The visual structure of planar semimodular lattices*

Gábor Czédli and E. Tamás Schmidt

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Cz+Ozsvárt+Udvari counted these lattices, and submitted this to EJC under the title "How many ways can two composition series intersect?".

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G. Grätzer, F. Wehrung, with several contributors: Lattice Theory: Special Topics and Applications. (in progress).

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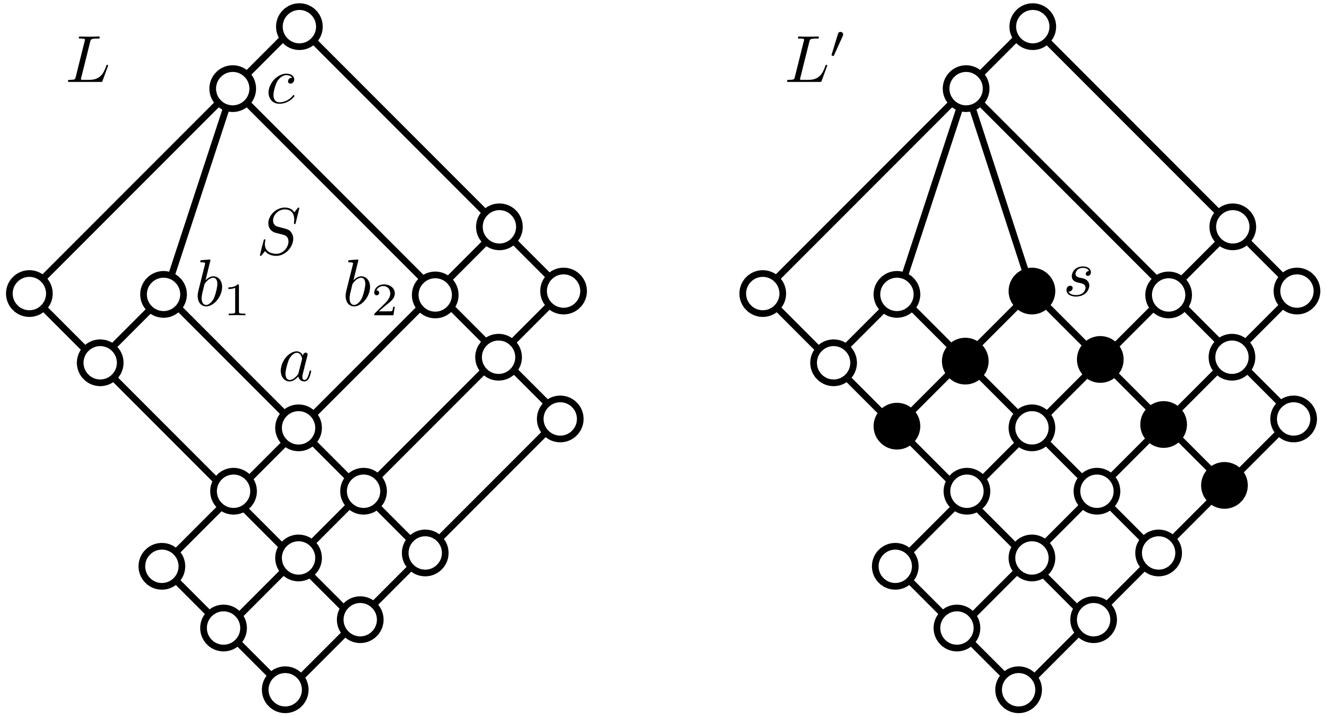
Now we return to the topic of the present talk: the visual structure.

