On the height of order ideals*

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Trivially, when $|\mathcal{F}|$ reaches its maximum, 2^m, then $\mathcal{F} = 2^A$ satisfies the averaged Frankl's property. Much less trivially:

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The proof in the above paper needed an answer to the following question: for a given n and a given finite Boolean lattice, what is the maximal height of n-element order ideals?

By a **greedy sequence of order ideals** of a poset P we mean:

$$I_1 \subset I_2 \subset \cdots \subset I_{|P|}$$

such that $|I_n| = n$ and, for each order ideal X,

$$h(X) \leq h(I_{|X|}).$$

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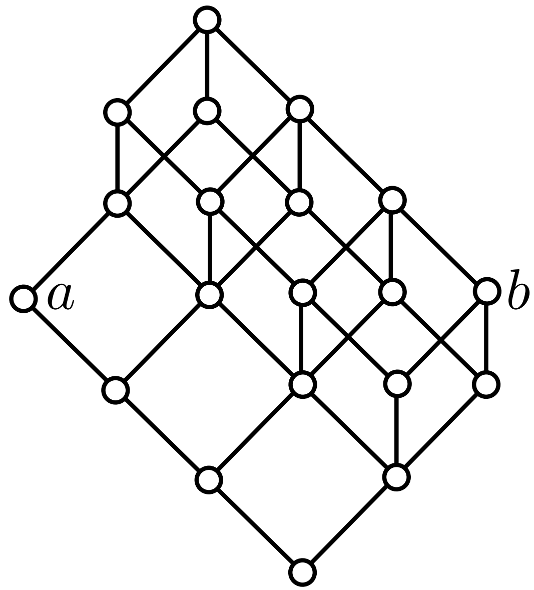
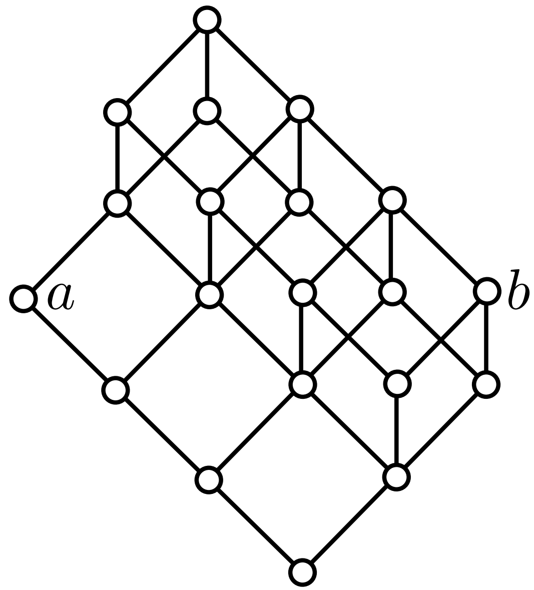
Greedy order ideal $=$ member of some greedy sequence of order ideals.



Here, among four-element order ideals, the height takes its maximum only at $\downarrow a$. Among five-element order ideals, the height takes its maximum only at $\downarrow b$.



Here, among four-element order ideals, the height takes its maximum only at $\downarrow a$. Among five-element order ideals, the height takes its maximum only at $\downarrow b$. Thus, there is no greedy sequence here! Even distributive lattices are difficult!

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• A consequence for digit-sum sequences.

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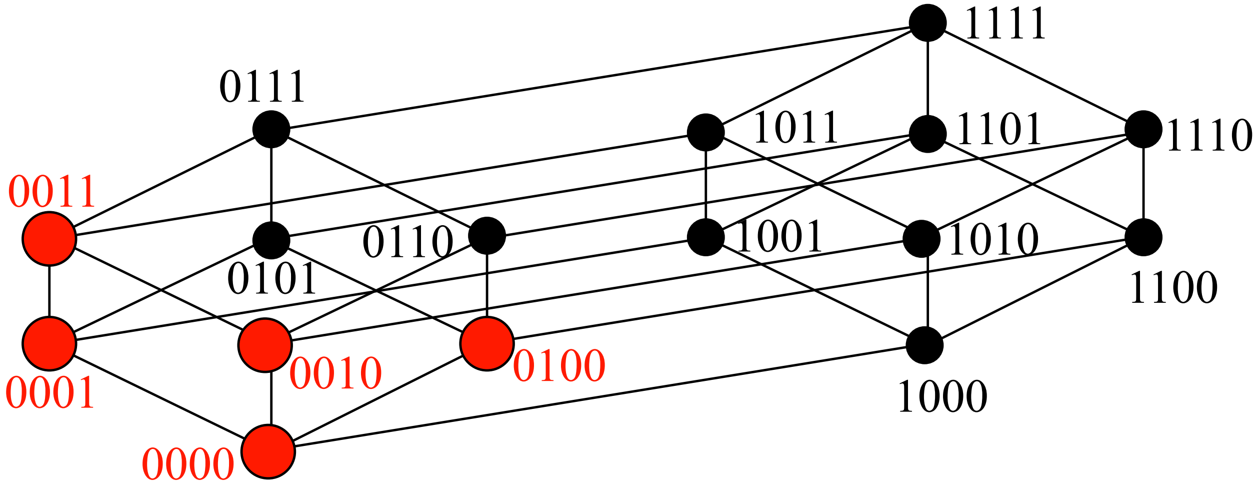
- We extend the results for direct products of chains.

- A consequence for digit-sum sequences.

- Some open problems.



