Planar Semimodular Lattices: Structure and Diagrams*

Gábor Czédli and George Grätzer

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*http://www.math.u-szeged.hu/∼czedli/
,
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Foreword

Advertisement: based on our joint chapter „Planar Semimodular Lattices: Structure and Diagrams" in

Lattice Theory: Special Topics and Applications, edited by **George Grätzer and Fred Wehrung**, to appear next year.

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The study of these easy mathematical objects: led to (at least) three (up to now) not-so-easy results; we mention three below.

1. A sharp result on congruence lattice representation For every result representing a finite distributive lattice D with n join-irreducible elements as the congruence lattice of a finite lattice L in some class $\mathbf{K}$ of lattices, the natural question arises: How small can we make L as a function of n and $\mathbf{K}$?

There are only two results of this type in the literature. For the first result, $\mathbf{K}$ is the class of all lattices (no restriction on L). It was proved in G. Grätzer, I. Rival, and N. Zaguia [1995] that $|L| = O(n^2)$ is best possible in this case.

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For the second result, that is the result relevant here, $\mathbf{K}$ is the class of rectangular lattices, to be defined later. Note that these lattices are planar and semimodular. For this case, it was proved in G. Grätzer and E. Knapp [2009, 2010] that $|L| = O(n^3)$ is best possible.

2. Jordan-Hölder theorem: Based on a proper understanding of planar semimodular lattices, we could strengthen the 140 year old Jordan-Hölder theorem for groups. Although this result is not as deep as the previous one, it sells (or should sell) well outside Lattice Theory. Details: later.

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3. Dropping planarity: Now that planar semimodular lattices are more or less understood, there is a hope that we can drop planarity from our assumptions. See E. T. Schmidt's home page, http://www.math.bme.hu/∼schmidt/, his „unpublished papers" there, for a lot of ideas. Also, we can mention G. Czédli , where the lattices corresponding to antimatroids and dually corresponding to convex geometries are coordinatized. This result leads to coordinatizations of antimatroids and convex geometries, so it may sell well even in Combinatorics. Details: not now. homepage, arXiv: please change to: "by $k-1$ permutation plus the number of superfluous points (= not belonging to any feasible set = belonging to the closure of $\emptyset$).

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The permutations on which our approach to the Jordan-Hölder theorem is based were discovered by R. P. Stanley [1972], and these permutations are in connection with geometry and Coxeter groups, see Armstrong [2009].

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Emphasis: **diagrams**. **Conventions** $\forall$ are **finite**. $\forall$ diagrams are **planar**. Properties diagrams (like semimodularity) = properties of the corresponding lattices.

D. Kelly and I. Rival [1975]. A *planar diagram* D of a finite lattice L is a pair $D = (\varphi, E)$ with the following three properties:

• φ is a one-to-one map of L into $\mathbf{R}^2$ such that if $a < b$ in L and $\varphi(a) = (a_1, a_2)$, $\varphi(b) = (b_1, b_2)$, then $a_2 < b_2$;

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• two distinct line segments of E are not incident except possibly at their endpoints.

This definition allows rigorous proofs.

For $D \in \mathsf{Dgr}(L)$: *left boundary chain* $\mathsf{C}_{\mathsf{l}}(D)$, *right boundary chain* $\mathsf{C}_{\mathsf{r}}(D)$, *boundary* $\mathsf{Bnd}(D) = \mathsf{C}_{\mathsf{l}}(D) \cup \mathsf{C}_{\mathsf{r}}(D)$.

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For a maximal chain C: *left side*, $\mathsf{LS}(C, D)$ ($\mathsf{LS}(C)$, for short), *right side* $\mathsf{RS}(C)$. Note $L = \mathsf{LS}(C) \cup \mathsf{RS}(C)$, $C = \mathsf{LS}(C) \cap \mathsf{RS}(C)$.

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Let $a \le b$, $D \in \mathrm{Dgr}(L)$, $D_{a,b}$ its restriction, $C_1 \subseteq \mathrm{LS}(C_2)$ and $C_2 \subseteq \mathrm{RS}(C_1)$ in $D_{a,b}$. Then $R = \mathrm{RS}(C_1) \cap \mathrm{LS}(C_2)$ is a **region** of D. It is a convex sublattice, $\mathsf{C_l}(R, D_{a,b}) = C_1$, $\mathsf{C_r}(R, D_{a,b}) = C_2$.

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Cell = minimal non-chain region. 4-**cell** = four-element cell. 4-cell $\Rightarrow$ **covering square**. If A is a 4-cell of D, then $A = \{0_A, 1_A, \mathrm{lc(A)}, \mathrm{rc(A)}\}$. (Lower case acronyms define elements.)

4-**cell diagram**, 4-**cell lattice**, $\exists \iff \forall$. $\mathrm{Ji}\, L = \mathrm{Ji}\, D$, $\mathrm{Mi}\, L$, $\mathrm{Di}\, L$.

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Kelly-Rival Corollary (1975). Let R be a region of $D \in \mathrm{Dgr}(L)$.

- $\mathrm{int}(R) \subseteq \mathrm{int}(L)$.
- If $u < v$ in L and $|R \cap \{u, v\}| = 1$, then $[u, v] \cap \mathrm{Bnd}(R) \neq \emptyset$.
- If $x \in \mathrm{int}(R)$, and $x \prec y$ or $y \prec x$ in L, then $y \in R$.

For $i \in \{1, 2\}$, let L_i be a planar lattice and let $D_i \in \mathrm{Dgr}(L_i)$. A bijection $\varphi\colon D_1 \to D_2$ is a *diagram isomorphism* if it is a lattice isomorphism. Equivalently, if $x \prec y$ iff $\varphi(x) \prec \varphi(y)$.

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A diagram isomorphism $\varphi\colon D_1 \to D_2$ is called a *similarity map* if for all $x, y, z \in D_1$ such that $x \prec y$ and $x \prec z$, y is to the left of z iff $\varphi(y)$ is to the left of $\varphi(z)$, and dually. D_1 and D_2 are **similar** lattice diagrams if there exists a similarity map $D_1 \to D_2$. We consider lattice **diagrams up to similarity**.

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The diagrams of a planar lattice L are **unique up to left-right symmetry** if for any $D_1, D_2 \in \mathrm{Dgr}(L)$, D_1 is similar to D_2 or to the vertical mirror image of D_2.

Lemma(Grätzer and Knapp, 2007) Let L be a planar lattice.

- If L is semimodular, then it is a 4-cell lattice. If $D \in \mathrm{Dgr}(L)$ and A, B are 4-cells of D with the same bottom, then these 4-cells have the same top.
- Conversely,

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Proof/ Part I.

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Proof/ Part I. Semimodularity is preserved by cp-sublattices. Hence D is a 4-cell diagram.

Let A and B be 4-cells with $0_A = 0_B$. Among lc(A), rc(A), lc(B), and rc(B), let x be the leftmost one and y be the rightmost one. Then $x \neq y$, and the interval $[0_A, x \vee y]$ is of length 2 by semimodularity. Hence

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Proof/ Part II. Wanted: if $z = x \wedge y \prec x$ and $z \prec y$, then $x \prec x \vee y = v$ and $y \prec v$. (This+finiteness imply semimodularity.) Assume the premise in $E \in \mathrm{Dgr}(L)$.

