

Representing finite convex geometries in lattice theoretical and geometrical ways Part I

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(This part: by **G. Czédli**)

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The study of congruence lattices led Grätzer and Knapp, in their 2007 Acta Sci. Math. paper, to the introduction of slim semimodular lattices. Some benefits of this concept are as follows.

- A uniqueness part to Jordan–Hölder theorem from 1870 (CzG & Schmidt, E.T., AU 2011)
- Representing $(0,1)$ -preserving lattice homomorphisms $D_1 \rightarrow D_2$ by the restriction of a rectangular lattice to a rectangular filter (CzG, AU 2012)
- Swing Lattice Game (CzG & Makay G.)
- New description of join-distributive lattices (CzG & Adaricheva, AU 2014)
- Circles and convex geometry (CzG & Kincses, 2016)
- Excursion to Combinatorics (CzG, Dékány T., Gyenizse G., Kulin, J., Ozsvárt L., Szakács N., Udvari Balázs, 2012–2016; $e^{-2} \cdot (n - 2)!/2$.)

Slim semimod. lattices \rightsquigarrow Swing Lattice Game (CzG-Makay, 2016)

First step towards circles: a referee of the CzG–Ozsvárt–Udvari paper (DM, 2012): swing semimodular lattices are *join-distributive*!

Many definition; the first one by Dilworth, 1940. Monjardet: "A use for frequently rediscovering a concept" (1985)

Definition

If $J_i L$ is the union of two chains, then L is *slim*. x^* stands for the join of all covers of $x \in L$. If, for all $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$, then L is *meet-semidistributive*.

The next statement is known and gives a good understanding of join-distributive lattices. For further characterizations, see S. P. Avann [8], which is recalled in P. H. Edelman [17, Theorem 1.1], and see also M. Stern [23, Theorem 7.2.27].

Proposition (for a finite lattice L , tfae:)

- L is join-distributive (semimodular & meet-semidistributive).
- L has unique meet-irreducible decompositions.
- For each $x \in L$, the interval $[x, x^*]$ is distributive.
- For each $x \in L$, the interval $[x, x^*]$ is boolean.
- The length of each maximal chain of L equals $|Mi L|$.
- L is **sm** (=semimodular) and diamond-free.
- L is sm with no cover-preserving diamond.
- L is a cover-pres. \vee -subsemilat. of a finite distributive lattice.
- $L^{\text{dual}} \cong \text{Sub}(\text{finite convex geometry})$ [\rightarrow Part II].
- **Adaricheva-CzG (2014):** L is **sm**, $\text{length}(L) < \infty$, and every trajectory intersects every maximal chain C **exactly once**.
- **Adaricheva-CzG (AU 2014):** as above with **at most once**.

Definition (Edelman [16]; Adaricheva, Gorbunov, Tumanov [?])

A pair $\langle E, \Phi \rangle$ is a *convex geometry*, also called *anti-exchange system*, if $\Phi: P(E) \rightarrow P(E)$ is a closure operator, $\Phi(\emptyset) = \emptyset$, and

- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, $x \neq y$, and $\Phi(A \cup \{x\}) = \Phi(A \cup \{y\})$, then $x = y$. (Anti-exchange property; x cannot be exchanged.)

Theorem (CzG (in DM, 2014))

Each slim semimodular $L \cong_{\text{dual}} \text{Sub}(a \text{ conv. geom. of circles})$.

Theorem (CzG 2016, <http://arxiv.org/abs/1611.09331>)

If K_0 is a compact convex subset of the plane \mathbb{R}^2 , then tfae:

- ① K_0 is a disk.
- ② For every $K_1 \subseteq \mathbb{R}^2$ and for arbitrary points $A_0, A_1, A_2 \in \mathbb{R}^2$, if K_1 is similar to K_0 and both K_0 and K_1 are included in the triangle $\text{Conv}(\{A_0, A_1, A_2\})$, then there exist a $j \in \{0, 1, 2\}$ and a $k \in \{0, 1\}$ such that K_{1-k} is included in $\text{Conv}(K_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\}))$.
- ③ The same as the second condition but “similar” is replaced by “isometric”.