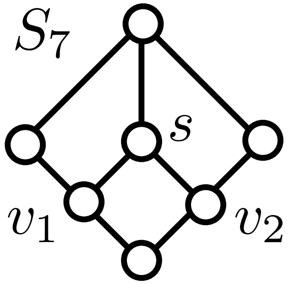
We want to derive our lattices from simpler ones. The main step is called *adding a fork to* L to obtain a new lattice L':



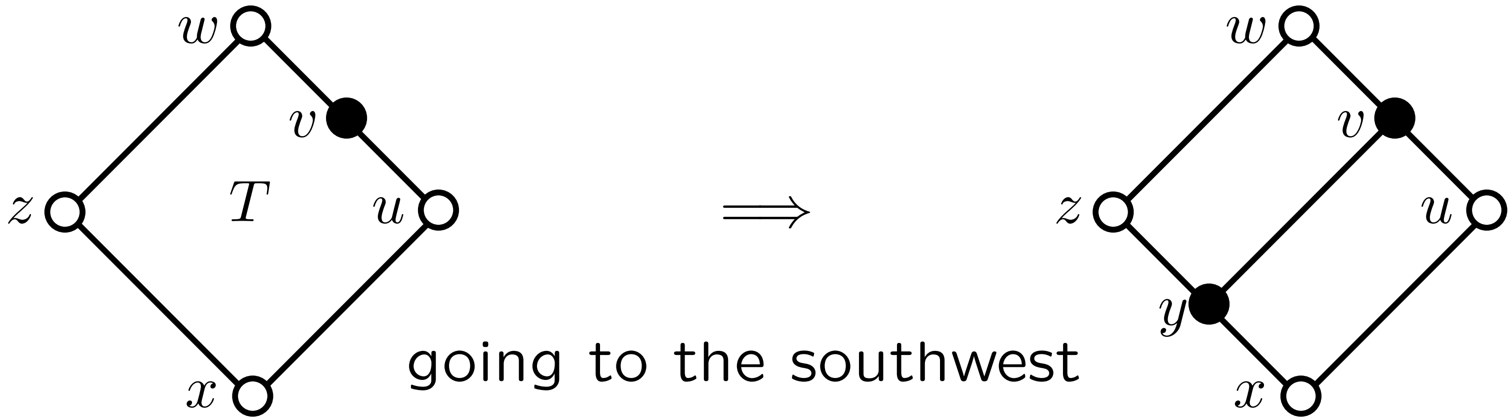
More exactly, we start from a covering square (= 4-cell) S.

Adding a fork, the details

We change S into an S_7:



Starting from s, we proceed to the southeast and the southwest to keep the "4-cell" property as long as we reach the boundary:



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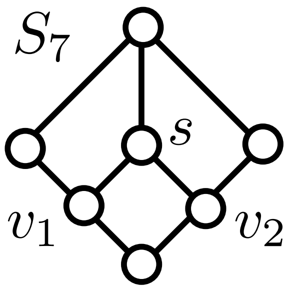

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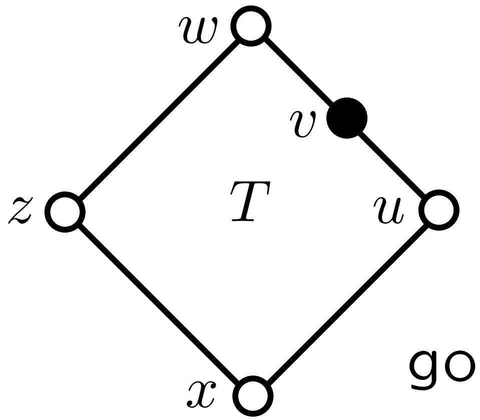

