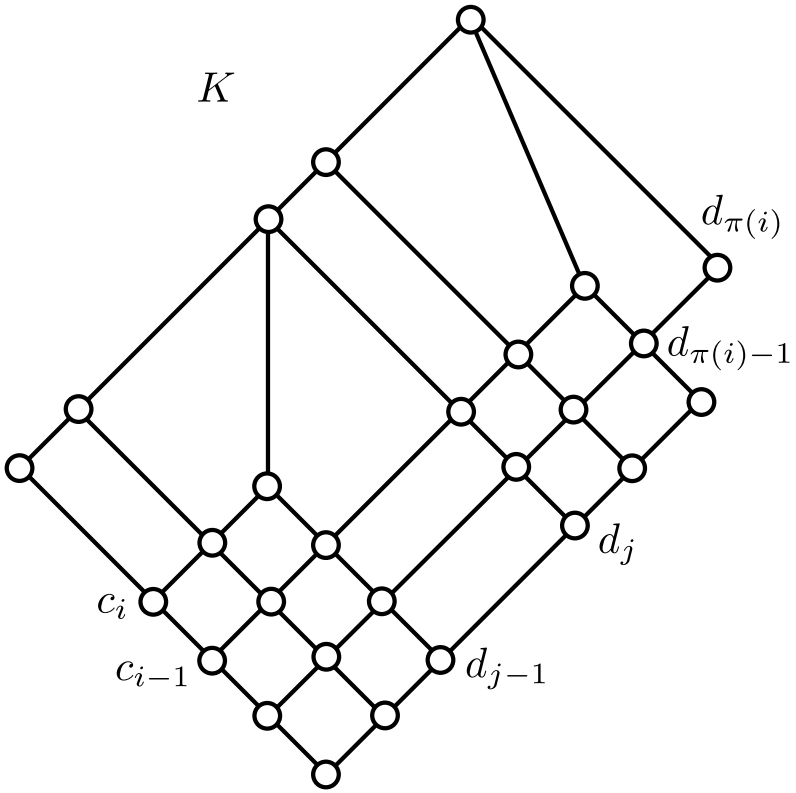
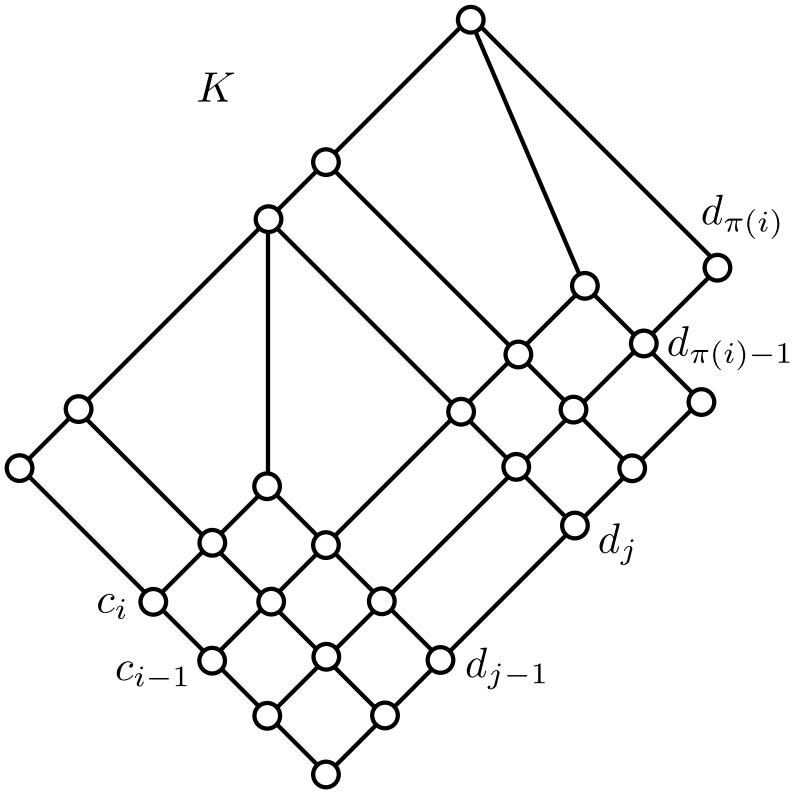
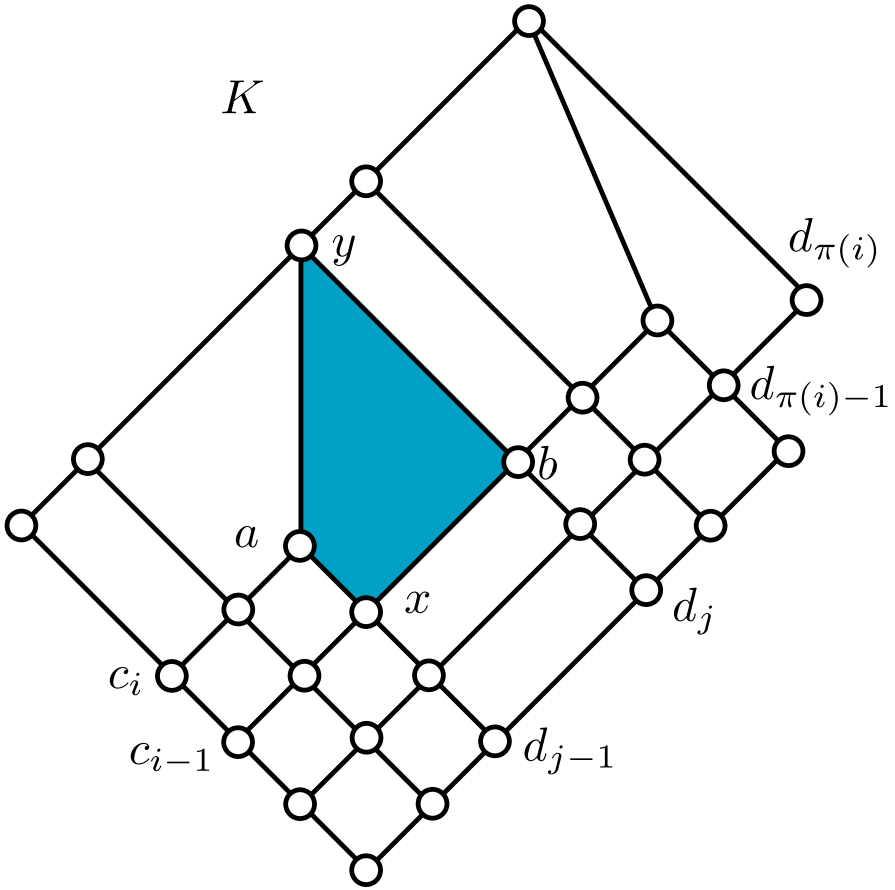


To show π is **highest matching**, that is, $[c_{i-1}, c_i] \not\searrow [d_{j-1}, d_j] \Rightarrow j \leq \pi(i)$, assume $j > \pi(i)$ (indirect assumption).



