Coordinatization of join-distributive lattices*

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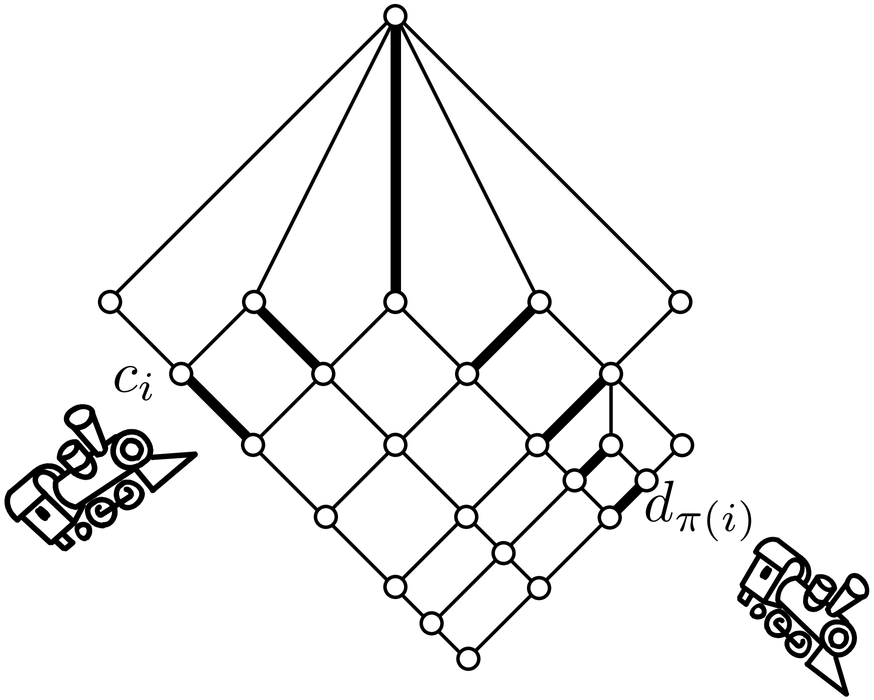
Coordinatization of join-distributive lattices

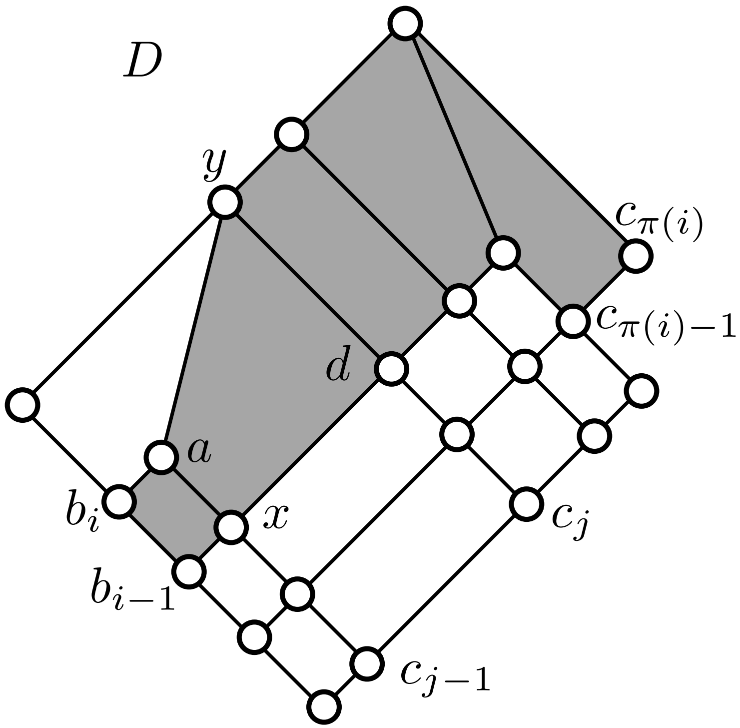
All lattices will be assumed to be **finite**!