$\Longrightarrow$

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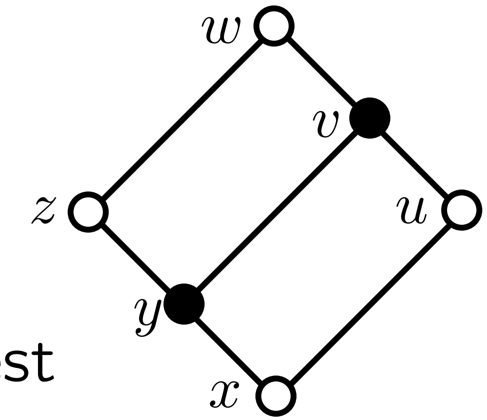
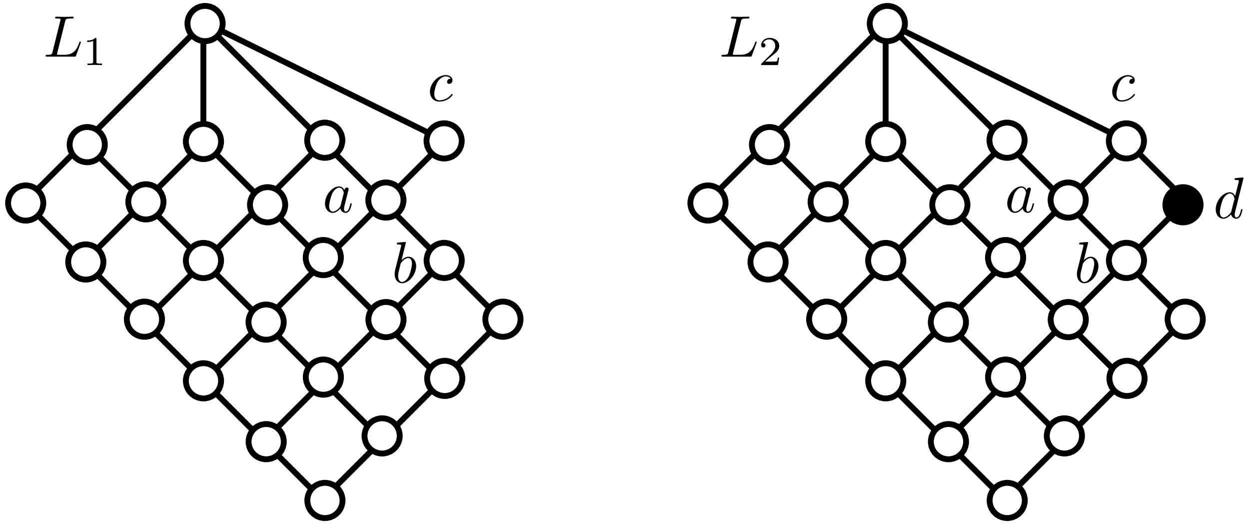


If L is a slim semimodular lattice, then so is L'.

Removing a strong corner d of L_2; we obtain L_1:



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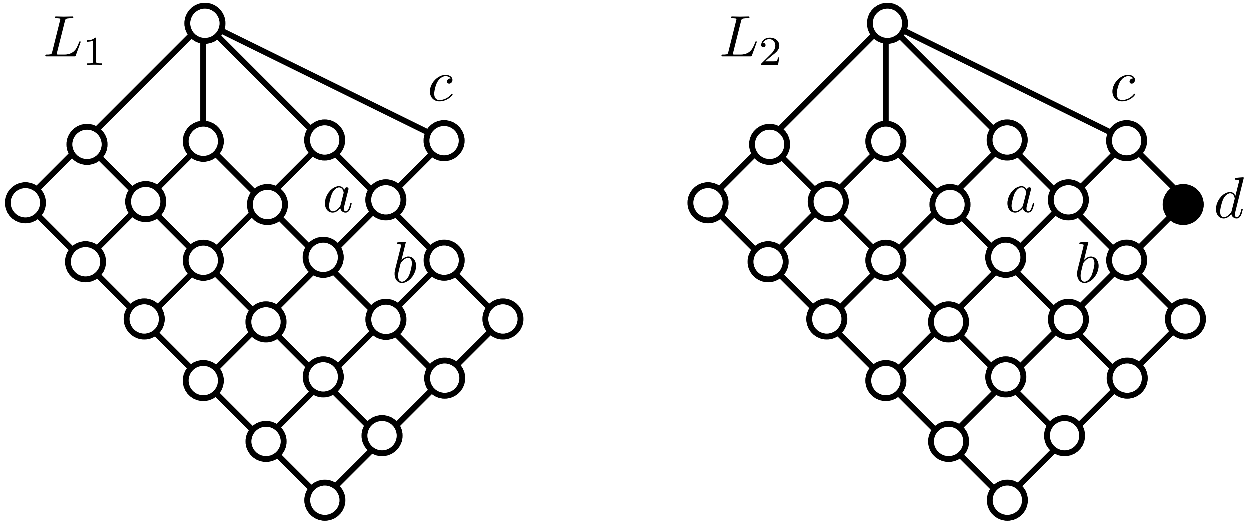