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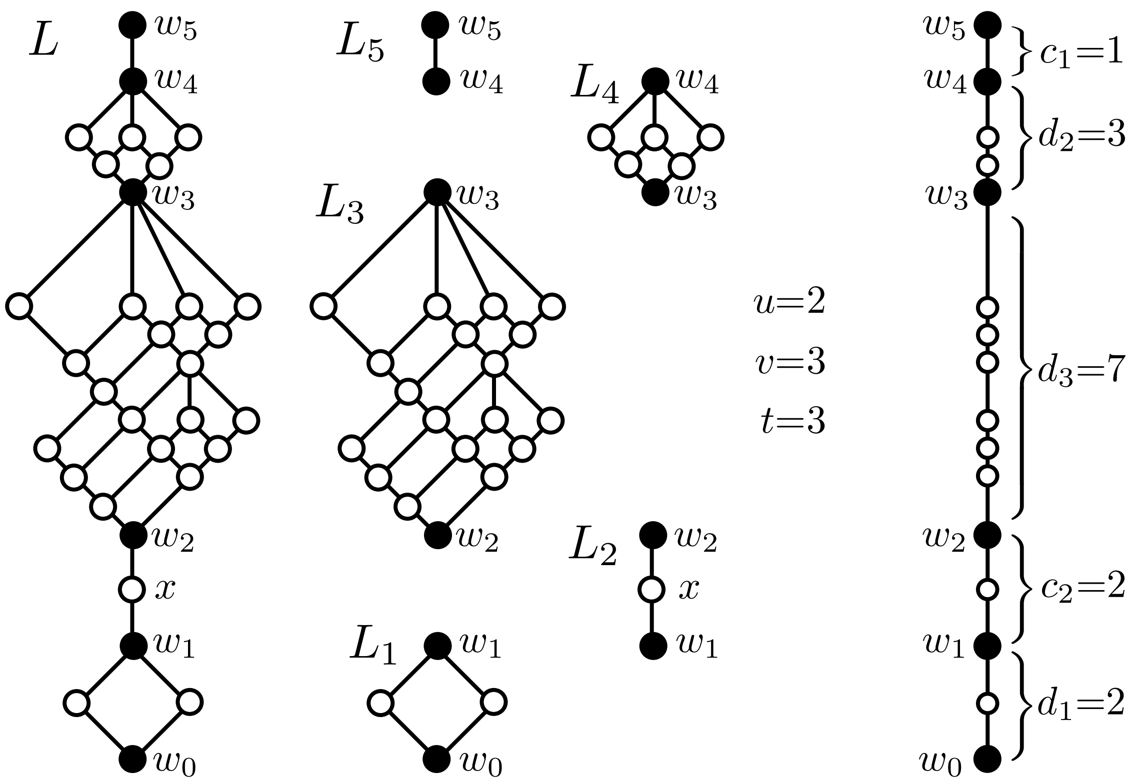
Grätzer and Knapp [2007], Czédli and Schmidt [2011]: a finite lattice L is called *slim* if Ji L contains no three-element antichain.
$\iff$ Ji L is the union of two chains.

Lemma Slim $\Rightarrow$ **planar** (even without semimodularity). $\sqrt{}$



Lemma A slim semimodular lattice can uniquely be decomposed into a glued sum of maximal chain intervals and indecomposable slim semimodular lattices.

Lemma (Grätzer and Knapp [2007], Czédli and Schmidt [2011]).

For a finite lattice L, the following are equivalent.

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- L is a slim semimodular lattice and a planar 4-cell lattice.

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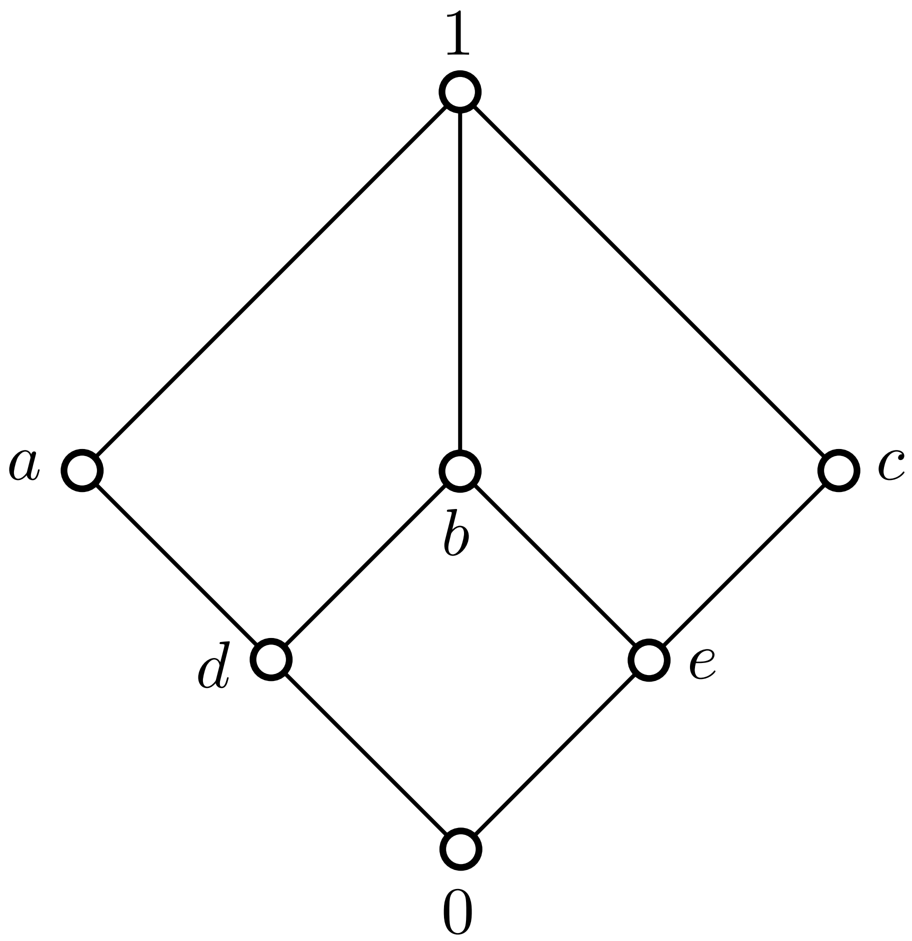
- All $D \in \mathrm{Dgr}(L)$ are 4-cell diagrams with no two distinct 4-cells having the same bottom.

Lemma (Grätzer and Knapp [2007], Czédli and Schmidt [2011]). A slim ($\Rightarrow$ planar) semimodular lattice is distributive iff N_7 (see below) is not a cover-preserving sublattice of L.



We generalize Czédli and Schmidt [2011] by dropping semimodularity:

Theorem. Let L be a slim lattice. Then we have:

- Bnd$(D) =$ Bnd(E) for $D, E \in$ Dgr(L) (that is, Bnd(L) does not depend on the diagram chosen).
- Ji $L \subseteq$ Bnd(L).
- If L is a glued sum indecomposable slim lattice, then its planar diagrams are unique up to left-right symmetry.

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• If L is a glued sum indecomposable slim lattice, then its planar diagrams are unique up to left-right symmetry.

Corollary. Let E_1 and E_2 be slim lattice diagrams, and let $\varphi\colon E_1 \to E_2$ be a diagram isomorphism (≈ lattice isomorphism). Then φ is a similarity map iff $\varphi(\mathsf{C}_{\mathsf{l}}(E_1)) = \mathsf{C}_{\mathsf{l}}(E_2)$ iff $\varphi(\mathsf{C}_{\mathsf{r}}(E_1)) = \mathsf{C}_{\mathsf{r}}(E_2)$.