(1)→(2): Adaricheva & Bolat (Sept.1, 2016) (Shorter: CzG, Oct.8)

Theorem (CzG 2016, <http://arxiv.org/abs/1611.09331>)

If K_0 is a compact convex subset of the plane \mathbb{R}^2 , then tfae:

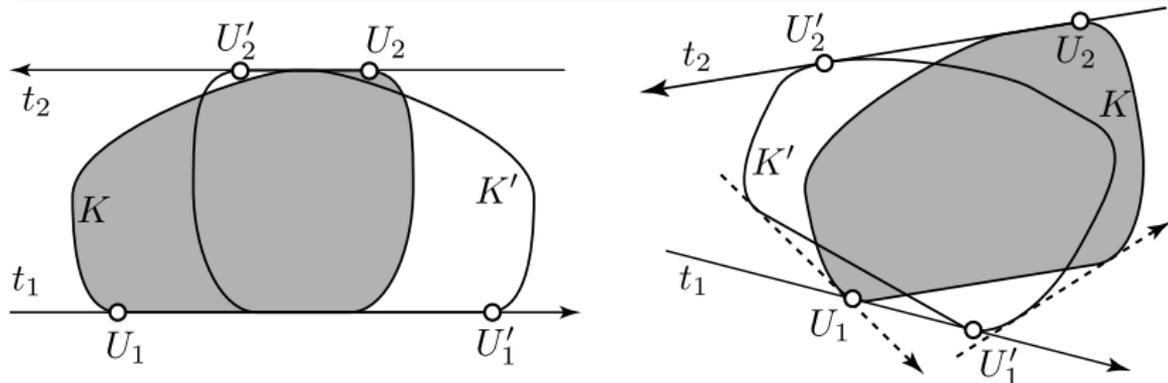
- ① K_0 is a disk.
- ② For every $K_1 \subseteq \mathbb{R}^2$ and for arbitrary points $A_0, A_1, A_2 \in \mathbb{R}^2$, if K_1 is similar to K_0 and both K_0 and K_1 are included in the triangle $\text{Conv}(\{A_0, A_1, A_2\})$, then there exist a $j \in \{0, 1, 2\}$ and a $k \in \{0, 1\}$ such that K_{1-k} is included in $\text{Conv}(K_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\}))$.
- ③ The same as the second condition but “similar” is replaced by “isometric”.

(2) \rightarrow (3): Trivial.

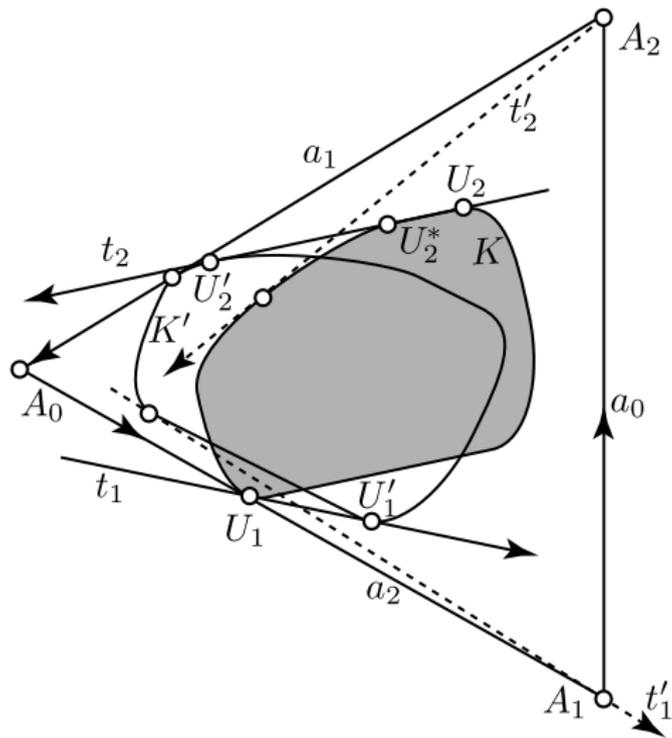
In what follows, $K := K_0 \models (3)$. K' or K^* stands for $K_1 \cong K$;