To show π is **highest matching**, that is, $[c_{i-1}, c_i] \not\searrow [d_{j-1}, d_j] \Rightarrow j \leq \pi(i)$, assume $j > \pi(i)$ (indirect assumption). Since K is governed by trajectories that say „only $\pi(i)$ is possible" and $j \neq \pi(i)$, we know that $[c_{i-1}, c_i] \not\searrow [d_{j-1}, d_j]$ holds only in L but not in K.



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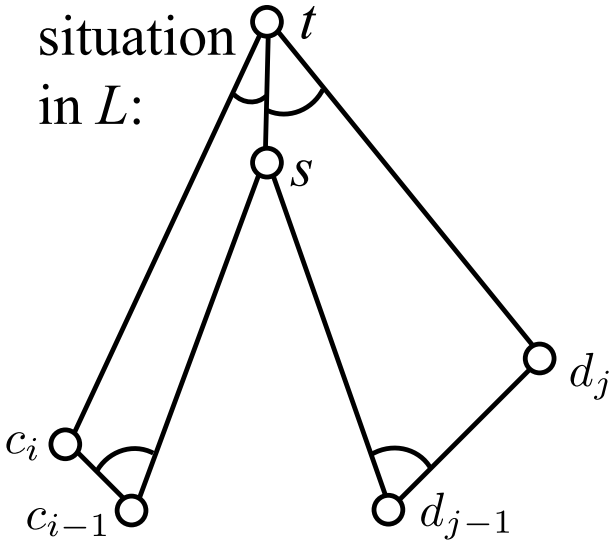
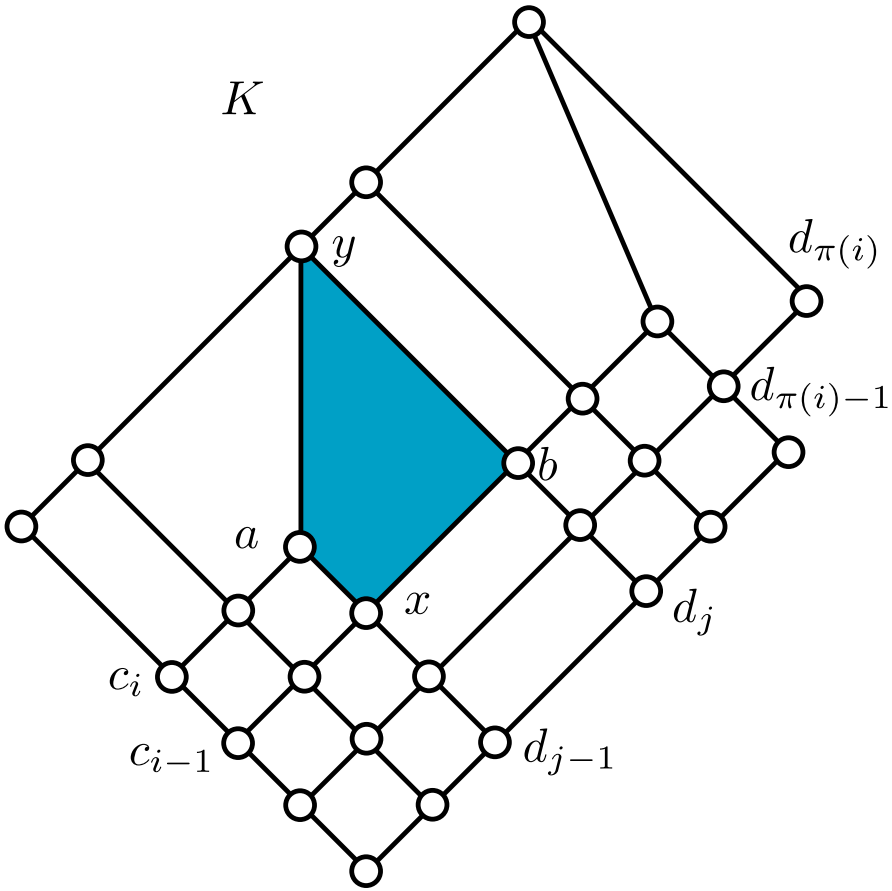


Denote the joins of c_{i-1} and c_i by d_{j-1}, d_j by x, y, a, b.



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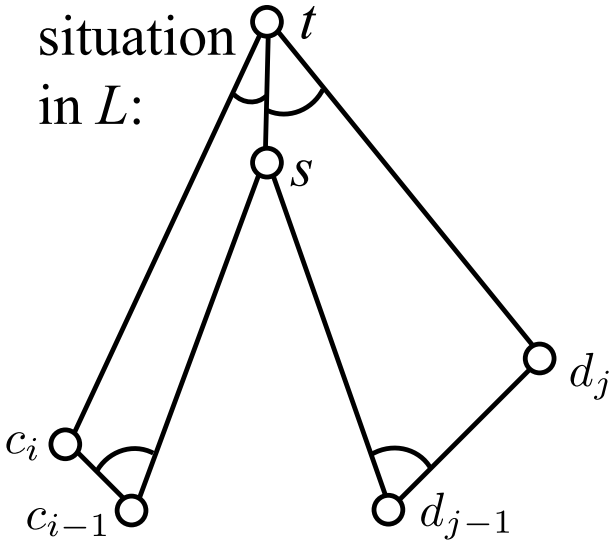