Semimodularity: $x \prec y$ implies $x \vee z \prec y \vee z$, for $\forall x, y, z \in L$.

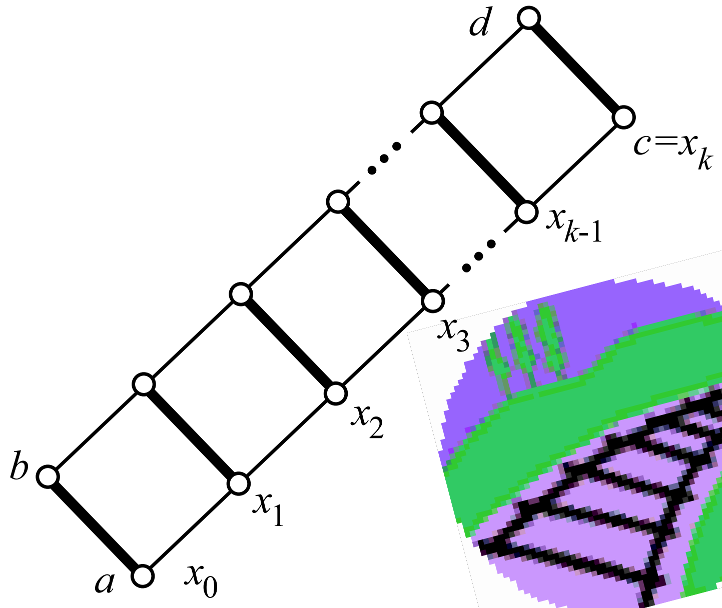
Slimness: $J(L)$ is the union of two chains.

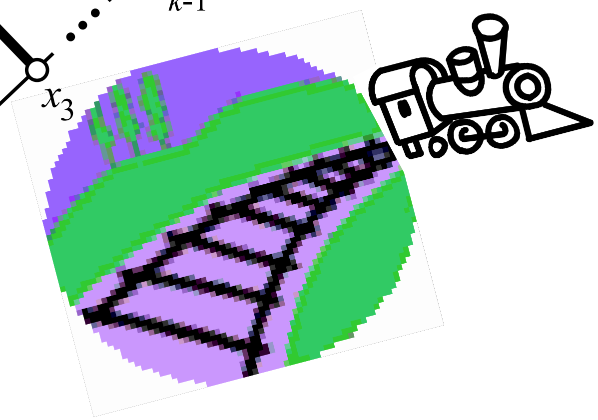
Fore example, two slim sm lattices (they are always planar):