Lemma

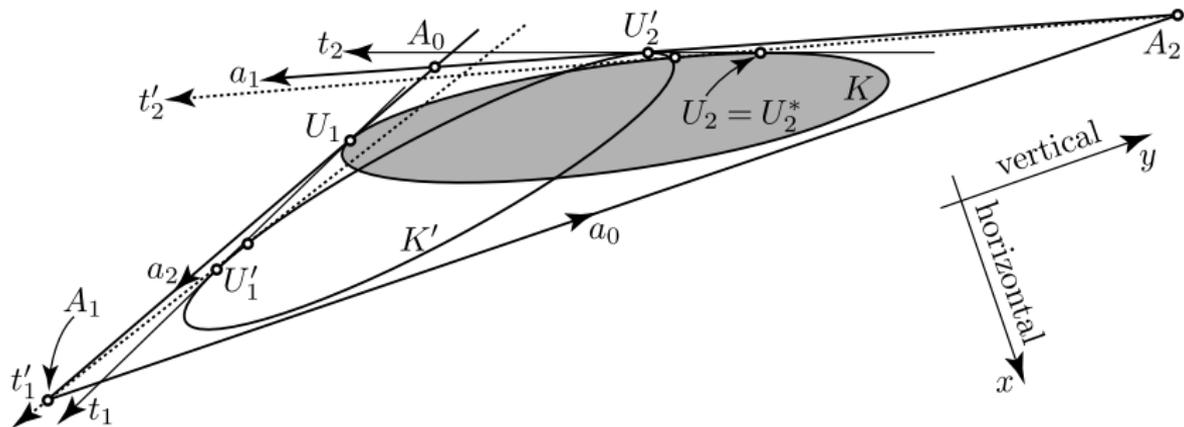
The figure below is impossible: K and K' cannot cross each other.



Proof: slide-turn the (common) supporting lines as indicated in the next two figures!



Slide-turn the (common) supporting lines!

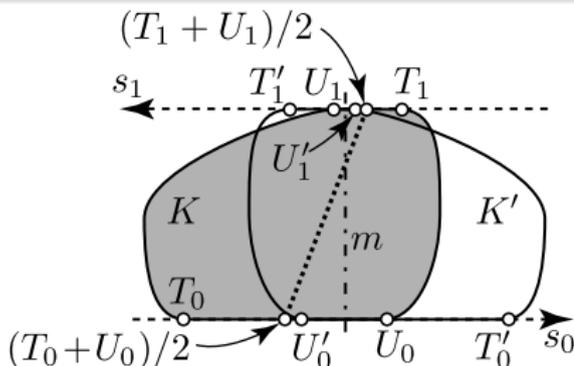
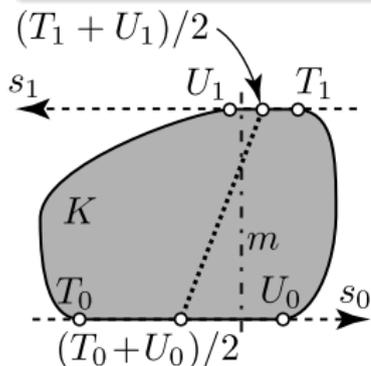


Lemma

The thick densely dotted line is perpendicular to s_0 and s_1 .

Proof.

Otherwise the Cross Lemma leads to a contradiction; see below. \square

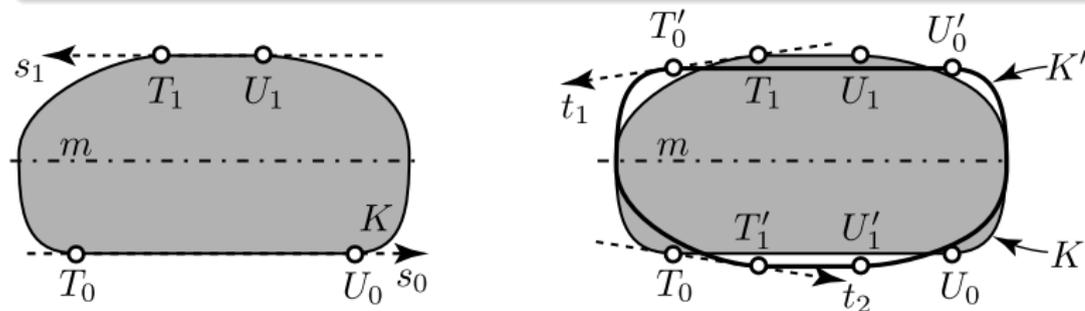


Lemma

For two parallel supporting lines, $\text{dist}(T_0, U_0) = \text{dist}(T_1, U_1)$.