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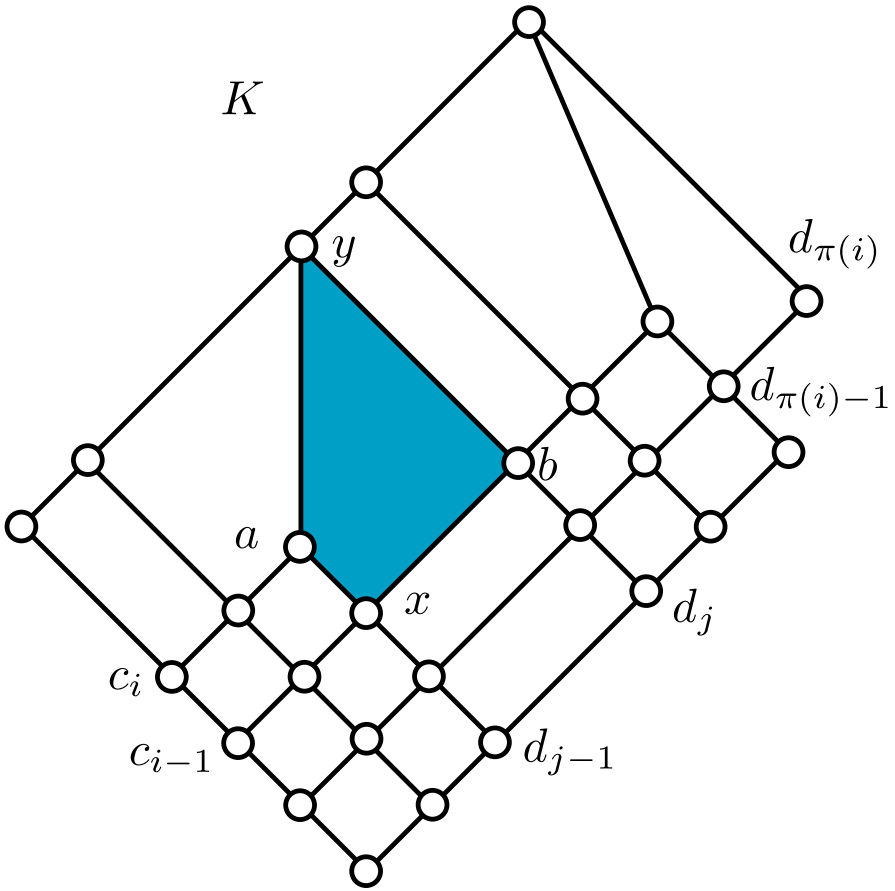
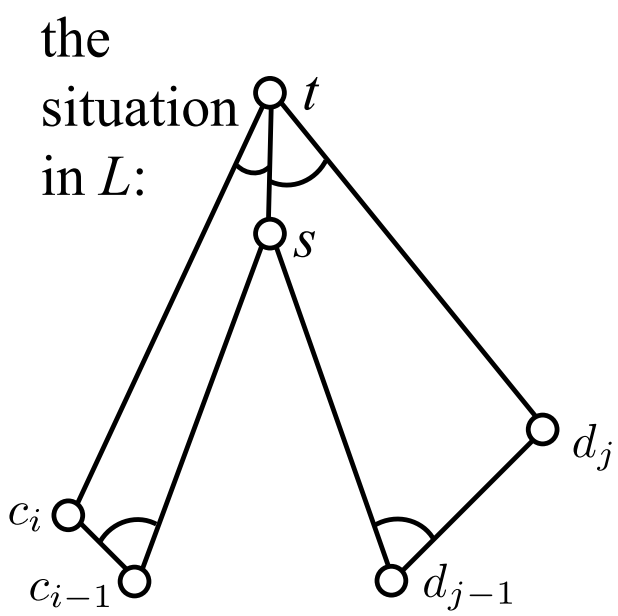
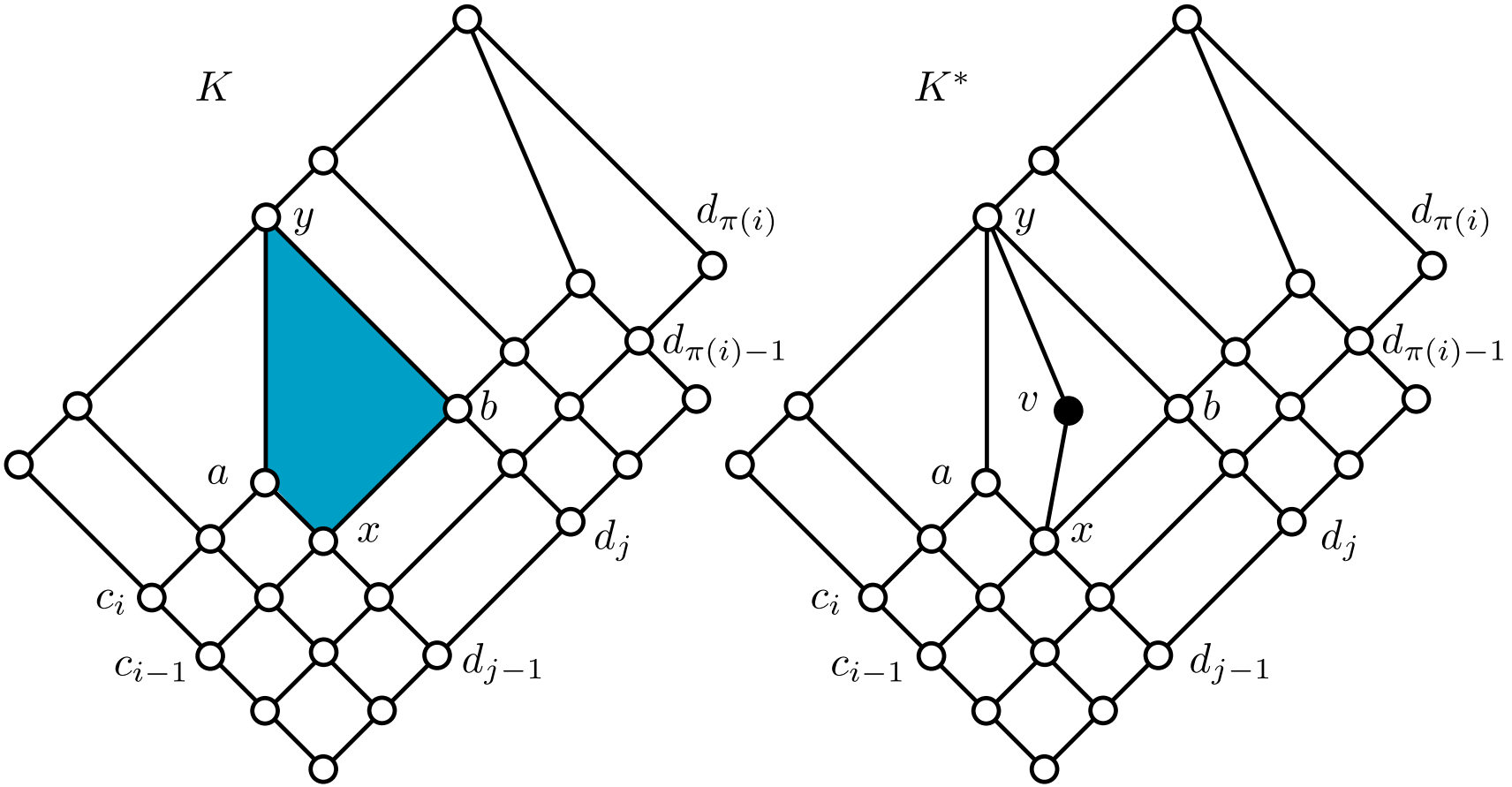