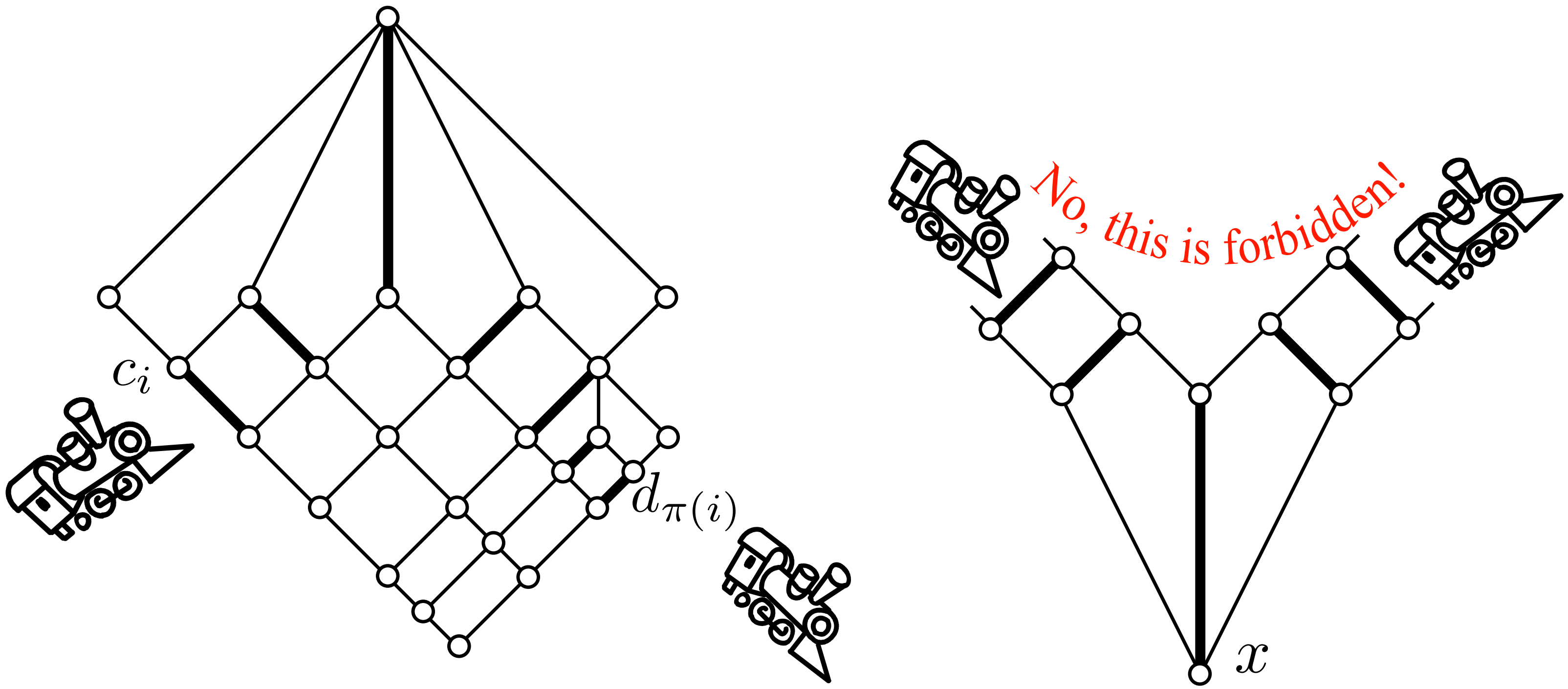
Trajectories of **the diagram of** a slim semimodular lattice: on the set of edges (=covering pairs), the "opposite sides of a covering square" generates an equivalence relation, whose classes are called *trajectories*.



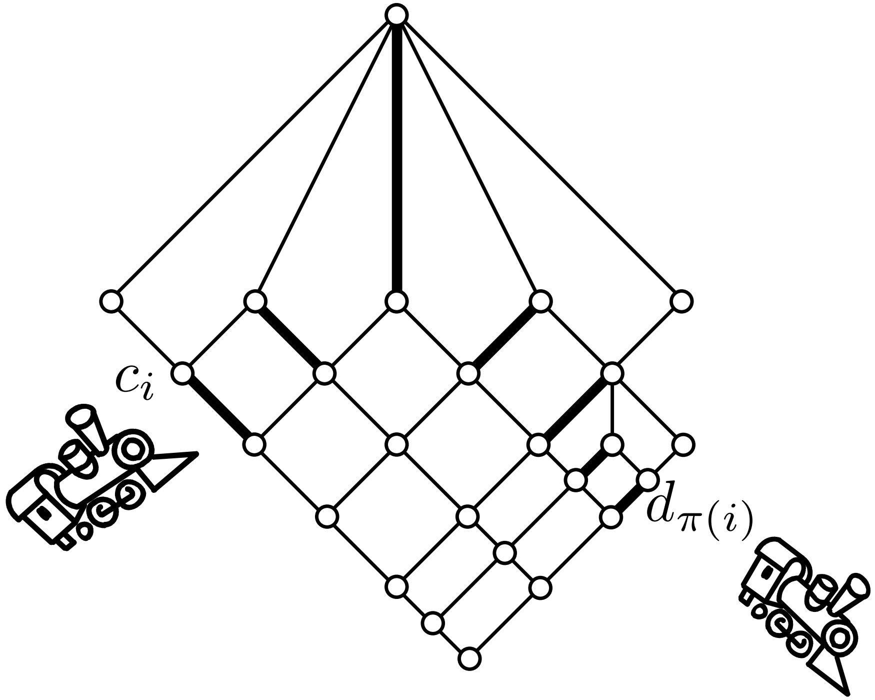


Trajectories were introduced by Czédli and E.T. Schmidt: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices; Algebra Universalis 66 (2011) 69–79.