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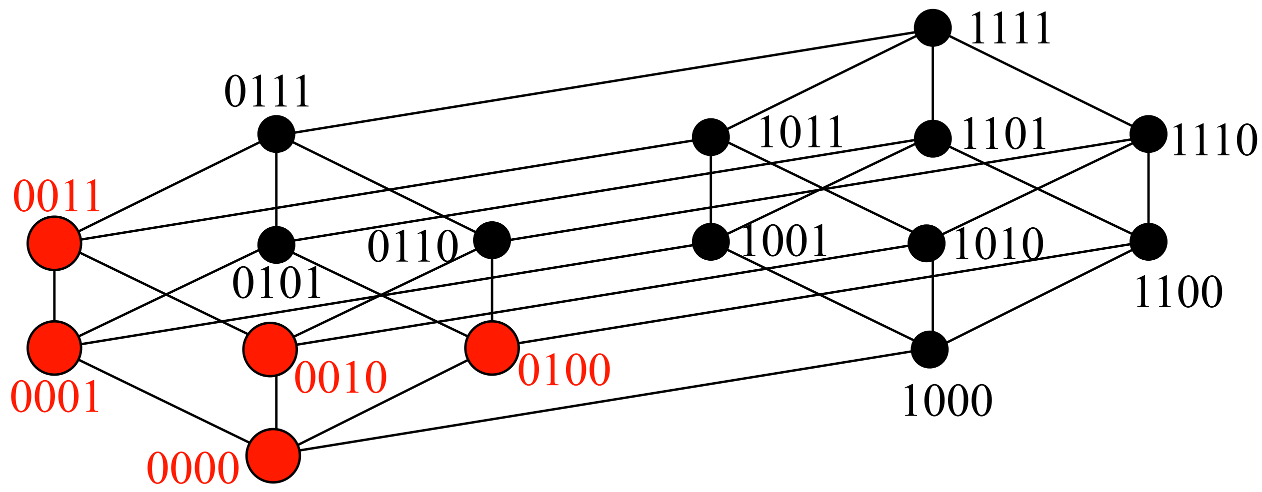
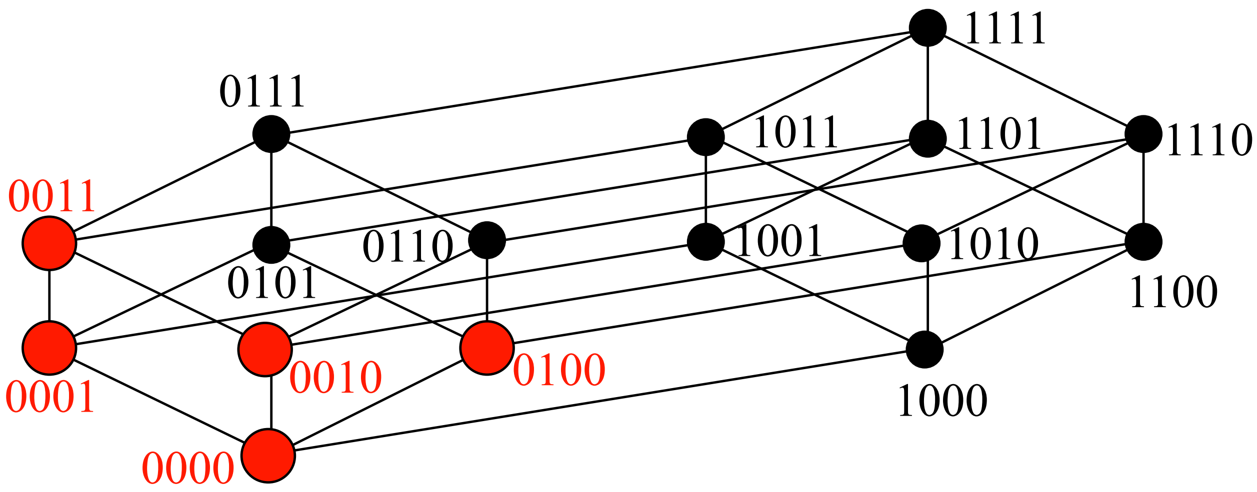


Label a finite Boolean lattice $P = B_m$ (which is a direct product) lexicographically, and define I_n as the n-element lexicographic initial segment. For example, I_5 in B_4 consists of the red bullets.



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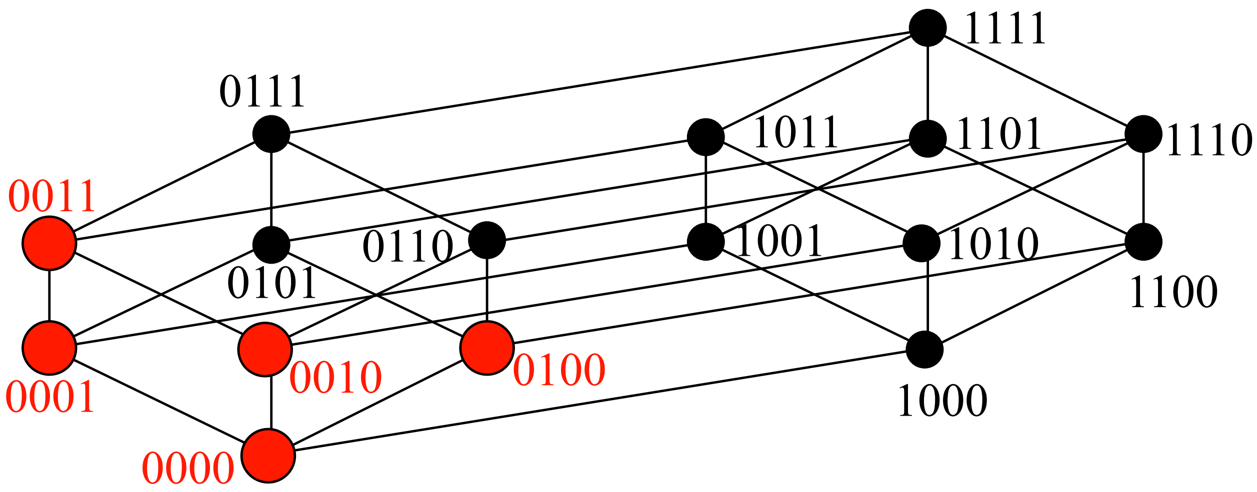

