

Tolerances as congruence images

Conference on Universal Algebra and Lattice Theory
Dedicated to the 80-th birthday of Béla Csákány

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Szeged, Hungary, June 21–25, 2012

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A Mal'tsev-like condition

Condition $M(n)$

For any pair (f, g) of $2n$ -ary terms such that the identity

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Theorem (Czédli, Kiss)

A variety satisfies **TImC** iff it satisfies $M(n)$ for every $n \geq 1$.

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Remark: No finite set of conditions $M(n)$ suffices.

Lattice varieties

Corollary

Every variety of lattices has TImC .

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If $f(\dots, x, x, \dots) \approx g(\dots, x, x, \dots)$ is a lattice identity, then let

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which is $g(\dots, x, y, \dots)$. Thus $M(n)$ holds. □

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Rules out possible generalizations.

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- Which important semigroup varieties have TImC ?
- Apply $M(n)$ to other famous Mal'tsev conditions, as above.

Other varieties without TImC

Theorem

If a congruence n -permutable variety has TImC , then it is congruence permutable.

Proof: by applying $M(n)$ to the Mal'tsev condition discovered by Hagemann and Mitschke.

Problems: discover positive and negative examples

- Which important semigroup varieties have TImC ?
- Apply $M(n)$ to other famous Mal'tsev conditions, as above.

Preprint

Czédli, Kiss: *Varieties whose tolerances are homomorphic images of their congruences*, <http://arxiv.org/pdf/1204.2228.pdf>.