Proof.

Otherwise the Cross Lemma leads to a contradiction; see below. \square

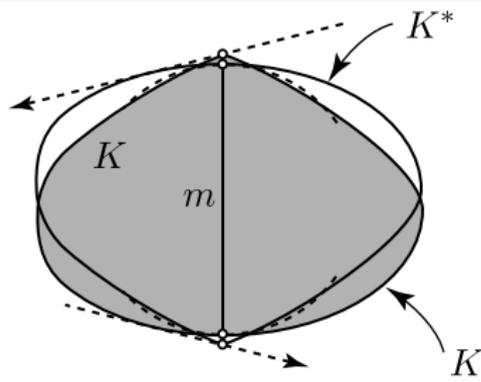
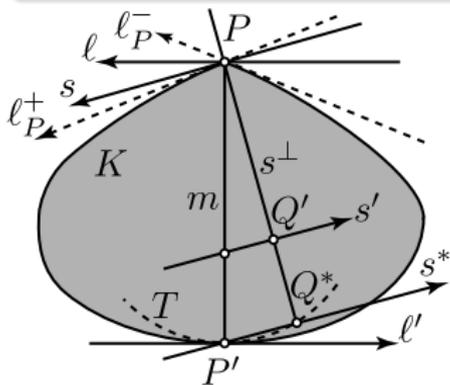


Lemma

There is no corner.

Proof.

Otherwise the Cross Lemma leads to a contradiction; see below. \square

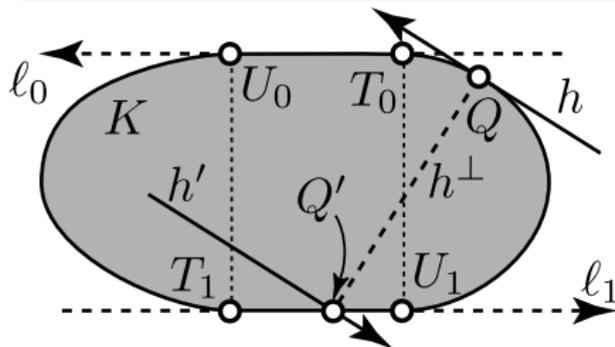


Lemma

No proper “tangent interval” is possible.

Proof.

Otherwise the Cross Lemma leads to a contradiction; see below. \square

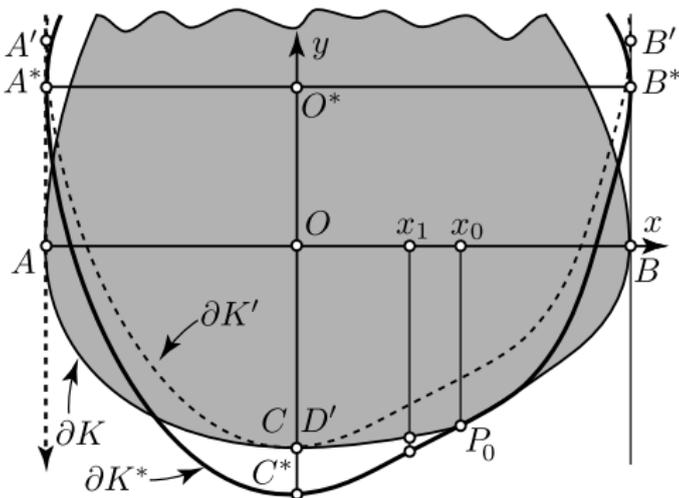
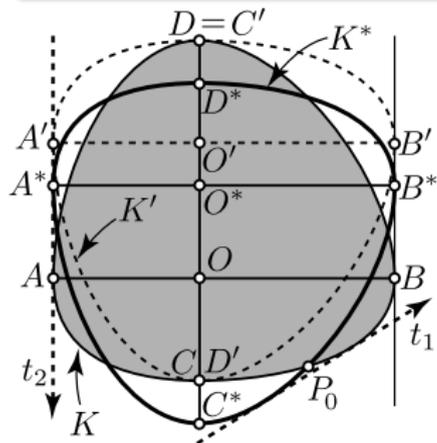


Now we have *opposite points*, which determine *diagonals*, and *tangent lines* that are **perpendicular to the corresponding diagonals**.