Let $D \in \mathrm{Dgr}(L)$, planar sm. If we omit all the „eyes" (= interior elements of D in all intervals of length two), then we get Slim D, the **full slimming** of D. The reverse procedure is **anti-slimming**. The full slimming is an operation for D, not for L. But

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Lemma. Let D_1 and D_2 be planar semimodular diagrams. If D_1 is isomorphic to D_2, then $\mathrm{Slim}\, D_1$ is isomorphic to $\mathrm{Slim}\, D_2$.

Lemma. A planar lattice is semimodular iff some (equivalently, all) of its full slimming sublattices is slim and semimodular.

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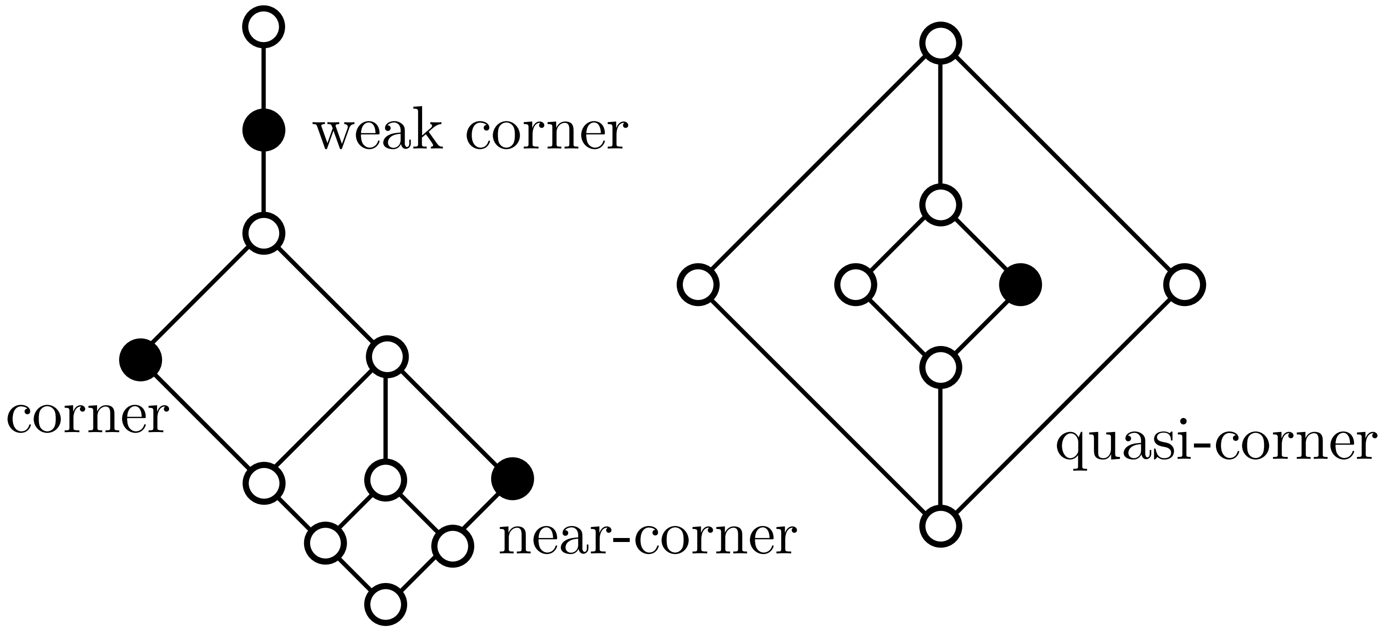
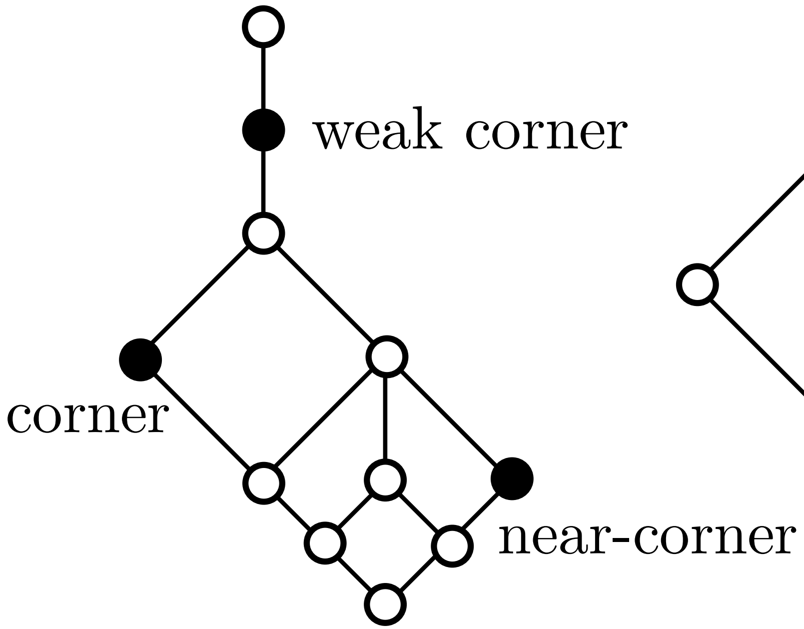
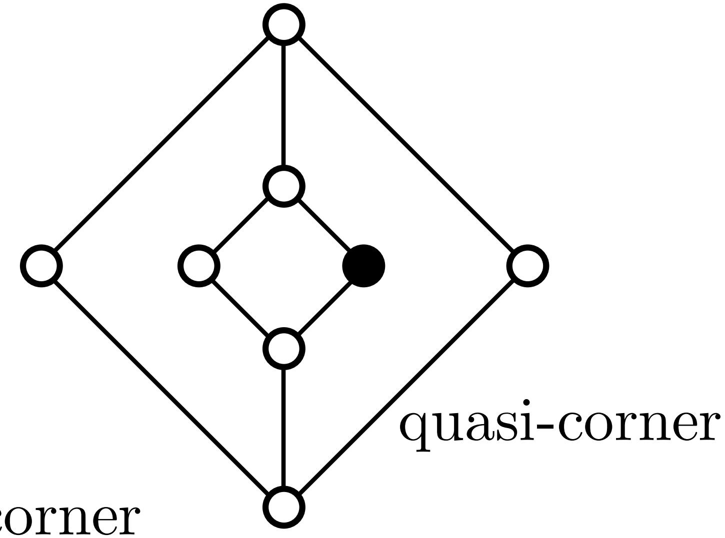
Proposition. A planar lattice is semimodular iff some (equivalently, all) of its full slimming sublattices is slim and semimodular.

$\Rightarrow$ Almost always, it suffices to deal with **slim** semimodular . . .