Denote the joins of c_{i-1} and c_i by d_{j-1}, d_j by x, y, a, b. The situation in L implies (very easy exercise) that $a \neq x \neq b$. Hence $\{x, a, b, y\}$ is a covering square by semimodularity.

v is the only ramification point in K^* Czédli-Schmidt, 2010 15′/5’



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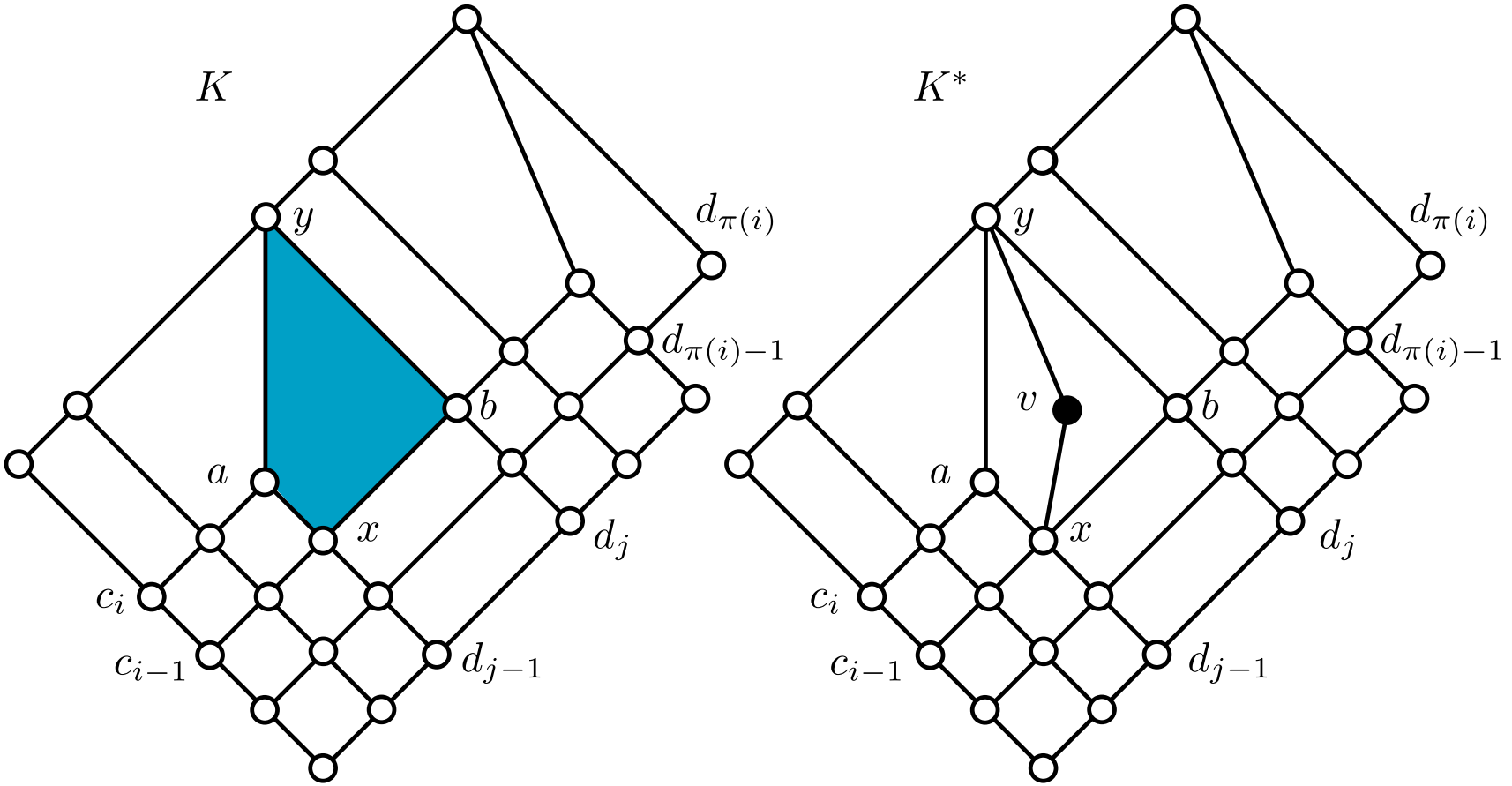
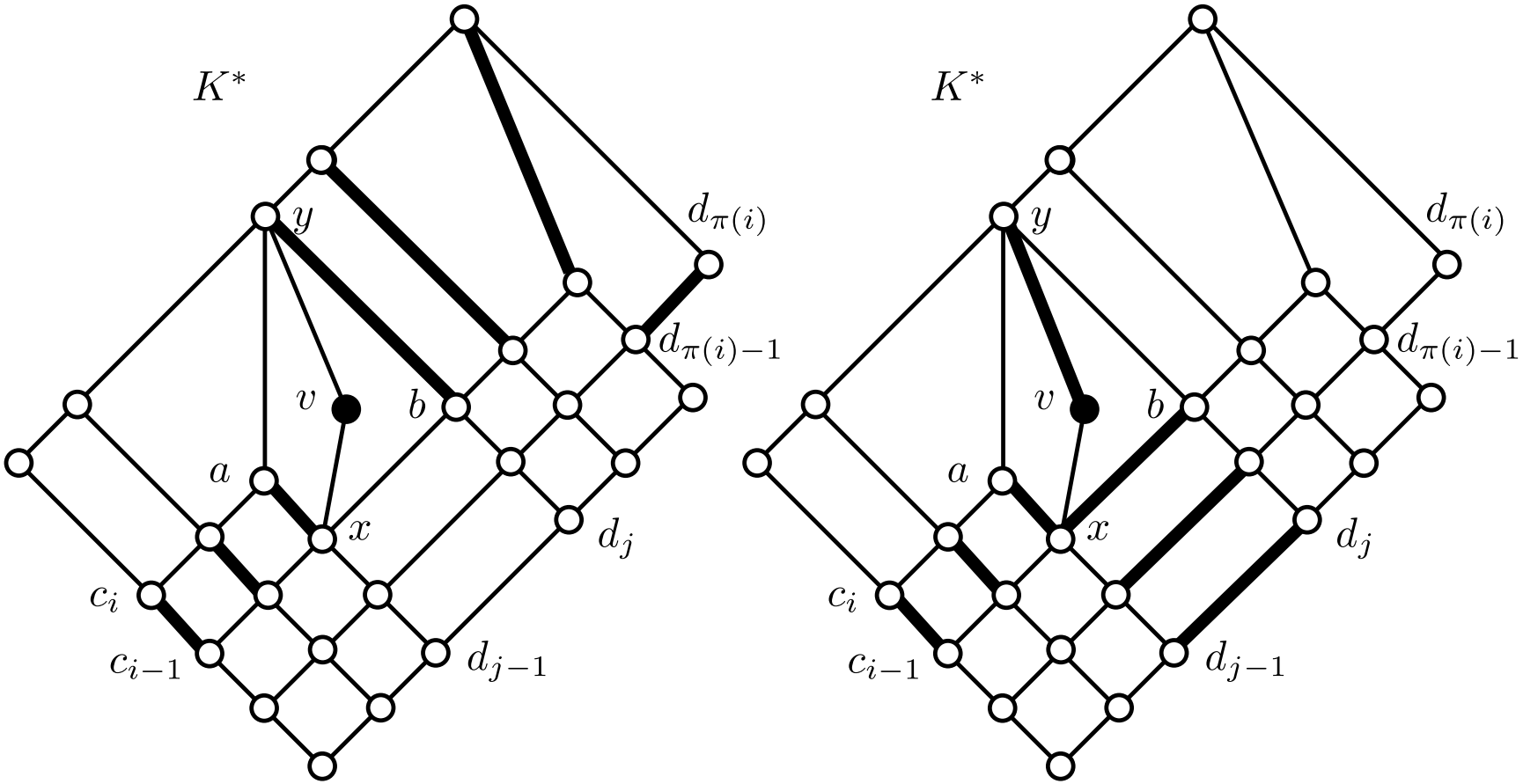
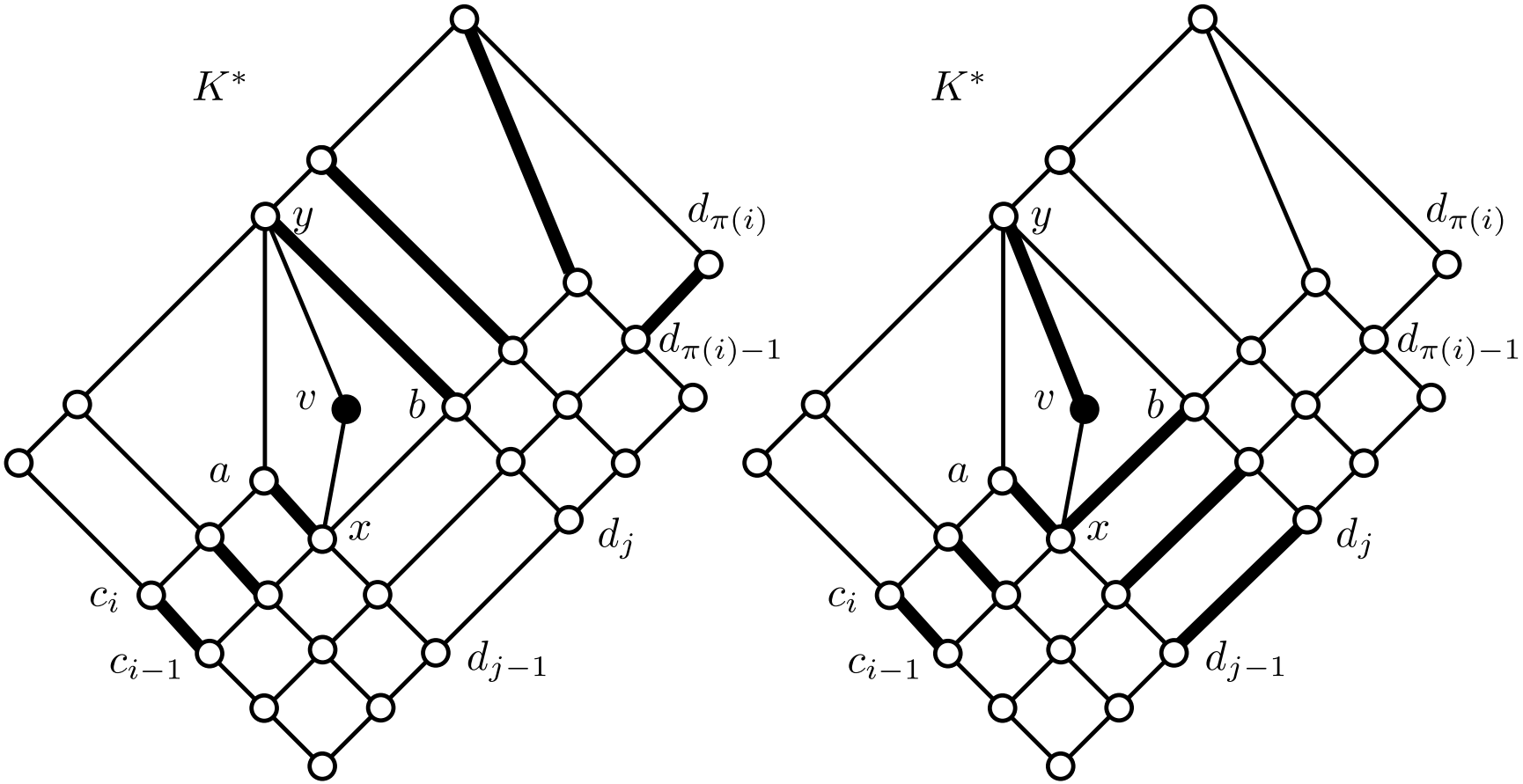