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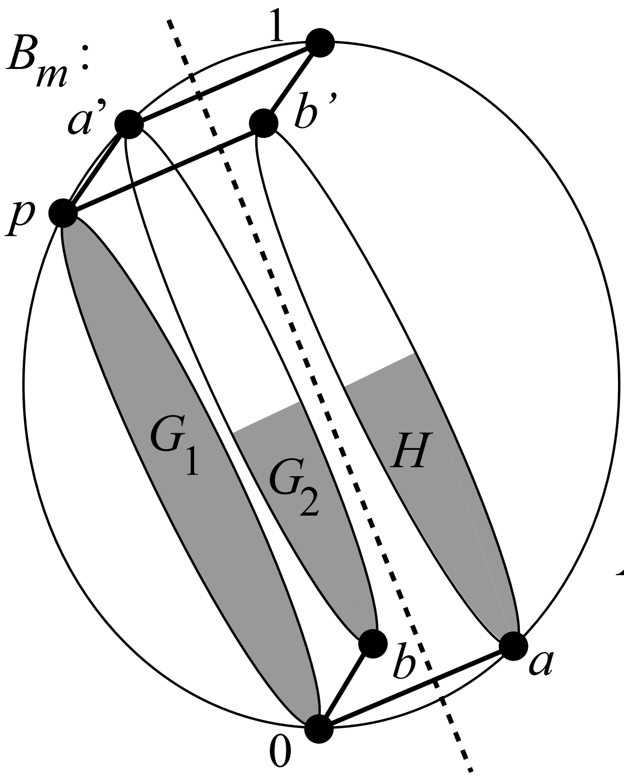
Hence we can assume that there is no such atom. I.e., all atoms $\in X$. Fix an atom a. In the figure, X is the grey subset.



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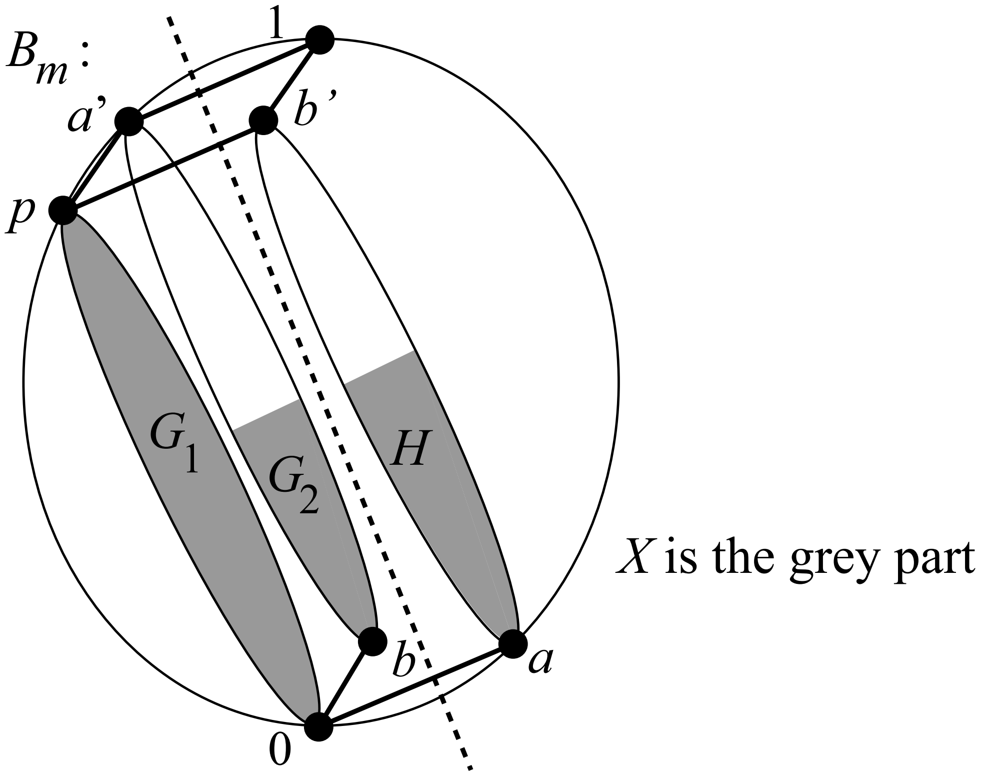
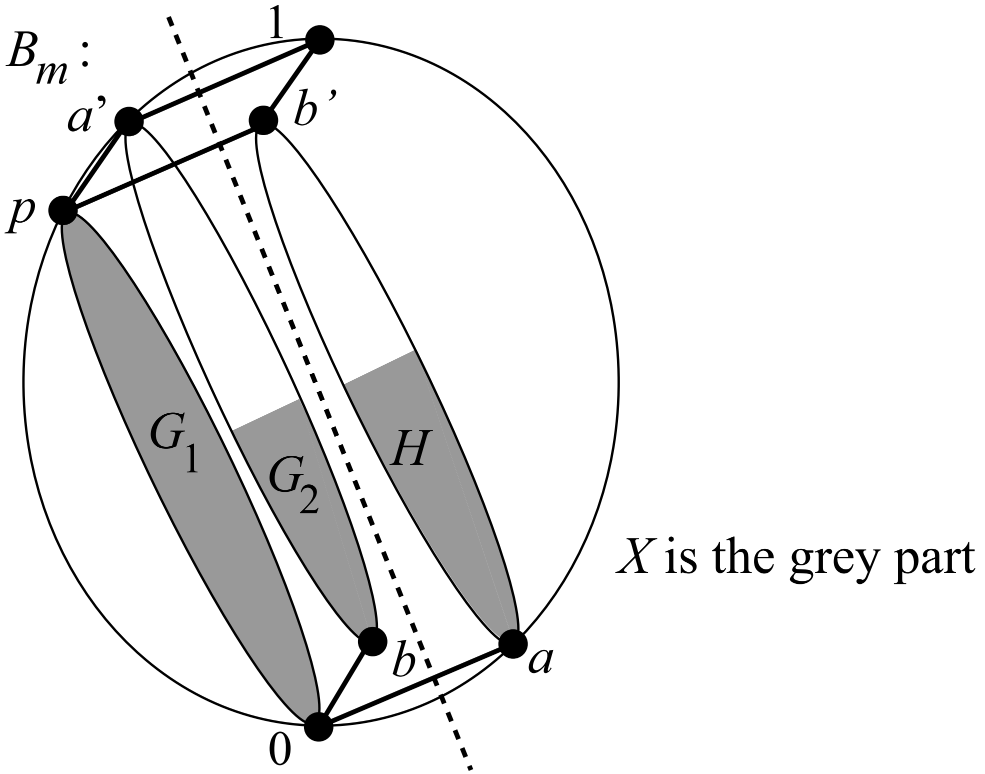