Weak corner $\in \mathsf{Bnd}(D) \cap \mathsf{Di}\, D$.
Near





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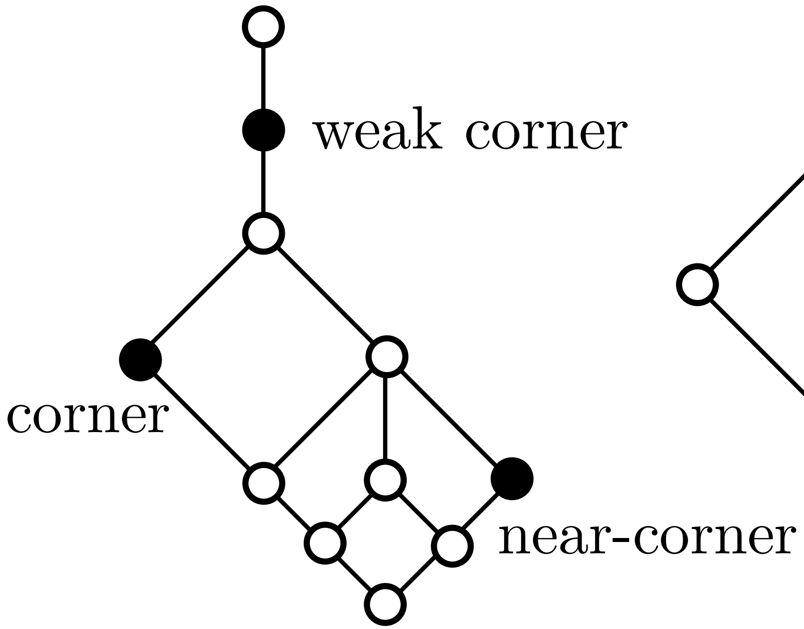
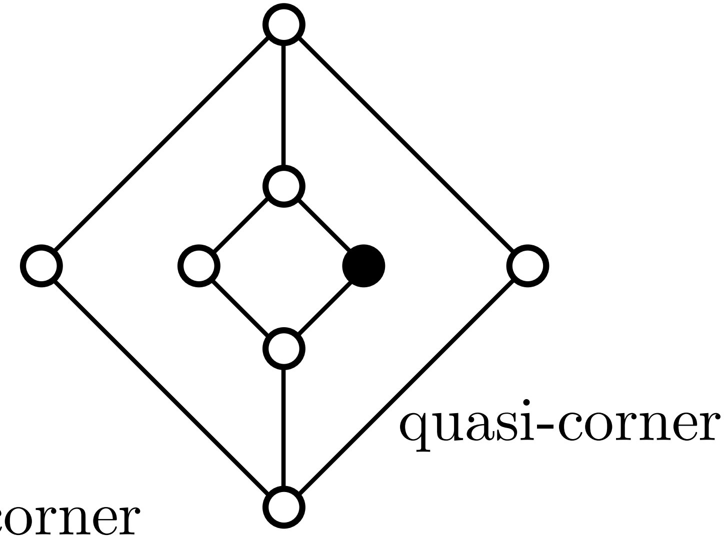
Near corner = a weak corner d such that d_* has exactly two covers and d^* has at least two lower covers.
Corner





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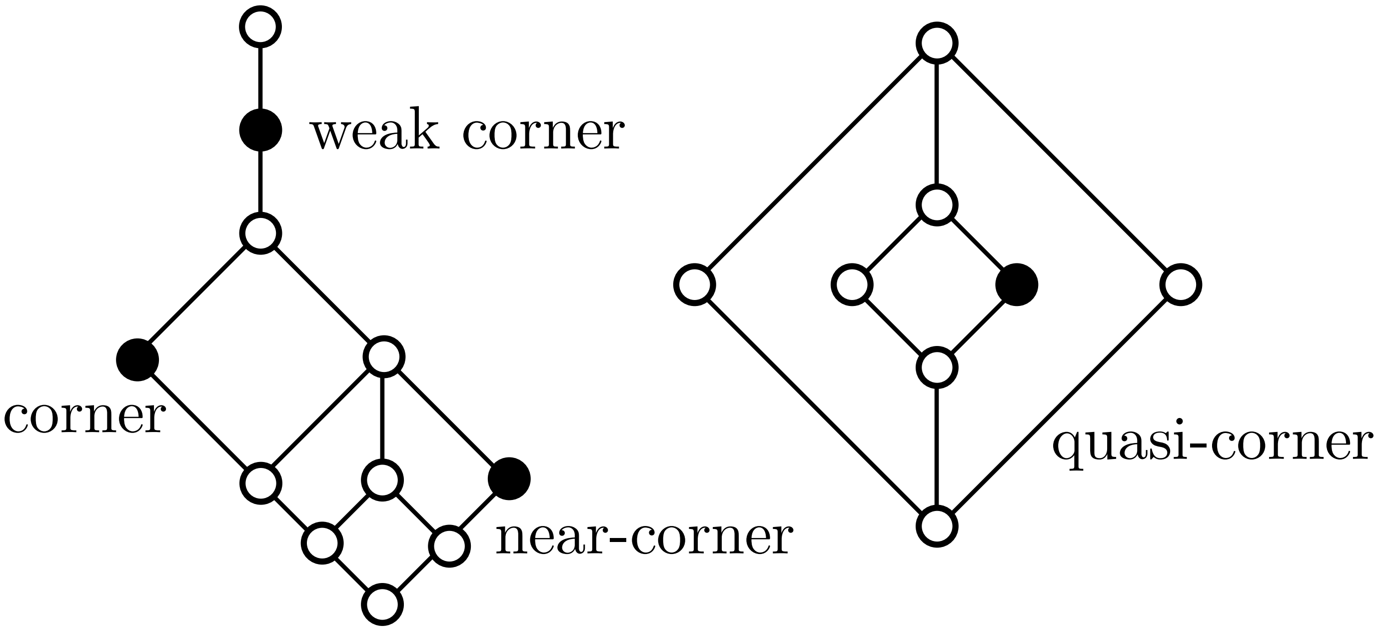