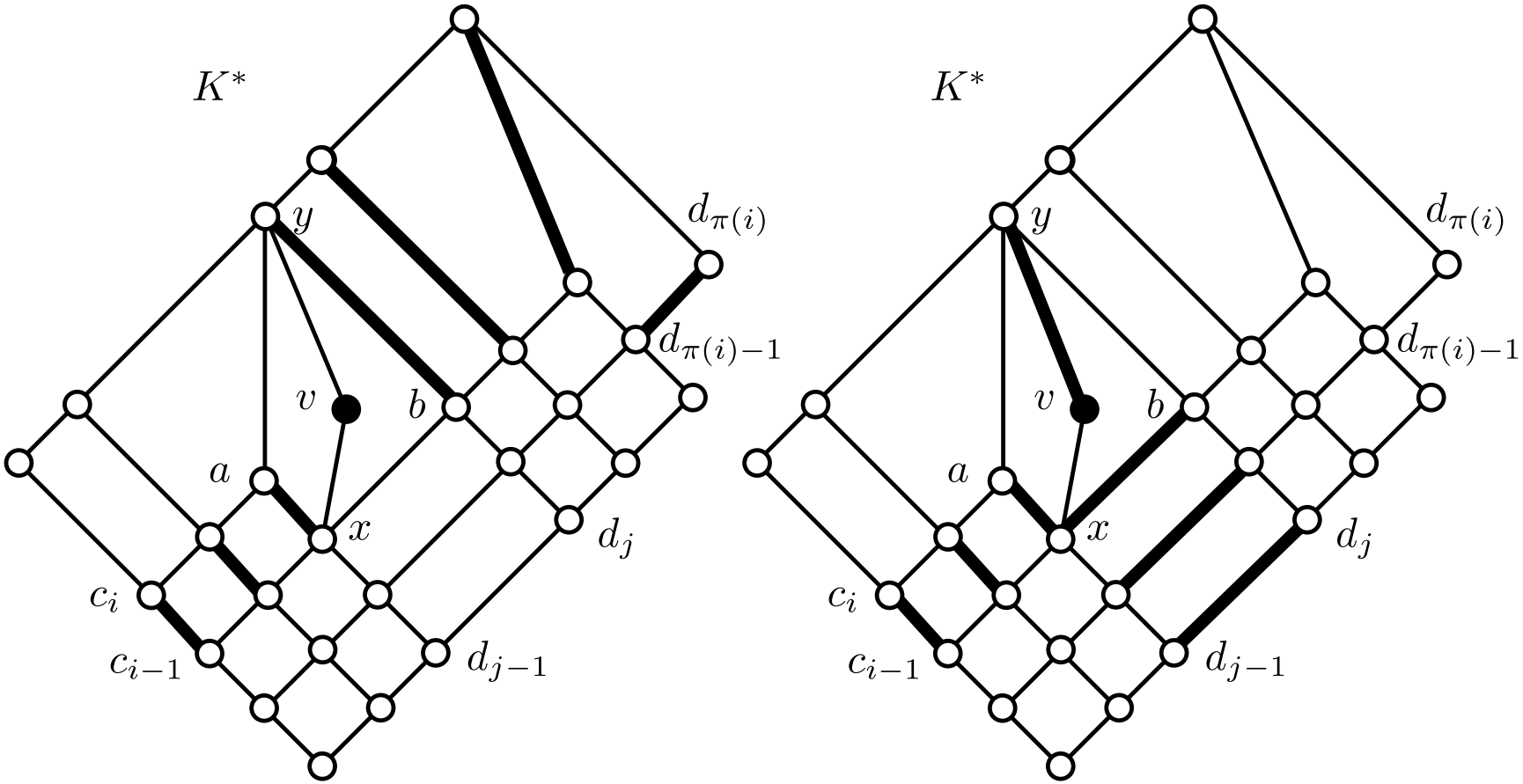
(Something unusual:) Insert a new element v into K; we get K^*. Note that v is not in L and K^* is **not a sublattice** of L, not even a join-subsemilattice of L, in general.