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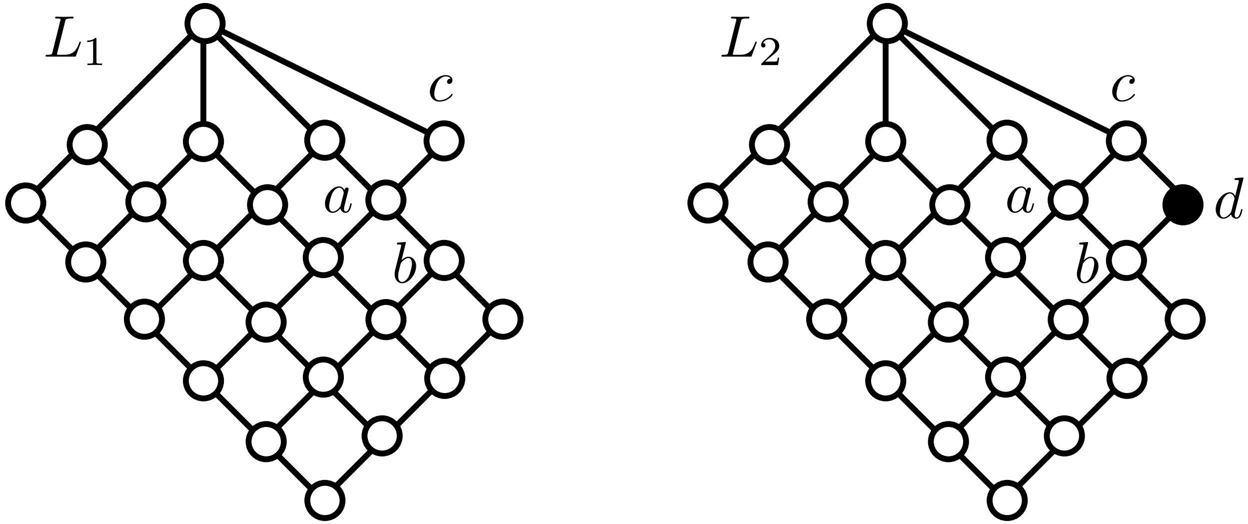

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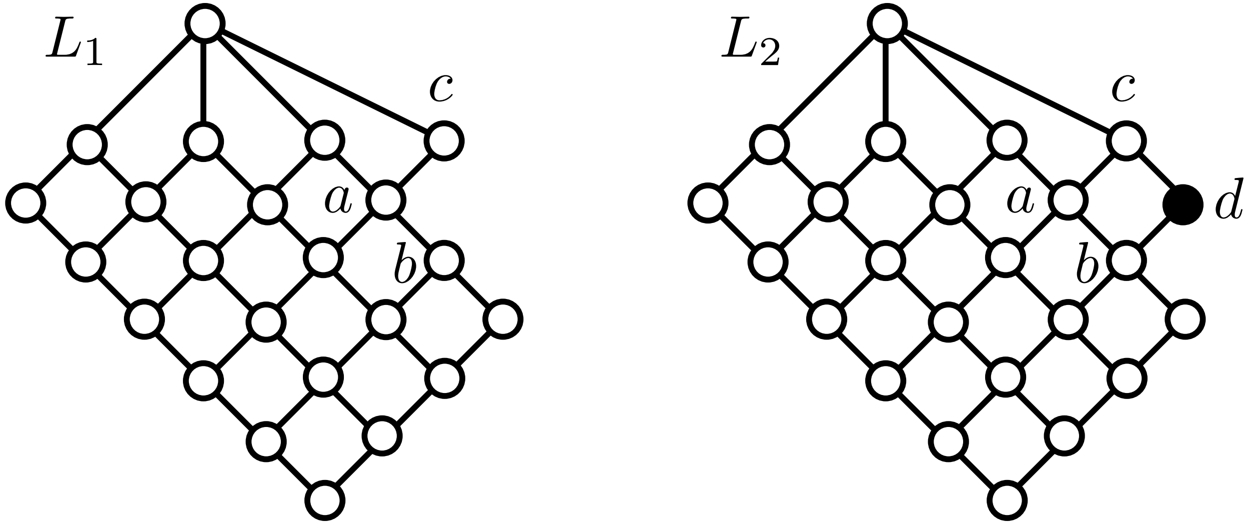