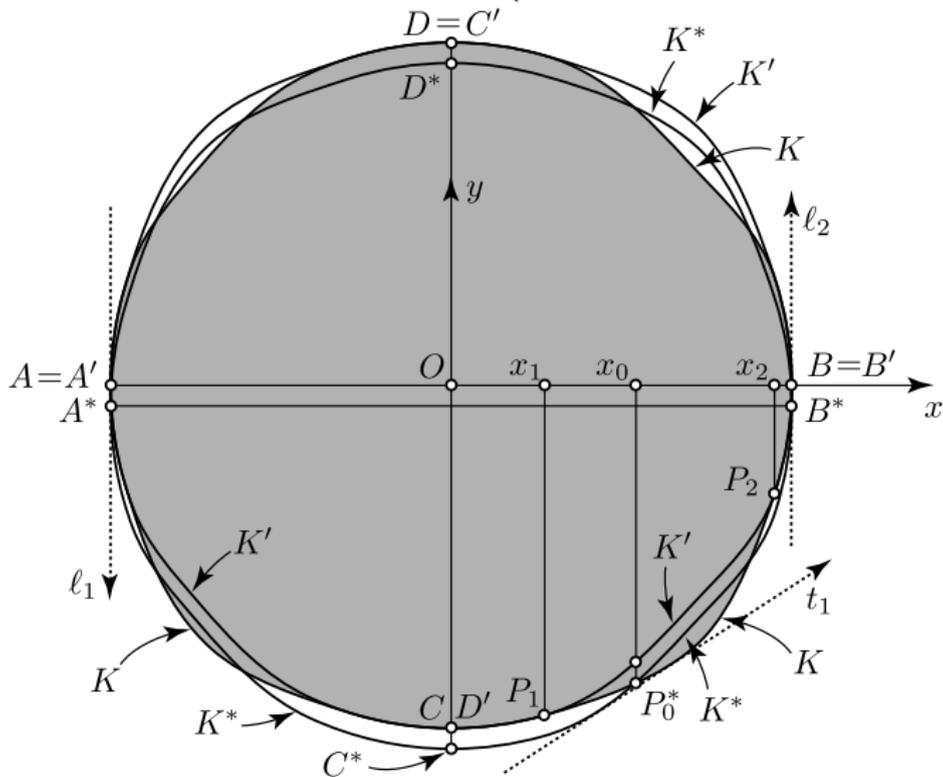
Cross Lemma yields:

Lemma

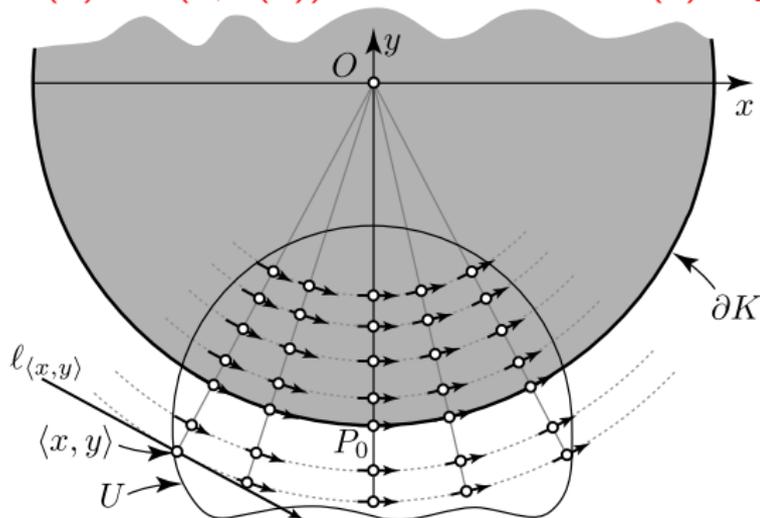
Any two perpendicular diagonal halve each other.



Otherwise use Cross Lemma (it suffices to prove axial symmetry):



If $f(x, y)$ and $\partial f / \partial y$ are continuous in U , then the solution of $u'(x) = f(x, u(x))$ with initial value $u(0) = y_0$ is unique. Q.E.D.



These slides: <http://www.math.u-szeged.hu/~czedli/>

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