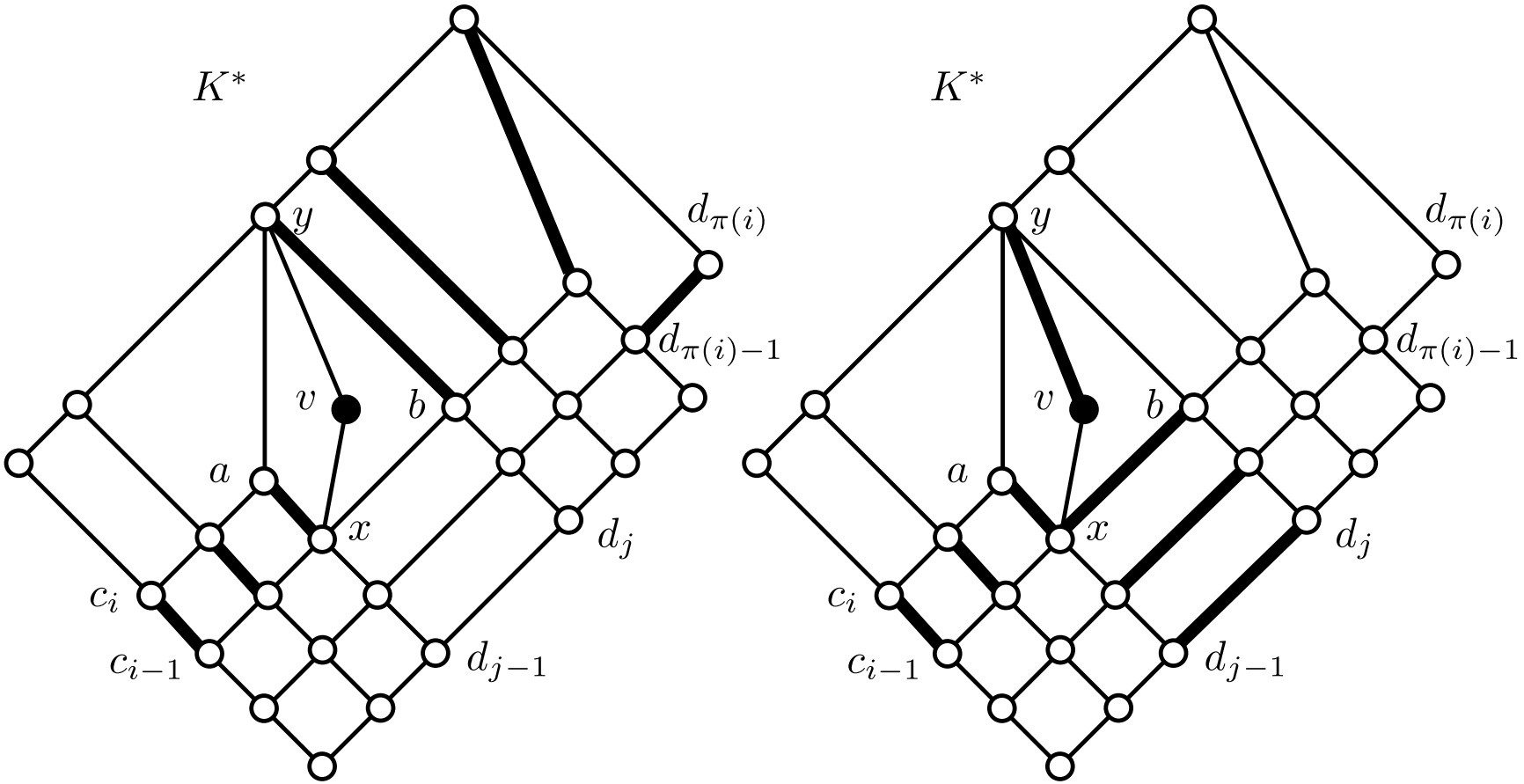
While the trajectories of K never ramify, the new element permits **exactly one** ramification (at v).



The old trajectory (on the left) pays no attention to v. It **keeps going straight** to the northeast for a while, then it may turn to the southeast, and arrives at the right (eastern) border at $[d_{\pi(i)-1}, d_{\pi(i)}]$.



The new trajectory (on the right) **turns** to the southeast **much sooner**; namely, already at v. Since it continues in K, it cannot turn to the northeast later. So, from v to the right boundary, it goes to the southeast, and finally stops at $[d_{j-1}, d_j]$



Since the new trajectory turns to the southeast sooner than the old one, it reaches the right boundary **lower** than the old one. Hence $j < \pi(i)$, contradicting the indirect assumption. Q.e.d.

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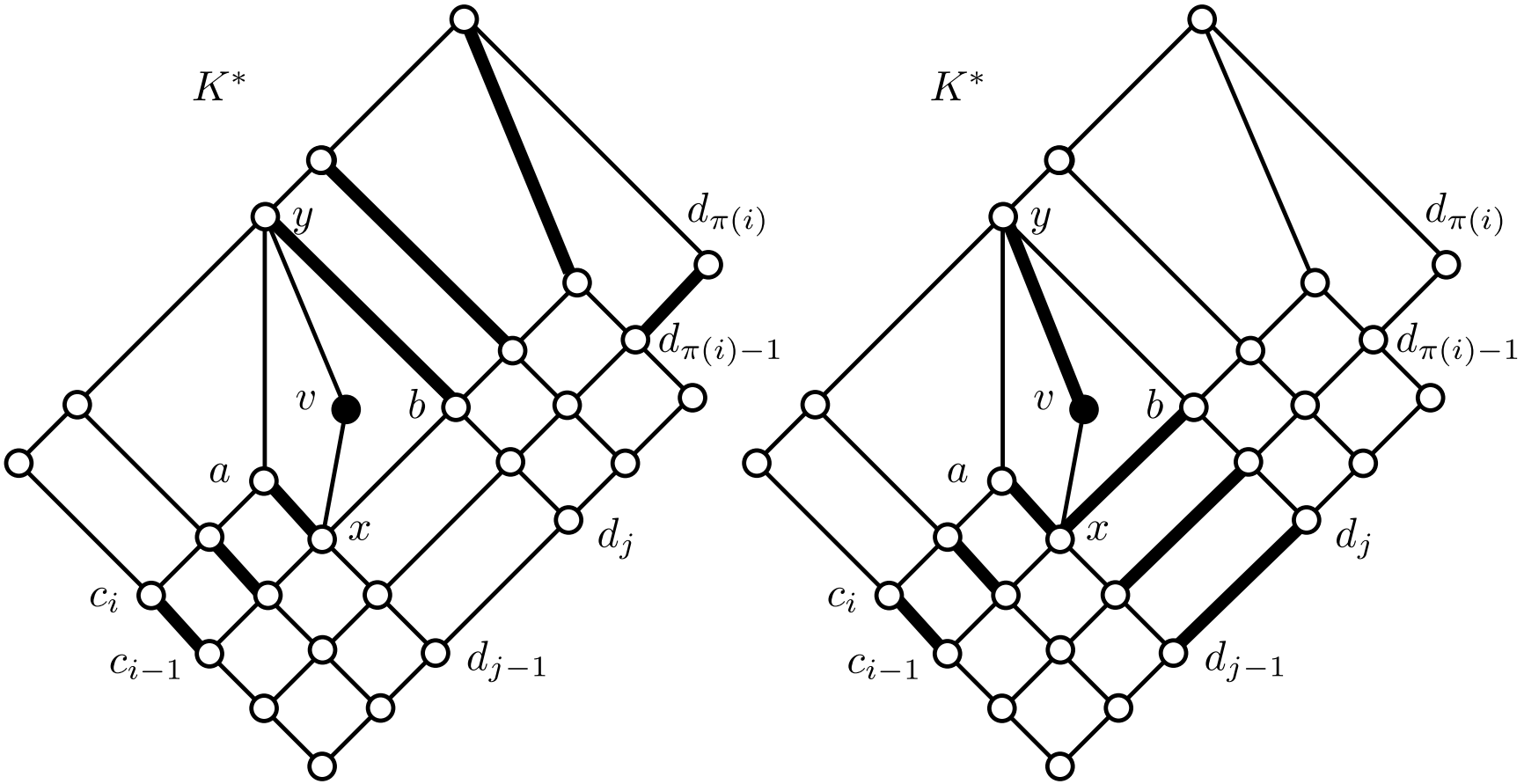