" *Traffic rules*" for trajectories (slim case):



Trajectories go from left to right, from the left boundary chain to the right one, they do not split, at most one turn and only from northeast to southeast is permitted.

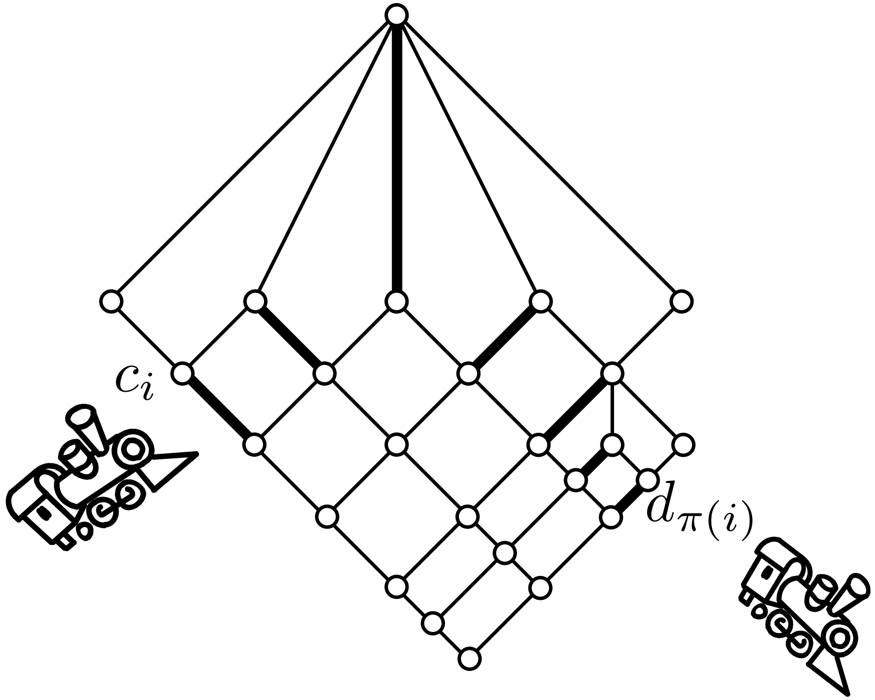


The *Jordan-Hölder permutation* of the **diagram** of a slim sm L: we (Cz–Schmidt, AU 2011) define $\pi \in S_n$ by trajectories, see on the left; n denotes length(L).

The old definitions of π_L: R.P. Stanley (1972, see also H. Abels 1991) are equivalent to ours; see Czédli and Schmidt (2013, Acta Sci. Math, to appear), where we prove that π **determines the diagram** and also the lattice (up to isomorphism)!

The advantage of trajectories: they are quite visual.

If L is slim sm, then, by the "traffic rules",



a maximal chain and a trajectory always have *exactly one common edge*.

Think of roads from north to south; the locomotive crosses each road exactly once.

Definition: **the trajectories of L are beautiful** iff each maximal chain and each trajectory have exactly one common edge. Finite lattices with this property are the lattices we deal with!
(We allow the case where trajectories split.)

It turns out: our lattices = {**join-distributive lattices**}. ≈ the most often discovered mathematical objects!

Meet-semidistributivity law: $x \wedge y = x \wedge z \Rightarrow x \wedge y = x \wedge (y \vee z)$.

We list some equivalent definition of join-distributive lattices.