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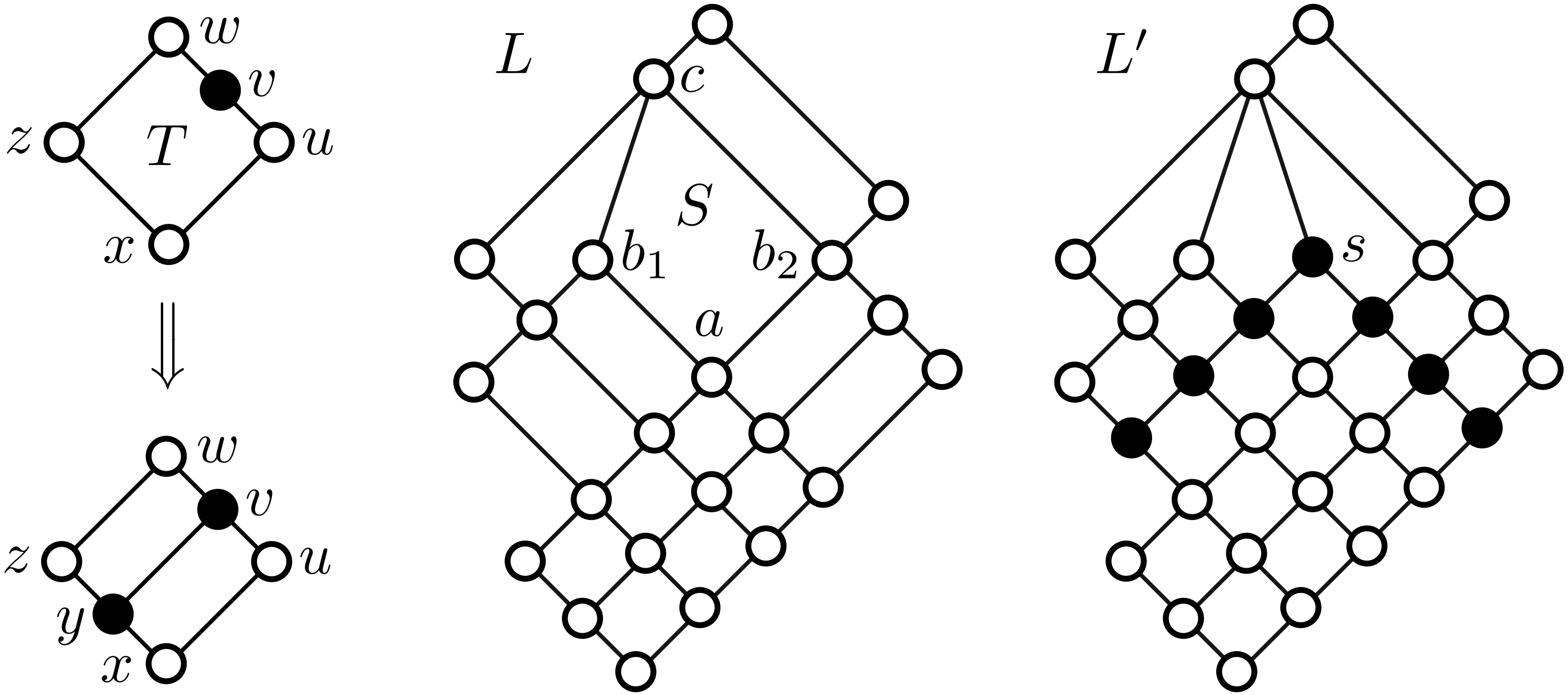
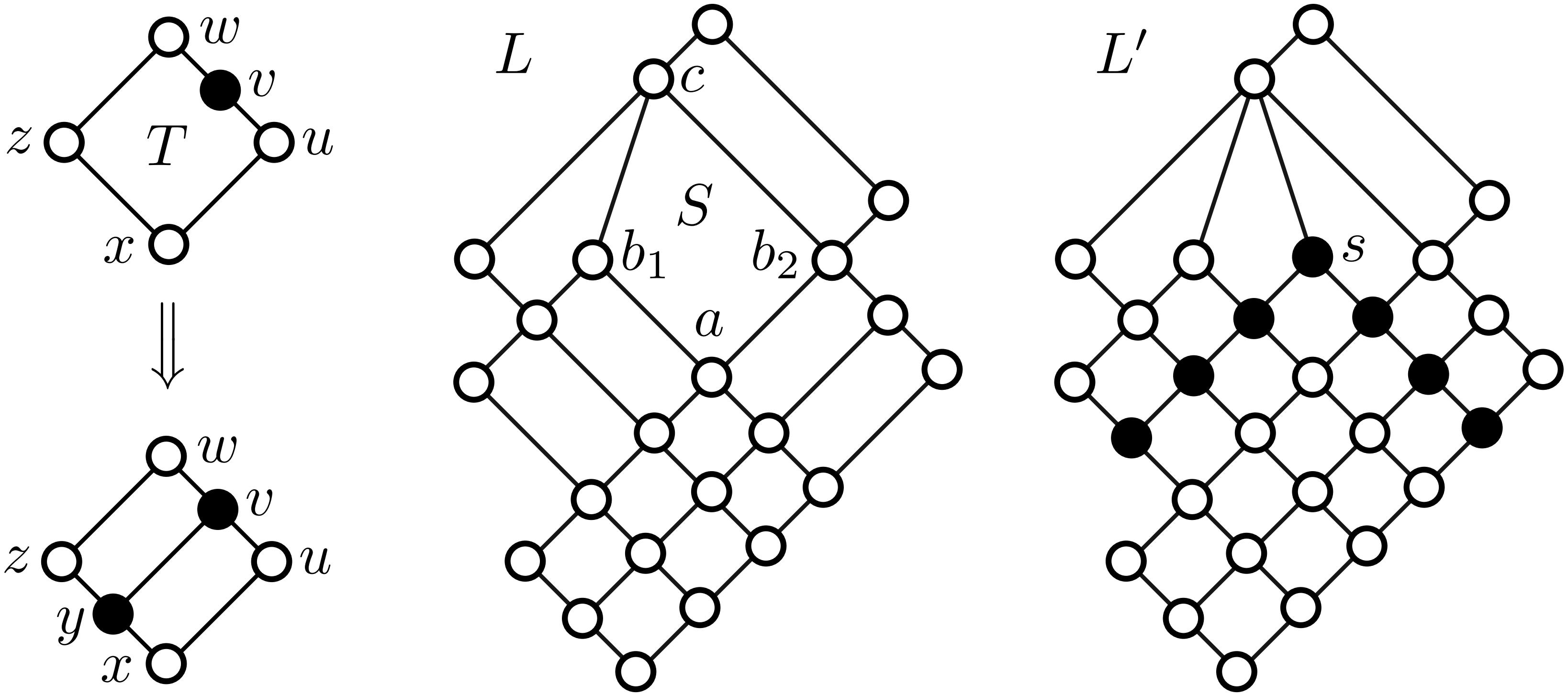
Each of the above can be **left** or **right**; and it can be **removed** or **added**. Adding or deleting a near corner **preserves** planarity and semimodularity. **Each** slim sm diagram can be obtained from a **chain** by adding near-corners, one-by-one (easy).

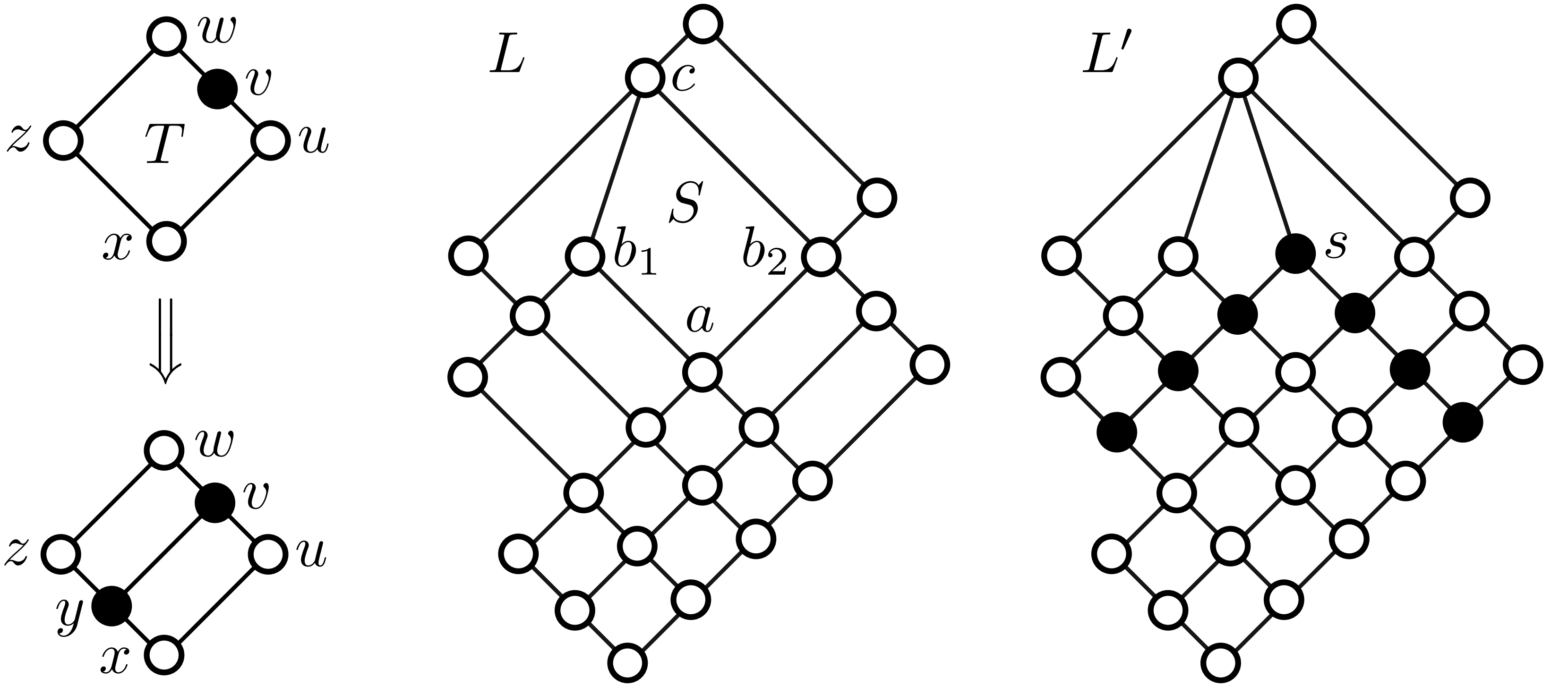


Adding a **fork** to D at a 4-cell S. P



Adding a **fork** to D at a 4-cell S. Preserves slimness and semimodularity.