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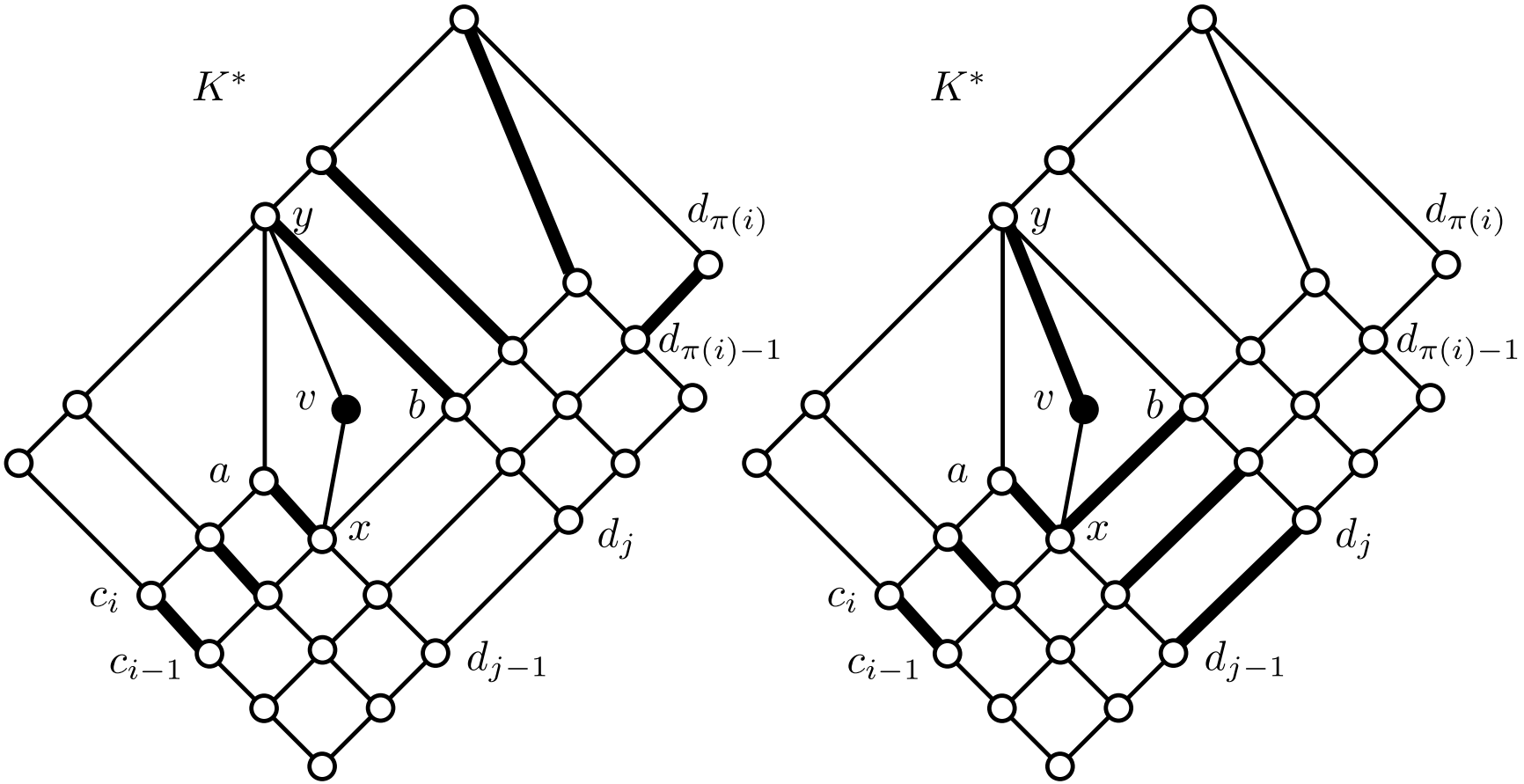

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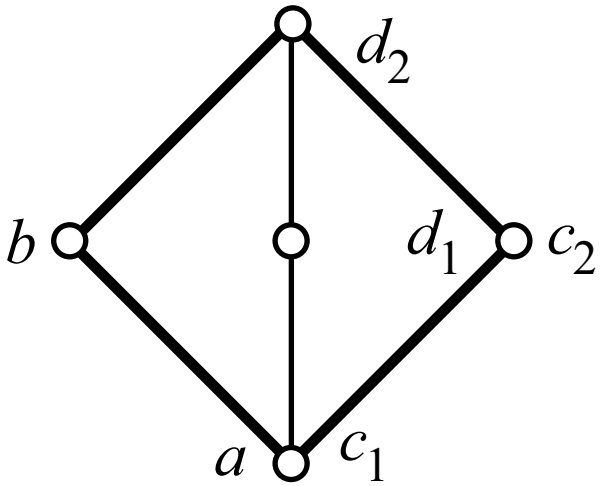
Remak For a single prime interval $[c_{i-1}, c_i] = [a, b]$, there is no uniqueness!



$[a, b] \not\searrow [c_j, d_j]$, for $j = 1, 2$.

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