Definition. A finite lattice L is **join-distributive** if one of the following twelve (equivalent) conditions hold:

- L is semimodular and meet-semidistributive. (Dilworth, 1940)
- L has unique meet-irreducible decompositions.
- For each $x \in L$, the interval $[x, x^*]$ is distributive.
- For each $x \in L$, the interval $[x, x^*]$ is boolean.
- The length of each maximal chain of L equals $|M(L)|$.
- L is semimodular and diamond-free (i.e., no M_3).
- L is semimodular and has no cover-preserving M_3 sublattice.
- L is a cover-preserving join-subsemilattice of a finite distributive lattice.
- $L \cong$ the lattice of open sets of a finite convex geometry.
- L is dually isomorphic to the lattice of closed sets of a finite **convex geometry**.
- $L \cong$ the lattice of feasible sets of a finite **antimatroid**.
- (Adaricheva–Czédli) L is semimodular with beautiful trajectories.

P.H. Edelman (1980): a pair $\langle E, \Phi\rangle$ is a **convex geometry**, if

- E is a finite set, and $\Phi\colon P(E) \to P(E)$ is a closure operator.
- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, $x \neq y$, and $x \in \Phi(A \cup \{y\})$, then $y \notin \Phi(A \cup \{x\})$. (This is the so-called *anti-exchange property*.)
- $\Phi(\emptyset) = \emptyset$.

R. E. Jamison-Waldner (1980): a pair $\langle E, \mathcal{F}\rangle$ is an **antimatroid** if E is a finite set, and $\emptyset \neq \mathcal{F} \subseteq P(E)$, $\mathcal{F}$ is union-closed, $\bigcup \mathcal{F} = E$, and for each nonempty $A \in \mathcal{F}$, $\exists x \in A$ with $A \setminus \{x\} \in \mathcal{F}$.

Complementary concepts; mutually determine each other.

Let L be a join-distributive lattice of length n. We say that $L^* = \langle L; C_1, \ldots, C_k \rangle$ is a a join-distributive lattice (of join-width at most k) with a k-dimensional **coordinate system** $\langle C_1, \ldots, C_k \rangle$ if the C_i are maximal chains such that $J(L) \subseteq C_1 \cup \ldots \cup C_k$.

• The trajectories are beautiful $\Rightarrow$ for each (say, the i-th) edge (=prime interval) of C_1 there exists a unique edge (say, the j-th) of C_t such that these two edges belong to the same trajectory. The rule $i \mapsto j$ defines a permutation $\pi_{1t} \in S_n$.

• The **coordinate structure** of L^* is $\vec{\pi} = \langle \pi_{12}, \ldots, \pi_{1k} \rangle \in S_n^{k-1}$. We denote $\vec{\pi}$ by $\xi(L^*)$. We say that S_n^{k-1} is the **set of k-dimensional coordinate structures**.

Main Theorem (Czédli, 2012) **The map $\xi\colon L^* \mapsto \vec{\pi}$ is a bijection from {join-distributive lattices with k-dimensional coordinate systems} to the set S_n^{k-1} of k-dimensional coordinate structures.**

Main Thm. $\xi\colon L^* \mapsto \vec\pi$ is a bijection.

Remark. The coordinate structure heavily depends on the coordinate system! If L is the 8-element boolean lattice with atoms a, b, c, then the coordinate system $C_1 = \{0, a, a \vee b, 1\}$, $C_2 = \{0, b, a \vee b, 1\}$, $C_3 = \{0, c, b \vee c, 1\}$ leads to $\pi_{12} = (12)$ and $\pi_{13} = (13)$ (two transpositions), while the choice $C'_1 = C_1$, $C'_2 = \{0, b, b \vee c, 1\}$, and $C'_3 = \{0, c, a \vee c, 1\}$ leads to $\pi'_{12} = (132)$ and $\pi'_{13} = (123)$ (two cycles of order 3).

Open problem: Give an elegant description for the pairs $\langle \vec\pi, \vec\sigma \rangle \in S_n^{k-1} \times S_n^{k-1}$ that come from the same lattice with appropriate choices of $\langle C_1, \ldots, C_k \rangle$. Solved only for $k = 2$ (the slim case).