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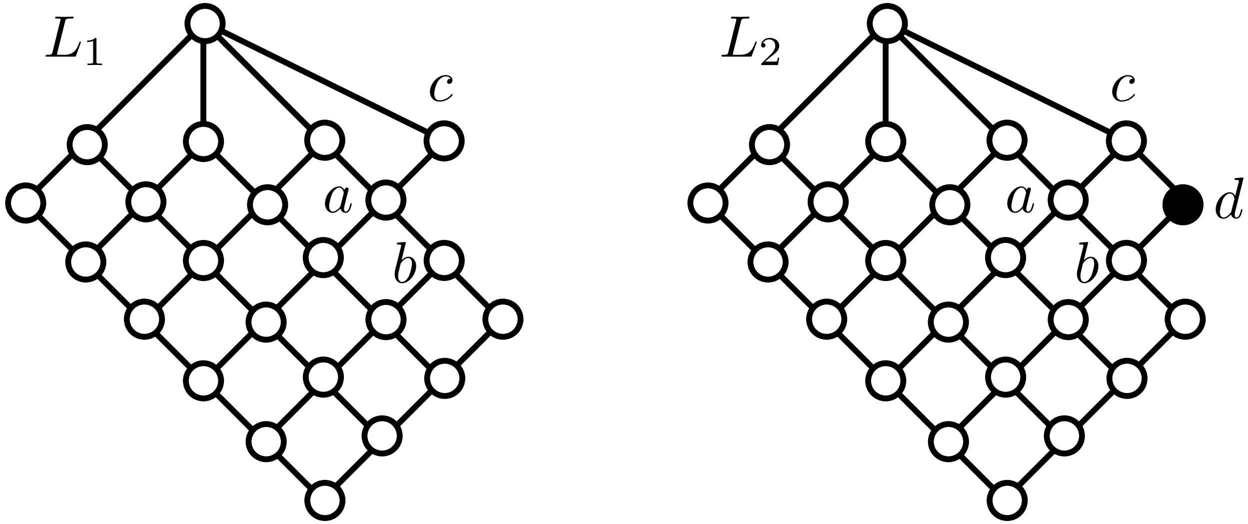

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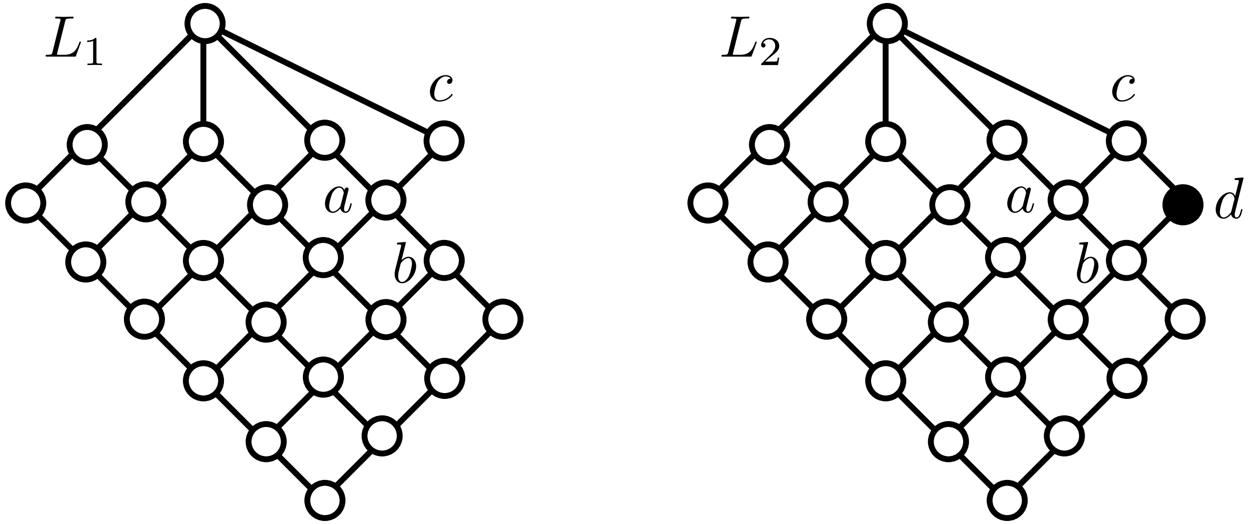

