

THE VISUAL STRUCTURE OF PLANAR SEMIMODULAR LATTICES

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A finite lattice L is called *slim* if no three join-irreducible elements of L form an antichain. Slim lattices are *planar*. So, they are relatively easy objects to understand. A lattice L is called (upper) *semimodular*, if $b \vee c$ covers or equals $a \vee c$ for all $a, b, c \in L$ such that b covers a .

Semimodular lattices have recently proved to be useful in strengthening a classical group theoretical result, namely, the Jordan-Hölder theorem. The recent story started with G. Grätzer and J. B. Nation [7], who proved that given two composition series of a group, there is a matching between their factors such that the corresponding factors are isomorphic for a very specific reason: they are related by the composite of a down-perspectivity with an up-perspectivity. In [2], this matching is shown to be unique. The main role in [2] is played by slim semimodular lattices (introduced in G. Grätzer and E. Knapp [4]); this is due to the fact that any two finite maximal chains of a semimodular lattice generate a join-subsemilattice that is a slim semimodular lattice. As it has been pointed out by Grätzer and Knapp [4], planar semimodular lattices can easily be obtained from slim ones. This way slim semimodular lattices play an important role in a series of papers [1], [8] and Grätzer and Knapp [4]–[6] on the Congruence Lattice Representation problem.

These developments motivate a deeper study of slim semimodular lattices. Our main results, the twin theorems below, are constructive visual structure theorems of these lattices. The relevant definitions will be explained by figures. While it seems to be difficult to provide many examples of small (and, preferably, planar) semimodular lattices when one is getting acquainted with Lattice Theory, this should not be a problem using the twins.

Theorem 1 ([3]). *Each slim semimodular lattice L can be obtained from a chain by using the operations (i) adding a fork and (ii) adding a strong corner, in a finite*

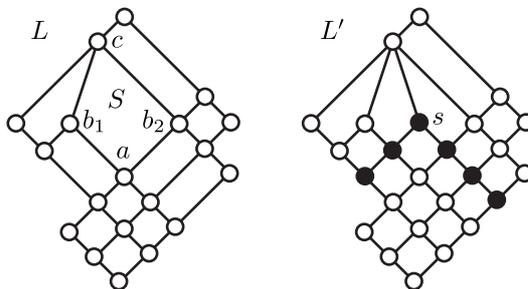


FIGURE 1. Adding a fork to L

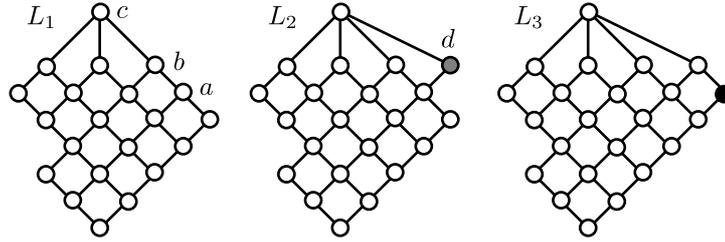


FIGURE 2. Adding a weak corner to L_1 , and a strong corner to L_2

number of steps. Furthermore (and almost evidently), the class of slim semimodular lattices is closed with respect to both operations.

Opposed to the above theorem, the next one is about *deleting* strong corners.

Theorem 2 ([3]). *Each slim semimodular lattice L can be obtained from the direct product of two nontrivial finite chains such that (i') first we add finitely many forks one by one, and then (ii') we remove strong corners, one by one. Again, the class of slim semimodular lattices is closed with respect to both operations, (i') and (ii').*

REFERENCES

- [1] G. Czédli : Representing homomorphisms of distributive lattices as restrictions of congruences of rectangular lattices, Algebra Universalis, submitted.
- [2] G. Czédli and E.T. Schmidt: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices, Algebra Universalis, to appear.
- [3] G. Czédli and E.T. Schmidt: Slim semimodular lattices. I. A visual approach, Order, submitted.
- [4] G. Grätzer and E. Knapp, Notes on planar semimodular lattices. I. Construction. Acta Sci. Math. (Szeged), **73** (2007), 445–462.
- [5] G. Grätzer and E. Knapp, Notes on planar semimodular lattices. III. Rectangular lattices. Acta Sci. Math. (Szeged) **75** (2009), 29–48.
- [6] G. Grätzer and E. Knapp, Notes on planar semimodular lattices. IV. The size of a minimal congruence lattice representation with rectangular lattices, Acta Sci. Math. (Szeged), **76** (2010), 3–26.
- [7] G. Grätzer and J. B. Nation, A new look at the Jordan-Hölder theorem for semimodular lattices. Algebra Universalis, published online , DOI 10.1007/s00012-011-0104-9
- [8] Schmidt, E.T.: Congruence lattices and cover preserving embeddings of finite length semimodular lattices. Acta Sci. Math.

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