Proposition (C



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Proposition (Czédli and Schmidt) ∀ slim semimodular diagram can be obtained from a chain by adding forks and corners.

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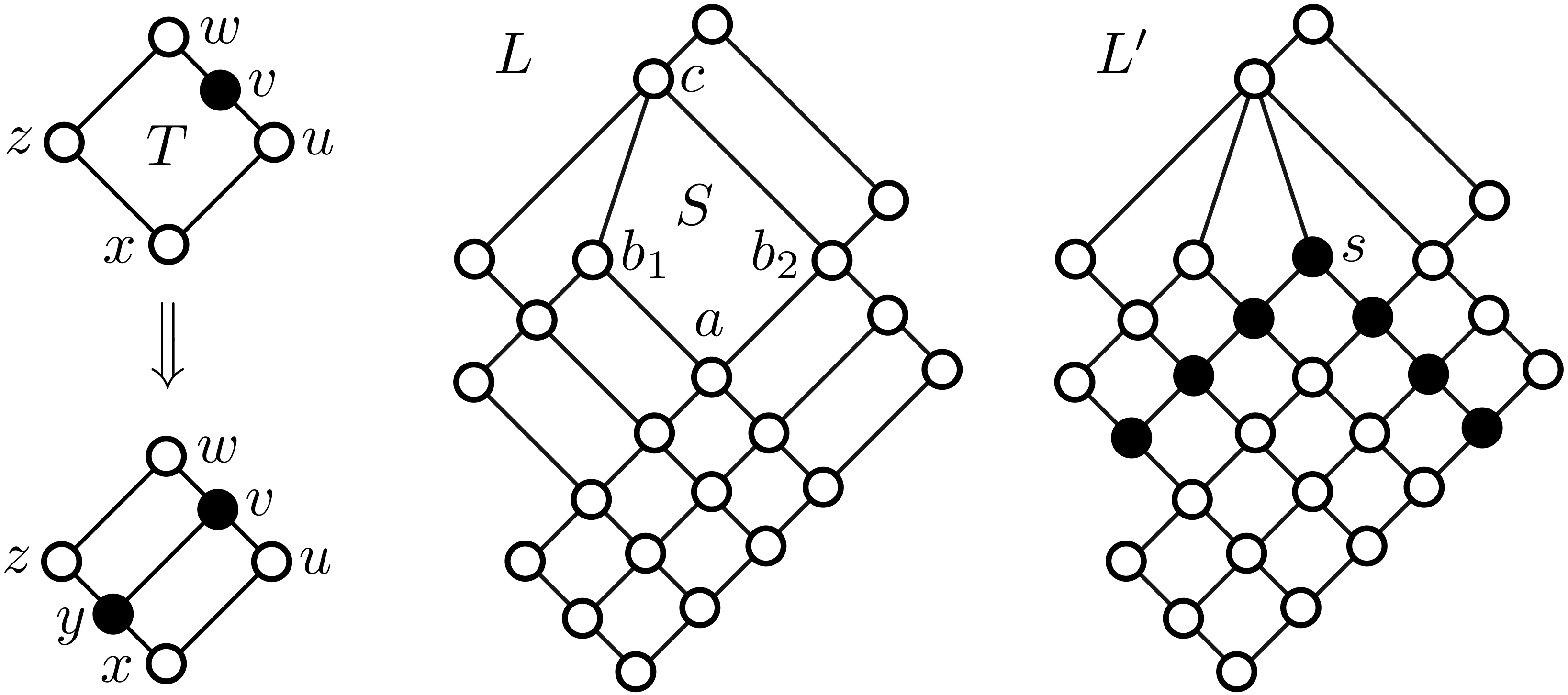
Forks



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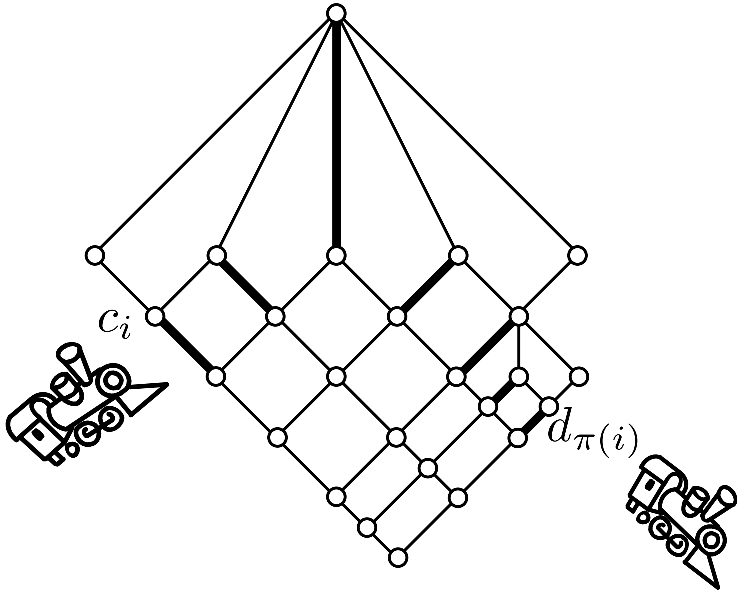
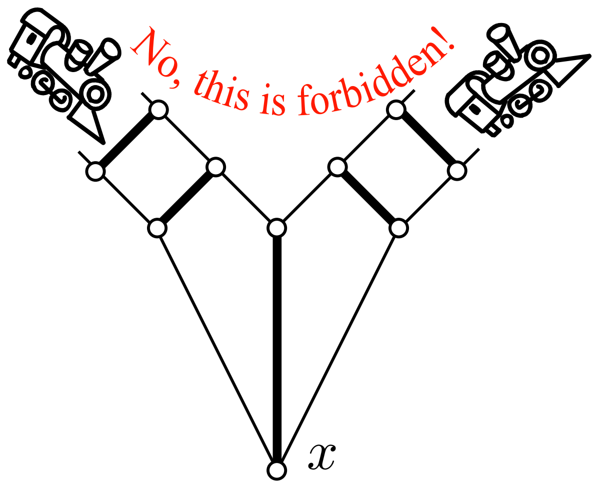
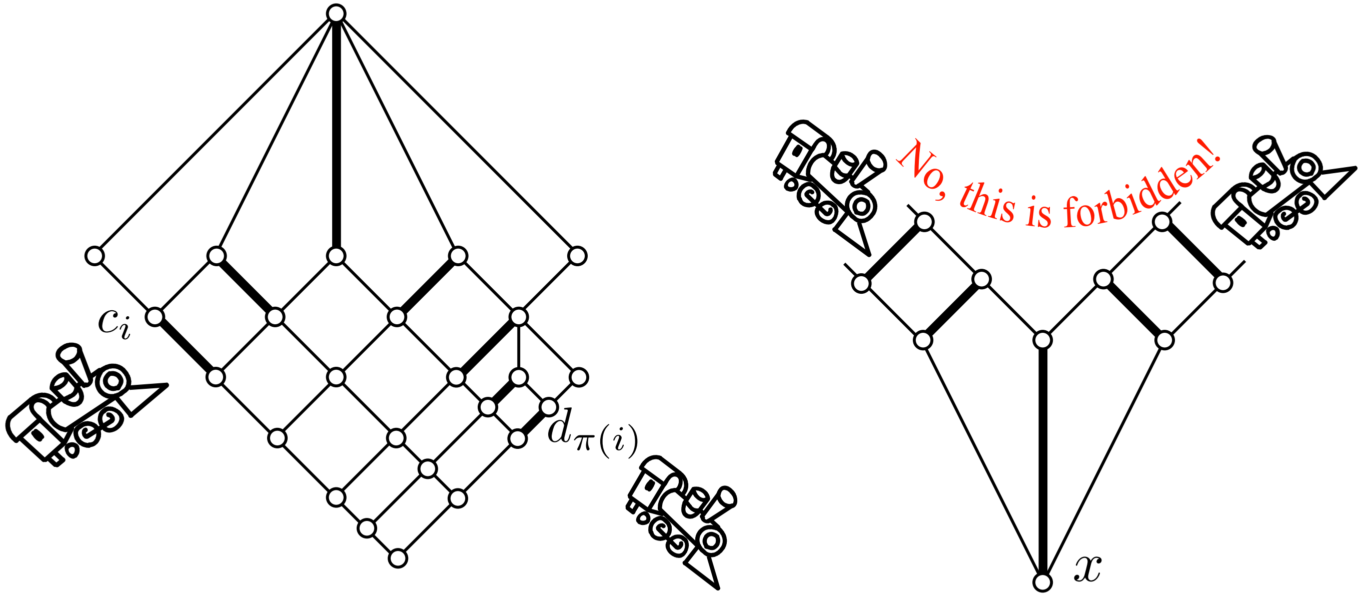
Proposition (Czédli and Schmidt) ∀ slim semimodular diagram can be obtained from a chain by adding forks and corners.