Let G be the „lower half” of X (below the dotted line), that is, $G = X \cap {\downarrow}a'$,



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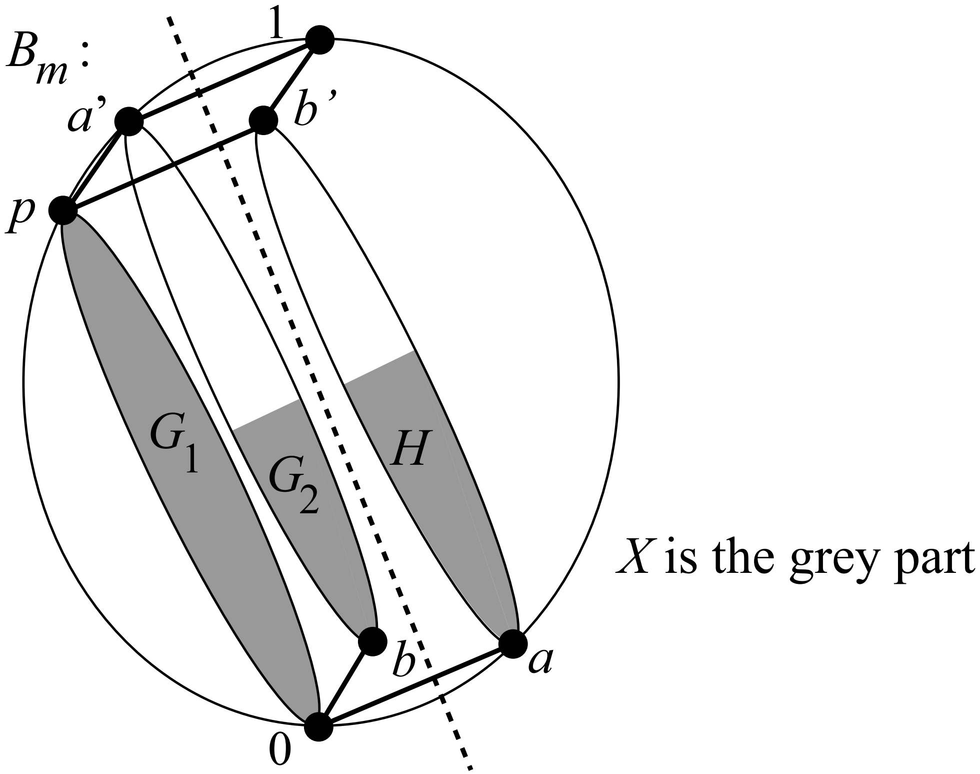

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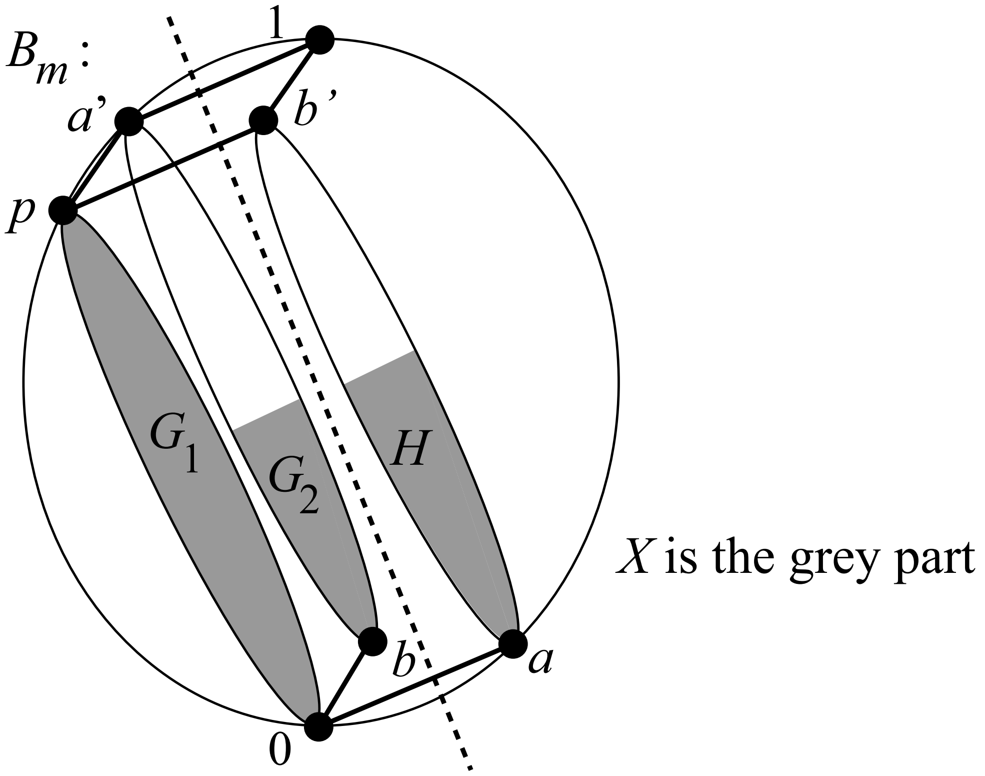