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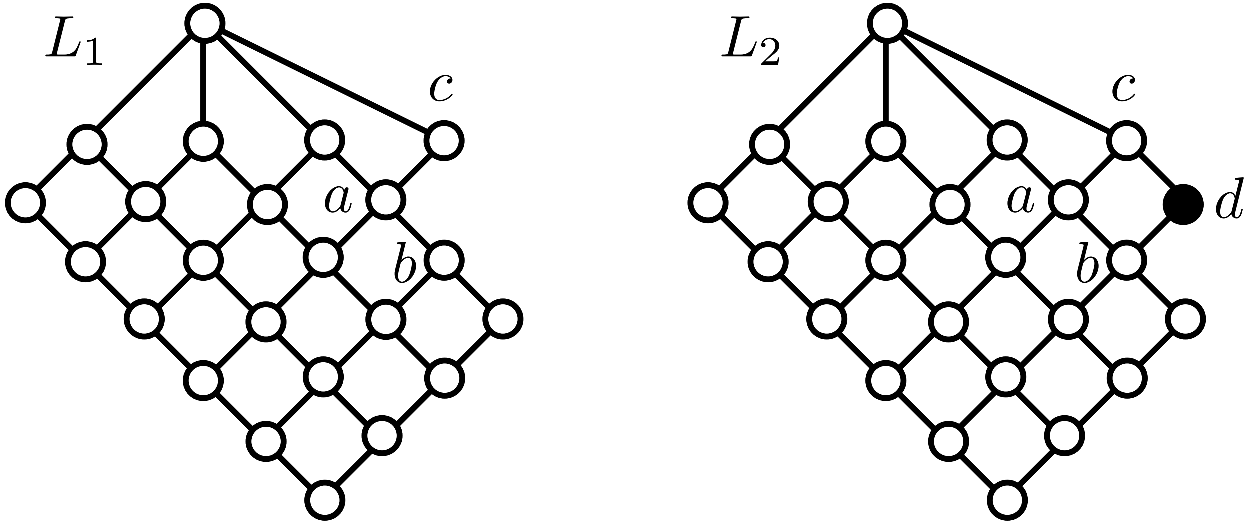

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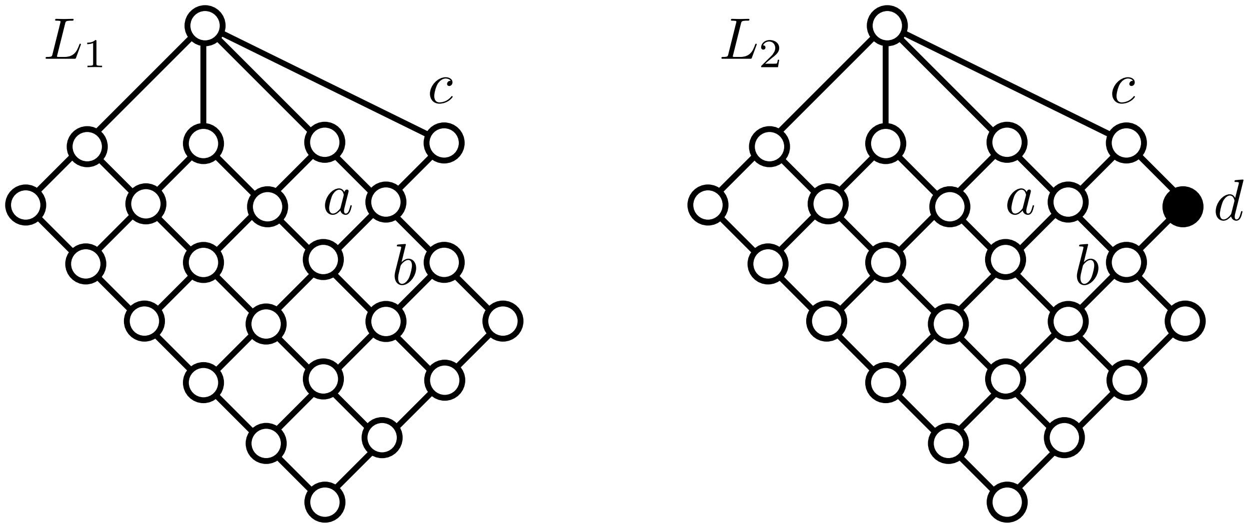