Theorem (Czédli and Schmidt) ∀ slim semimodular diagram (or lattice) with at least three elements can be obtained from a grid (= chain × chain) by adding forks, and **then** removing corners.





Definition (Czédli and Schmidt 2011) Two prime intervals of D are *consecutive* if they are opposite sides of a 4-cell. The blocks of the equivalence generated by the consecutivity relation are called $\mathbf{C}_2$**-trajectories**. *Pr*



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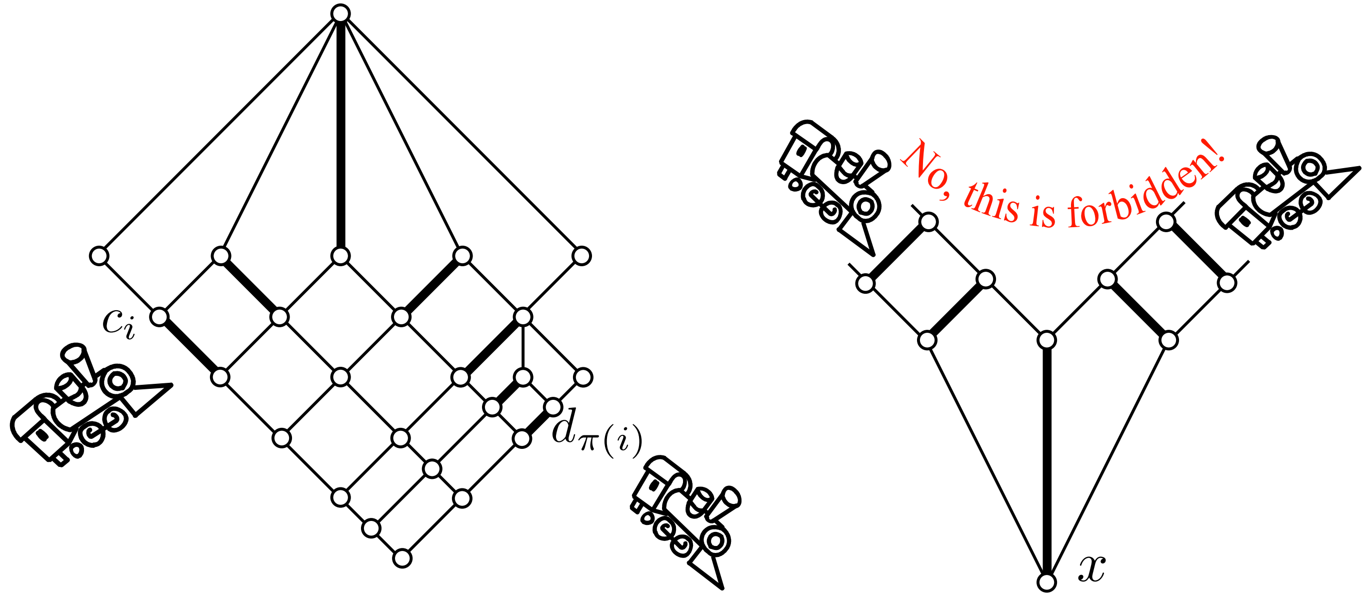
$\mathbf{C}_3$