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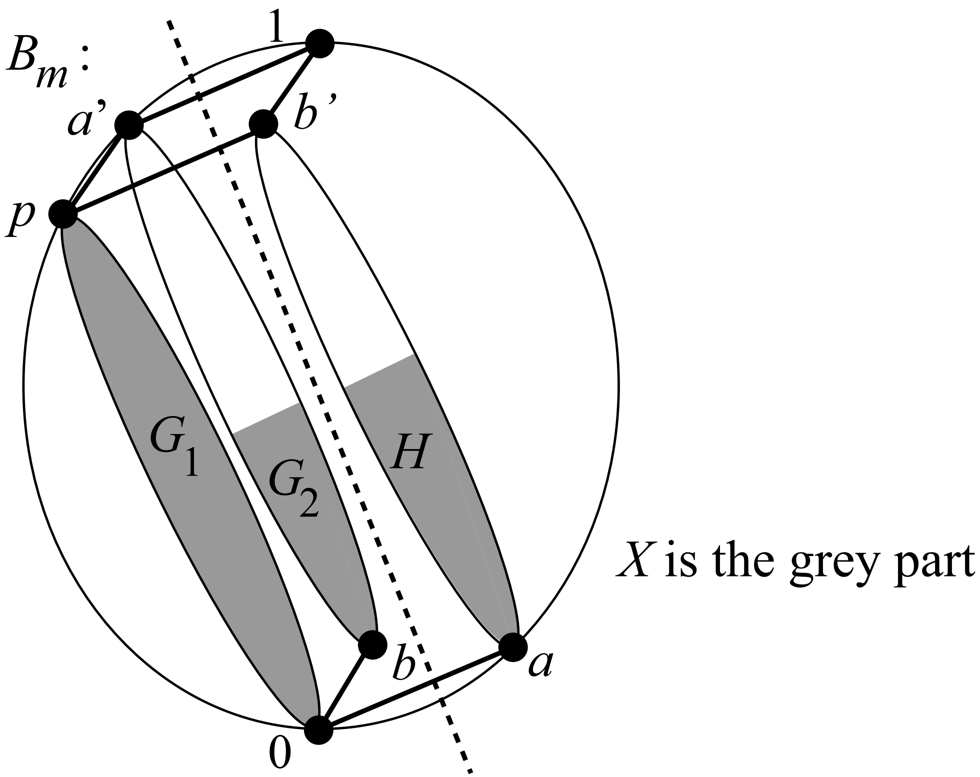
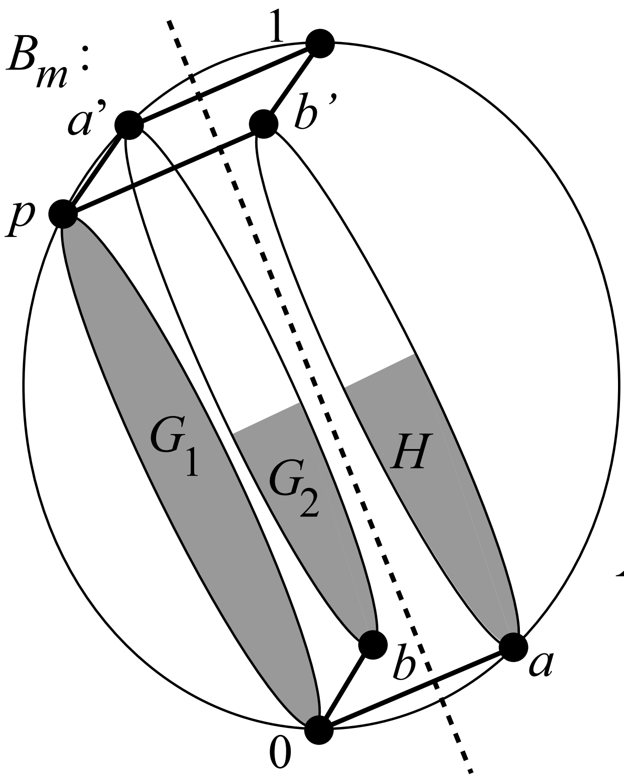


We can change G into greedy in $\downarrow a'$.



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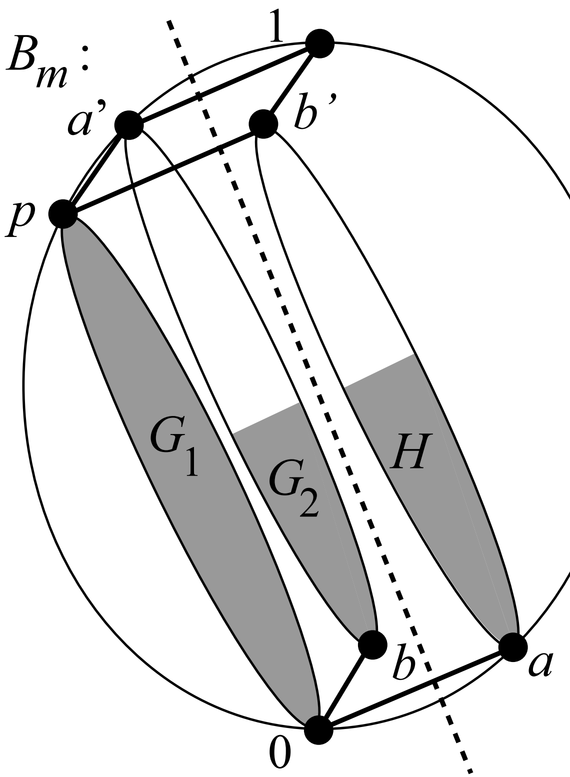

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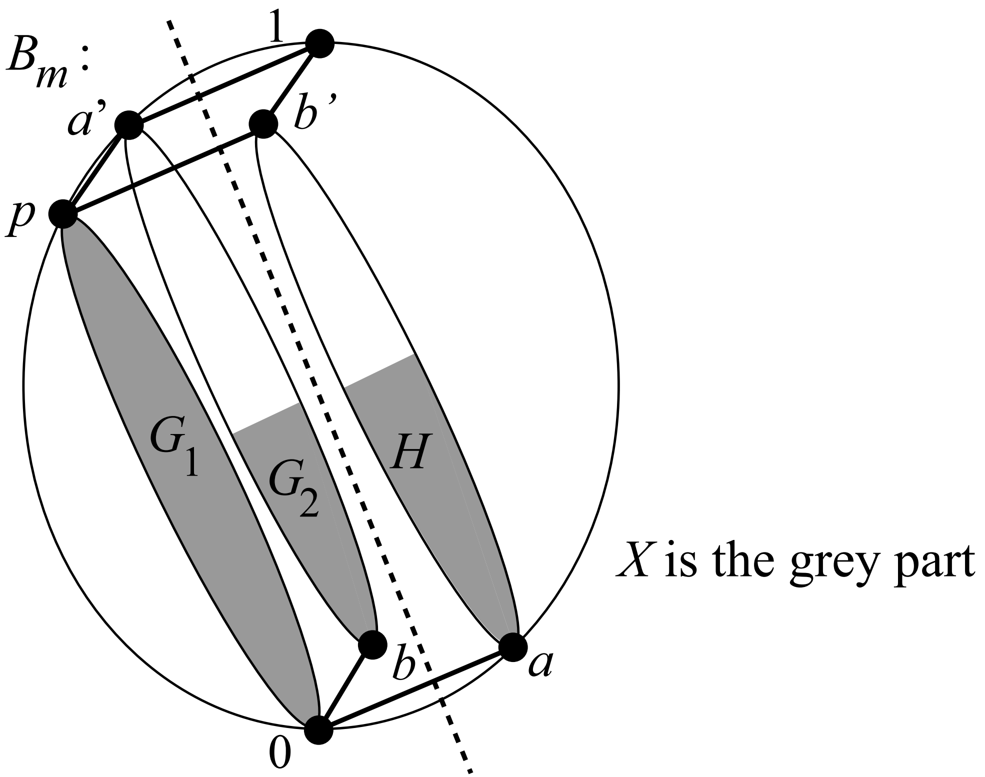