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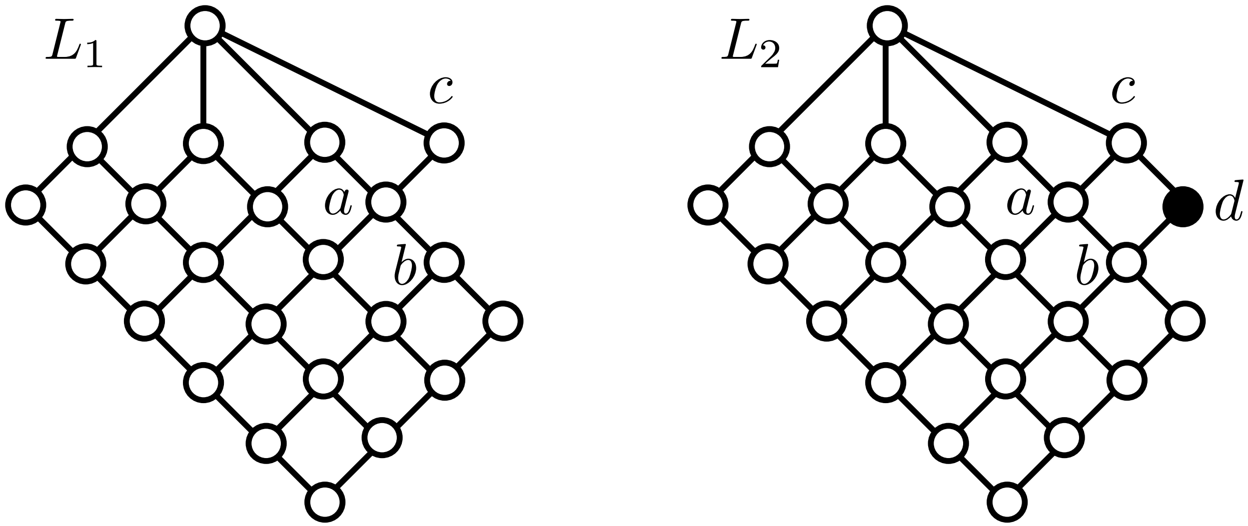

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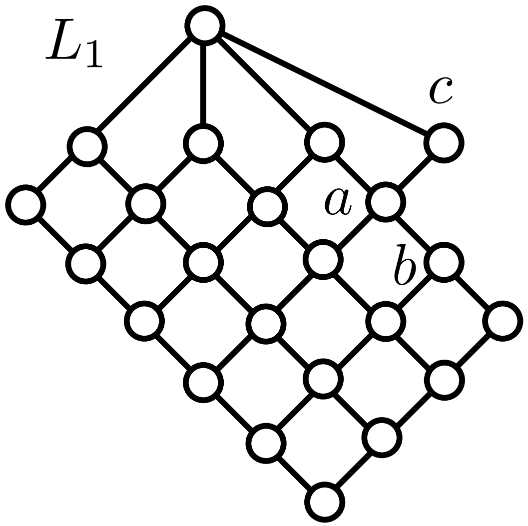

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Next, we formulate a pair of “twin theorems”.

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Again, we have already mentioned that the class of slim semimodular lattices is closed with respect to both operations.