Main Thm. $\xi\colon L^* \mapsto \vec{\pi}$ is a bijection.

What about the coordinates of the *elements* of L?

To answer this question, let $\eta = \xi^{-1}$; we shall describe η.

Main Thm. $\xi\colon L^* \mapsto \vec{\pi}$ is a bijection. $\eta := \xi^{-1}$.

For $\vec{\pi} \in S_n^{k-1}$, we define $\eta(\vec{\pi}) = L^*(\vec{\pi}) = \langle L(\vec{\pi}); C_1(\vec{\pi}), \dots, C_k(\vec{\pi})\rangle$.

It is convenient to define $\pi_{jt}(i) = \pi_{1t}(\pi_{1j}^{-1}(i))$. Note that in the model $\langle L; C_1, \dots, C_k\rangle$, π_{jt} is what the trajectories define between the chains C_j and C_t.

By an **eligible $\vec{\pi}$-tuple** we mean a k-tuple $\vec{x} = \langle x_1, \dots, x_k\rangle \in \{0, 1, \dots, n\}^k$ such that $\pi_{ij}(x_i + 1) \geq x_j + 1$ holds for all $i, j \in \{1, \dots, k\}$ such that $x_i < n$. (Roughly saying: if we enlarge a component of $\vec{x}$ by 1, then its images will be big.)

Main Thm. $\xi\colon L^* \mapsto \vec\pi$ is a bijection. $\eta(\vec\pi) = \xi^{-1}(\vec\pi) = L^*(\vec\pi)$.
$\vec x \in \{0, \ldots, n-1\}^k$ is eligible $\iff$ $\pi_{ij}(x_i + 1) \geq x_j + 1$ if $x_i < n$.

Definition. Let $L(\vec\pi) := \{$eligible $\vec\pi$-tuples$\}$ with the componentwise ordering. We have defined the lattice; the elements are coordinatized by eligible $\vec\pi$-tuples.

For $i \in \{1, \ldots, k\}$, an eligible $\vec\pi$-tuple $\vec x$ is i-**minimal** if for all $\vec y \in L(\pi)$, $x_i = y_i$ implies $\vec x \leq \vec y$. Let $C_i(\vec\pi)$ be the set of all i-minimal eligible $\vec\pi$-tuples.

We have defined $\eta(\vec\pi) = L^*(\vec\pi) = \langle L(\vec\pi); C_1(\vec\pi), \ldots, C_k(\vec\pi)\rangle$.
(One has to prove that this construct works and $\eta = \xi^{-1}$.)

Main Thm. $\xi\colon L^* \mapsto \vec\pi$ is a bijection. $\vec x$ is eligible iff $\pi_{ij}(x_i+1) \geq x_j + 1$. $\xi^{-1}(\vec\pi) = \langle\{\text{eligibles}, 1\text{-minimals}, \ldots, k\text{-minimals}\}\rangle$.

http://www.math.u-szeged.hu/∼czedli/ or arxiv.org/1208.3517; 20 pages. Later, **Kira Adaricheva** pointed out that my Main Theorem is closely related to an old result of P. H. Edelman and R. E. Jamison (1985) on convex geometries.

This connection is analyzed in a joint paper by Adaricheva and Czédli [arxiv.org/1210.3376 or my web site]. In this paper, we show that my Main Theorem and the Edelman-Jamison description can mutually be derived from each other in less than a page.

Although the lattice-theoretical approach is somewhat longer, it makes sense by the following reasons.

1st, it exemplifies how **Lattice Theory can be applied** to other fields of mathematics.

2nd, not only our methods and the motivations are different from that of Edelman and Jamison, the **two results are not exactly the same** even if the latter is translated to lattice theory.

3rd, trajectories led to a **new characterization of join-distributive lattices**.

4th, it is not yet clear which approach can be used to attack the open problem mentioned before. Thank you for your attention!

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