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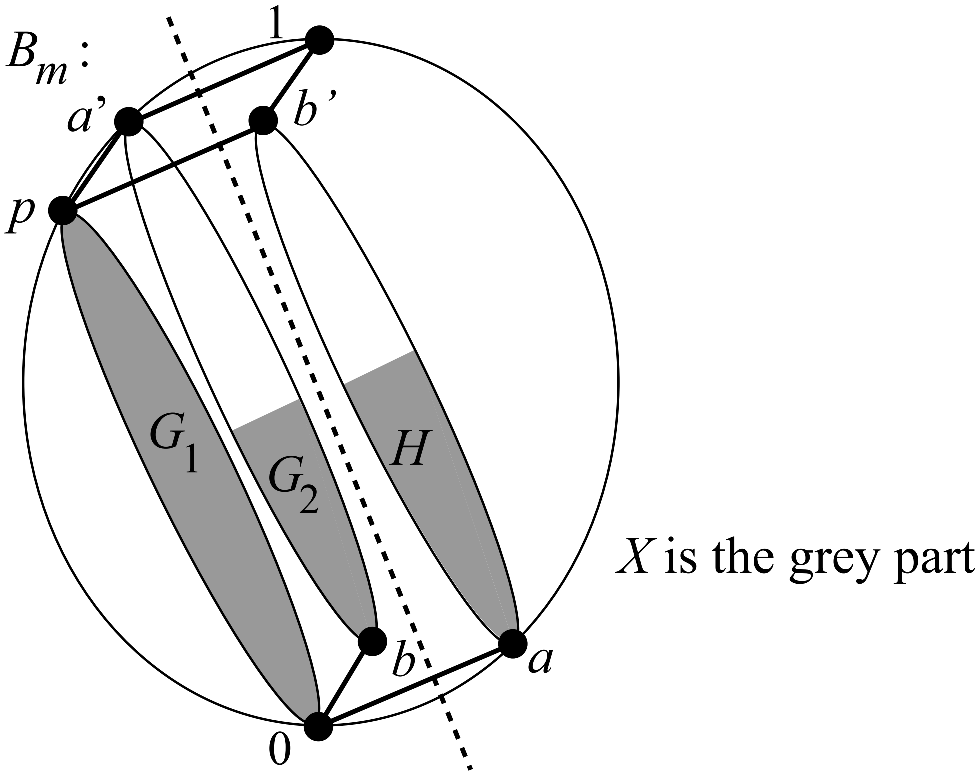

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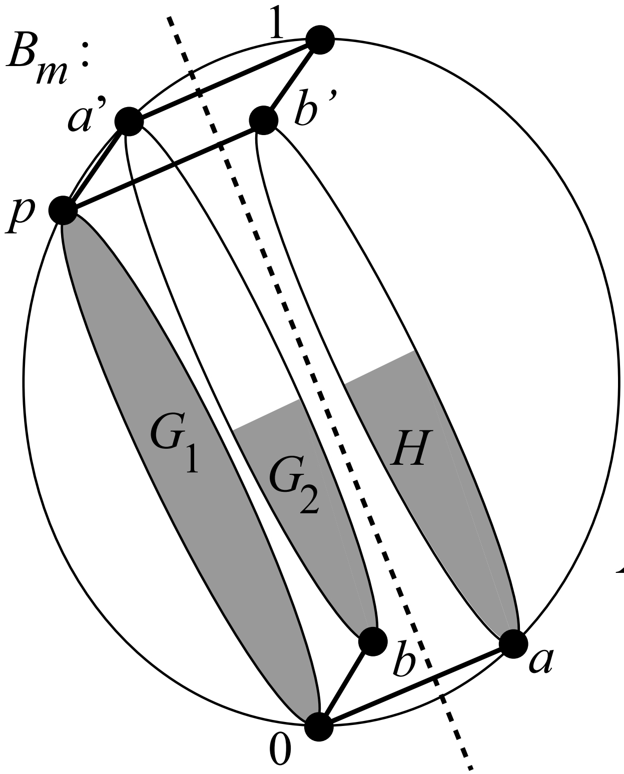

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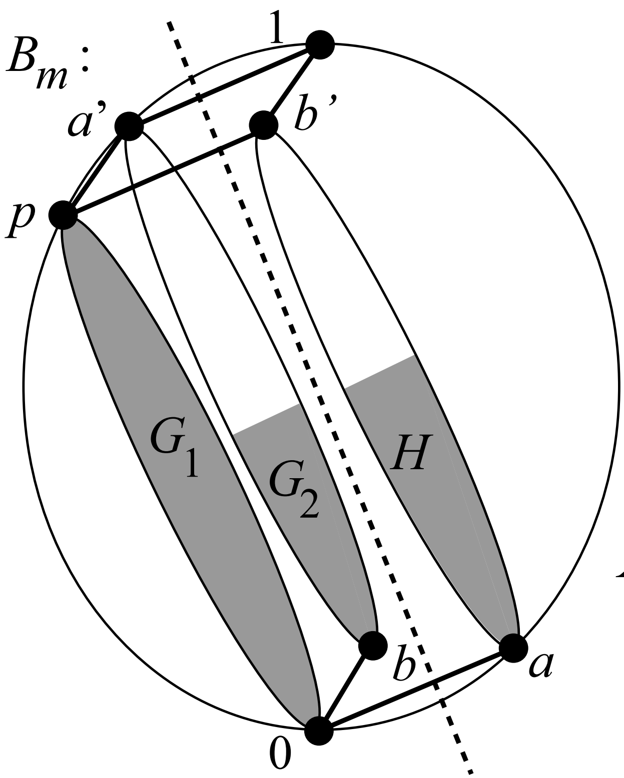