We conclude the talk with some of the lemmas needed in the proof.

Lemma 1 (*mainly Grätzer and Knapp*) *For every finite lattice* L*, the following five conditions are equivalent:*

(i) *L is a slim semimodular lattice;*

(ii) *L is a slim semimodular* 4*-cell lattice;*

(iii) *L is a planar semimodular lattice without cover-preserving* M_3*-sublattices;*

(iv) *L is a planar semimodular lattice in which* 4*-cells and covering squares are the same;*

(v) *L is a* 4*-cell lattice in which no two distinct* 4*-cells have the same bottom.*

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Lemma 4 *If L is a planar lattice, a and b belong to the same boundary chain of L and $a \prec b$, then either a is meet-irreducible, or b is join-irreducible.*

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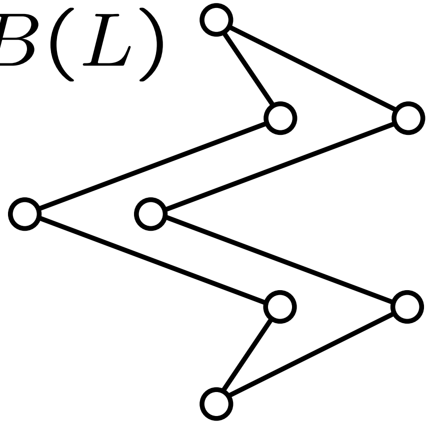

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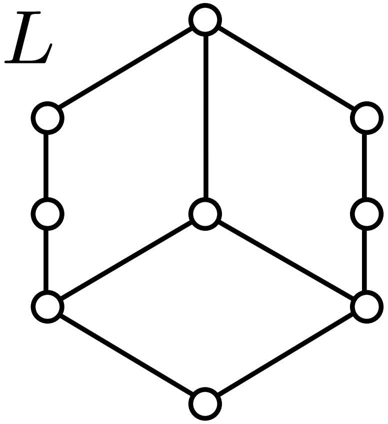

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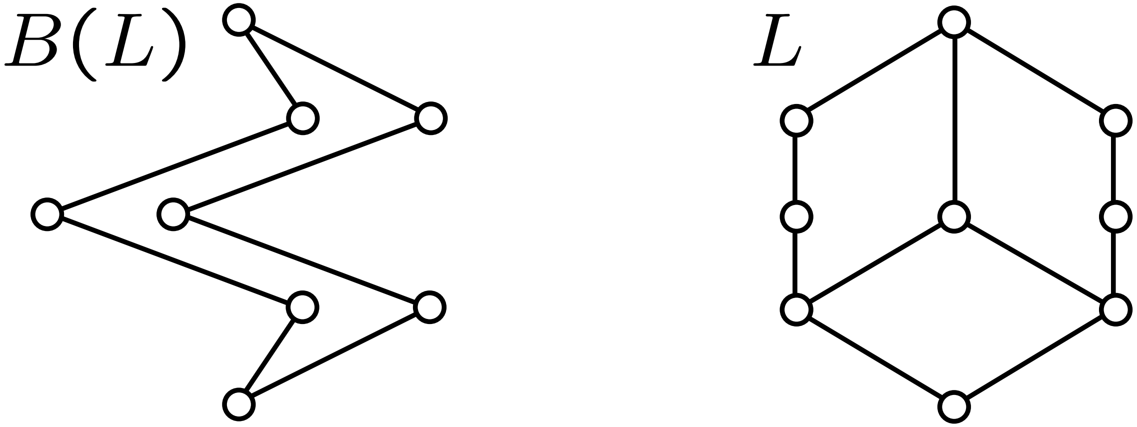

Lemma 7 *The contour of a slim* *semimodular* *lattice is arbitrary.*

(In fact, instead of semimodularity, it suffices to assume that L satisfies the Jordan-Hölder chain condition.)

Our **papers** are available at:

http://www.math.u-szeged.hu/~czedli/

and

http://www.math.bme.hu/~schmidt/

This file will be available at

http://www.math.u-szeged.hu/~czedli/

from the next week.

Thank you for your attention!