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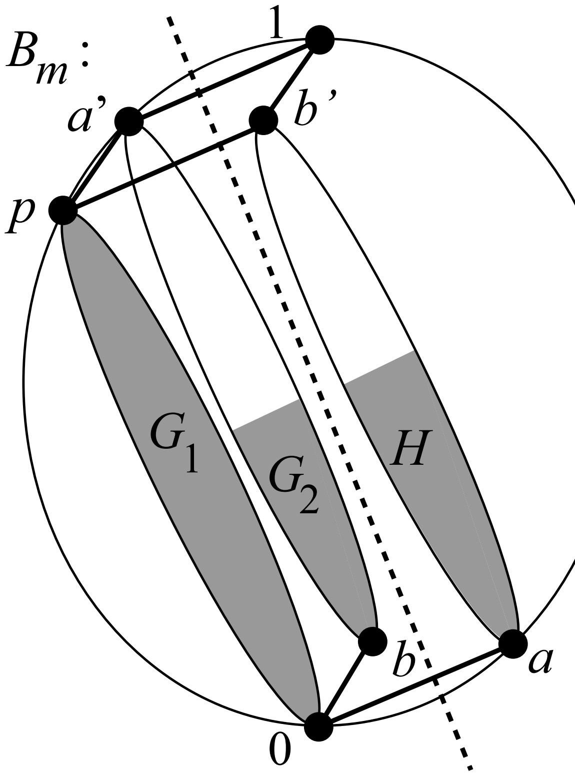
Now $G \cup H$ is an order ideal (that is why we chose $v = b'$), and $h(X) \leq h(G \cup H)$. But it is not necessarily greedy. Since $G = G_1 \cup G_2$ is greedy, G_2 is greedy in $\uparrow b$, and also in $[b, a']$.



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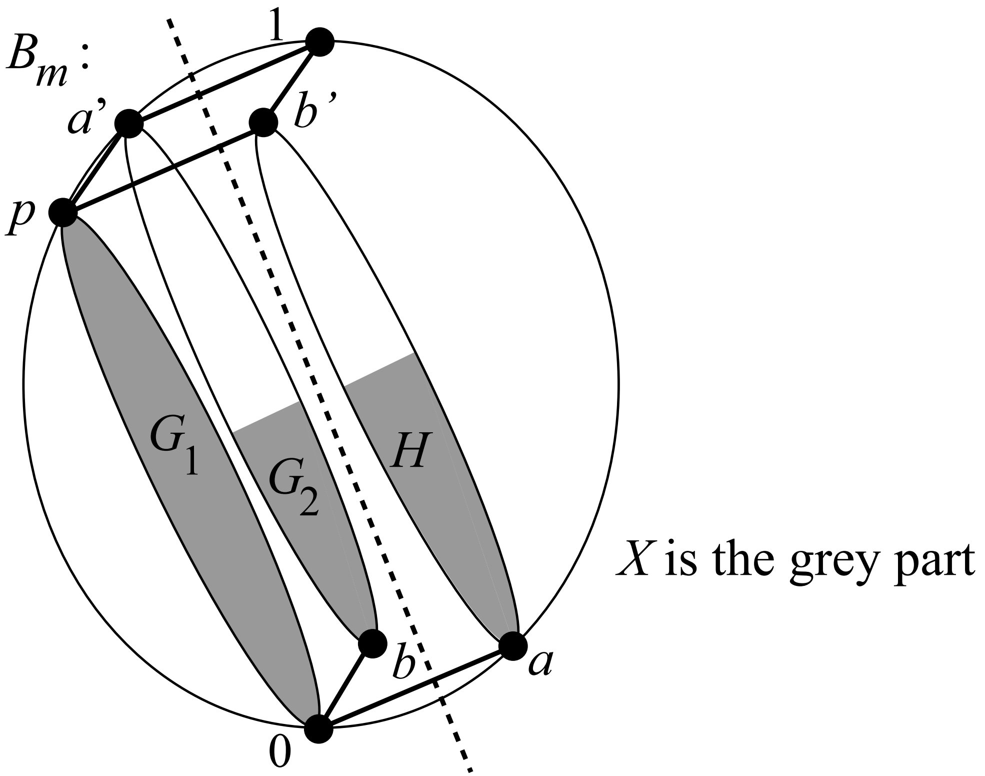

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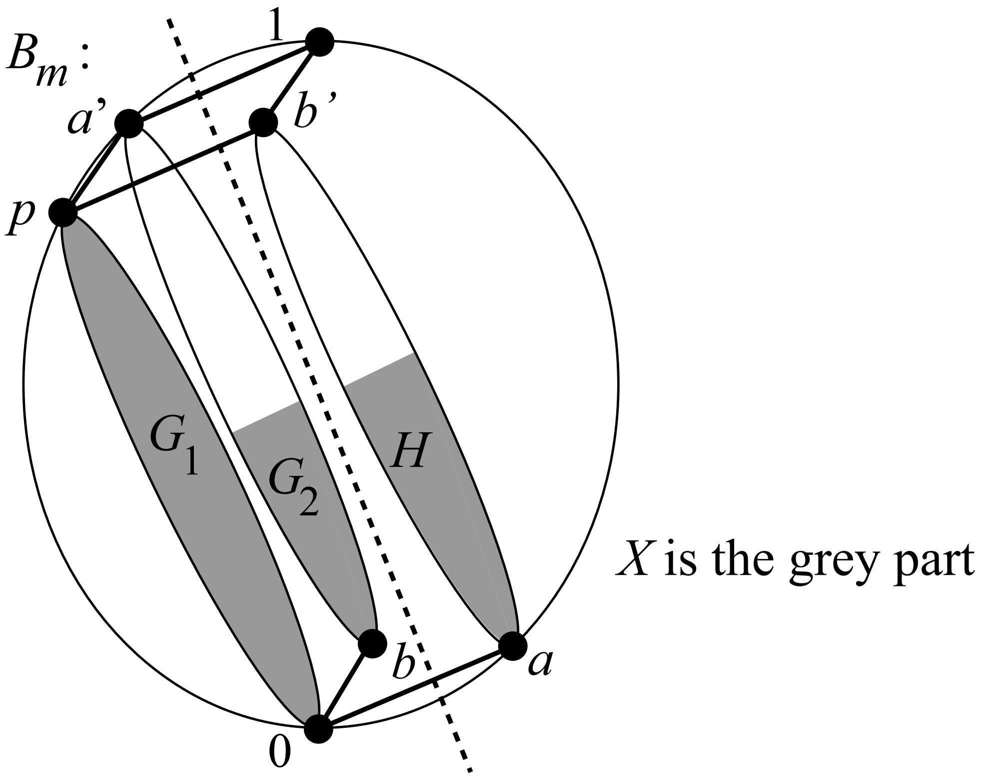

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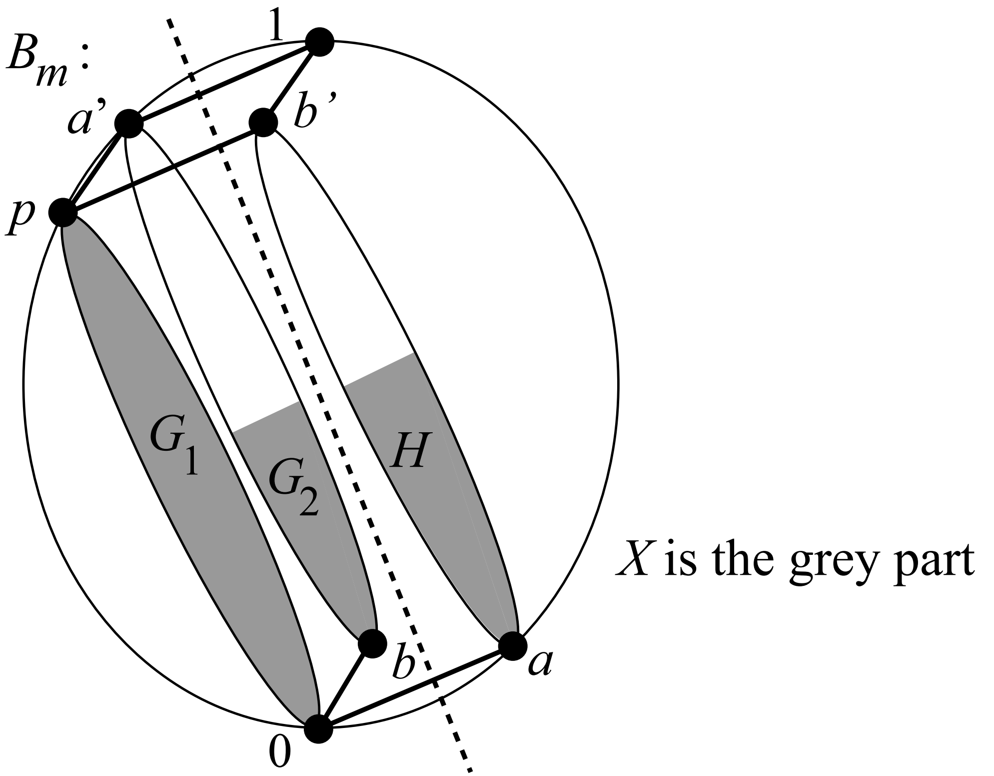

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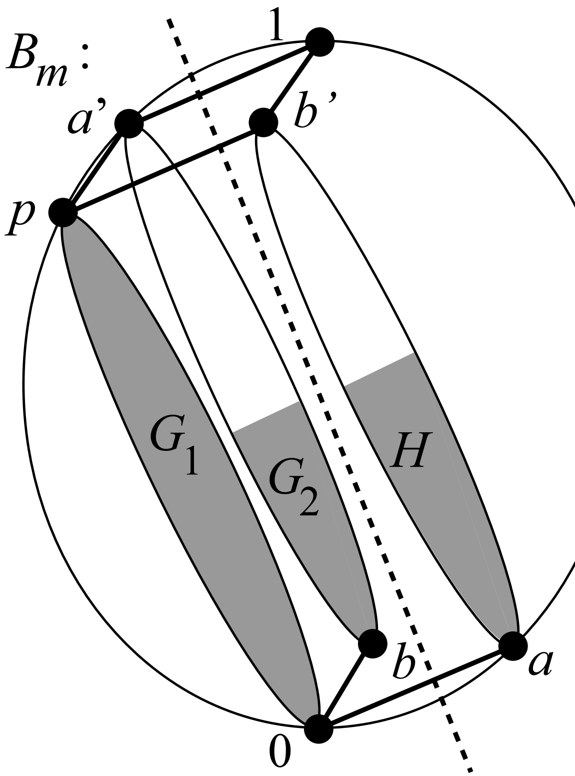

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Shifting lemma for Boolean lattices: Let (X_1, X_2) and (Y_1, Y_2) be pairs of greedy order ideals of B_m such that $|X_1| + |X_2| = |Y_1| + |Y_2|$, and either $Y_1 = B_m$ or $Y_2 = \emptyset$.

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We have outlined the Boolean case.

Direct product of chains.

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There is no rotational symmetry $\Longrightarrow$ the proof is much more complex. For example, it is not sufficient to formulate the shifting lemma for *pairs* of greedy order ideals.

The **Shifting Lemma** holds in D, but we have to formulate it for t-tuples of greedy order ideals rather than for pairs of them.

Namely, if $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ are greedy order ideals of D such that

$$\sum_{i=1}^{m} |X_i| = \sum_{i=1}^{m} |Y_i|$$

and there exists $1 \le i \le m$ such that

$$Y_1 = \cdots = Y_{i-1} = B_m, \quad Y_{i+1} = \cdots = Y_m = \emptyset,$$

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Open problems:

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