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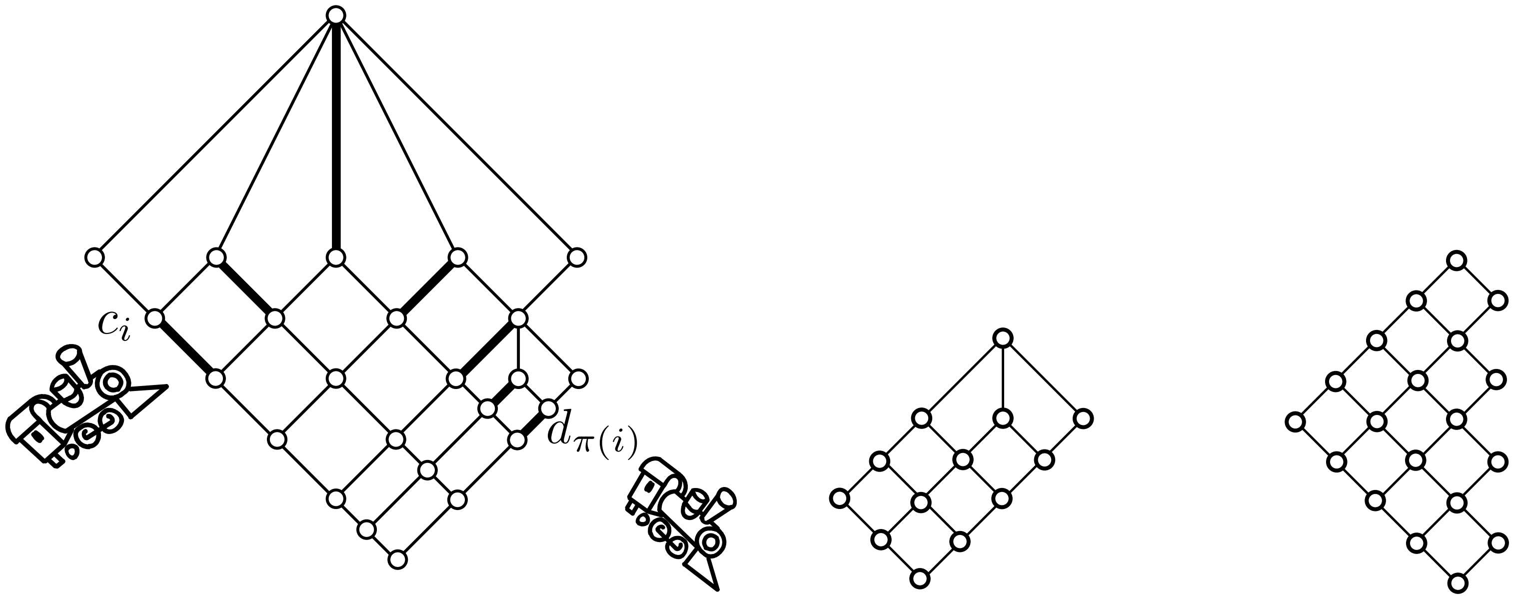
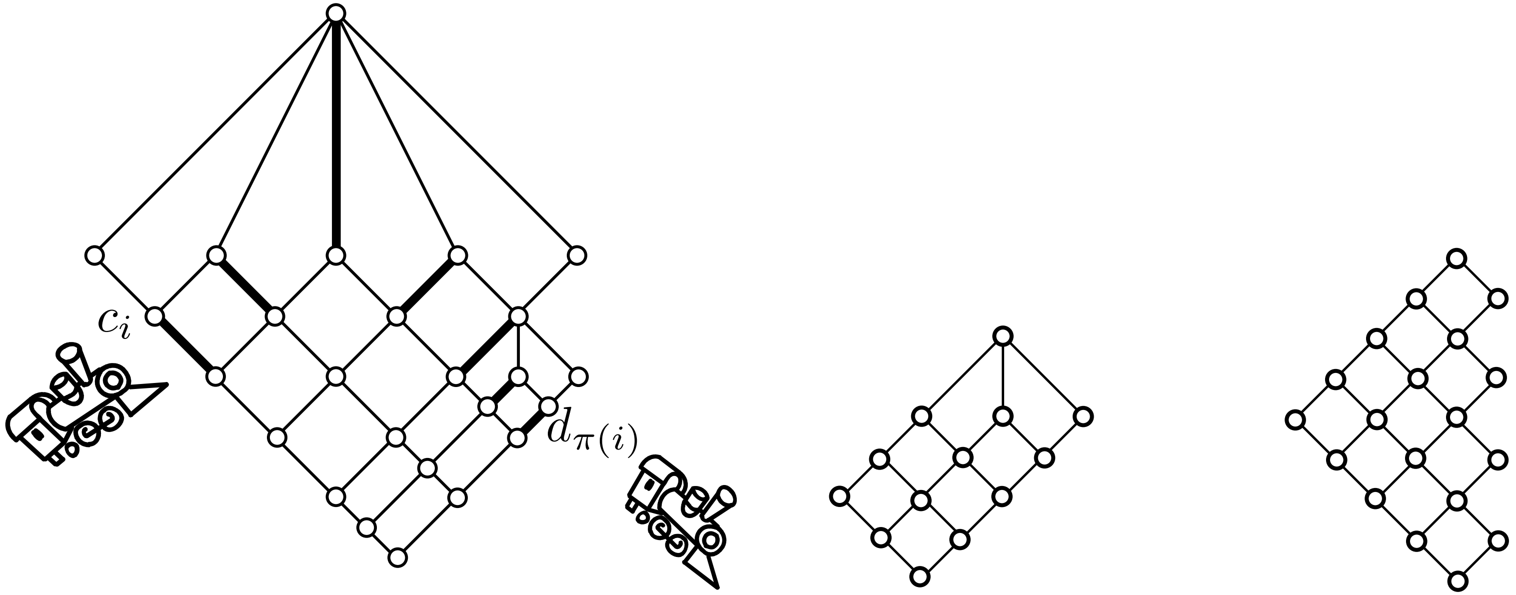
C_3-trajectory Two cover-preserving C_3 are consecutive $\iff$ opposite sides of a cover-preserving $C_3 \times C_2$. Same properties.



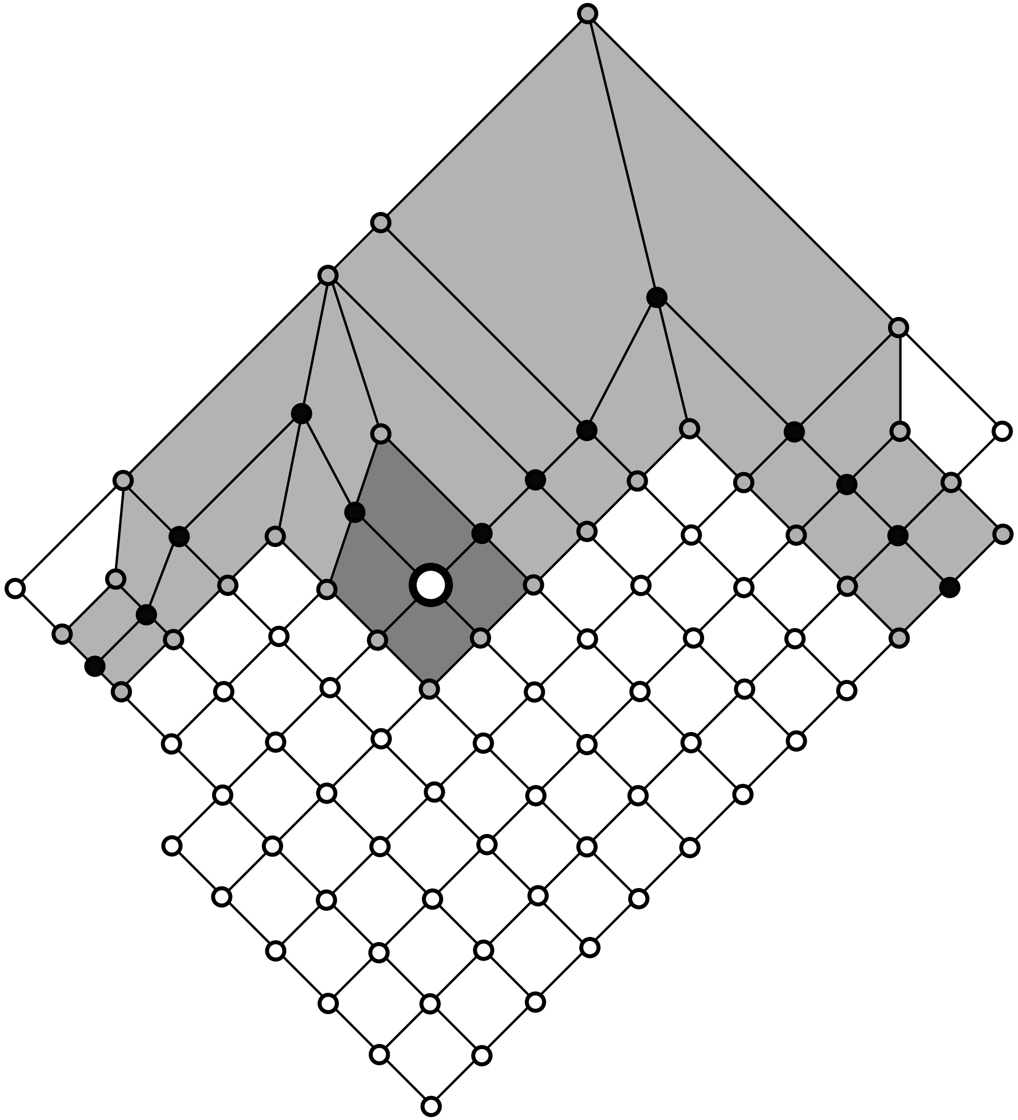
The left and right ends of a C_2-trajectory are on the boundary.

The *elements*

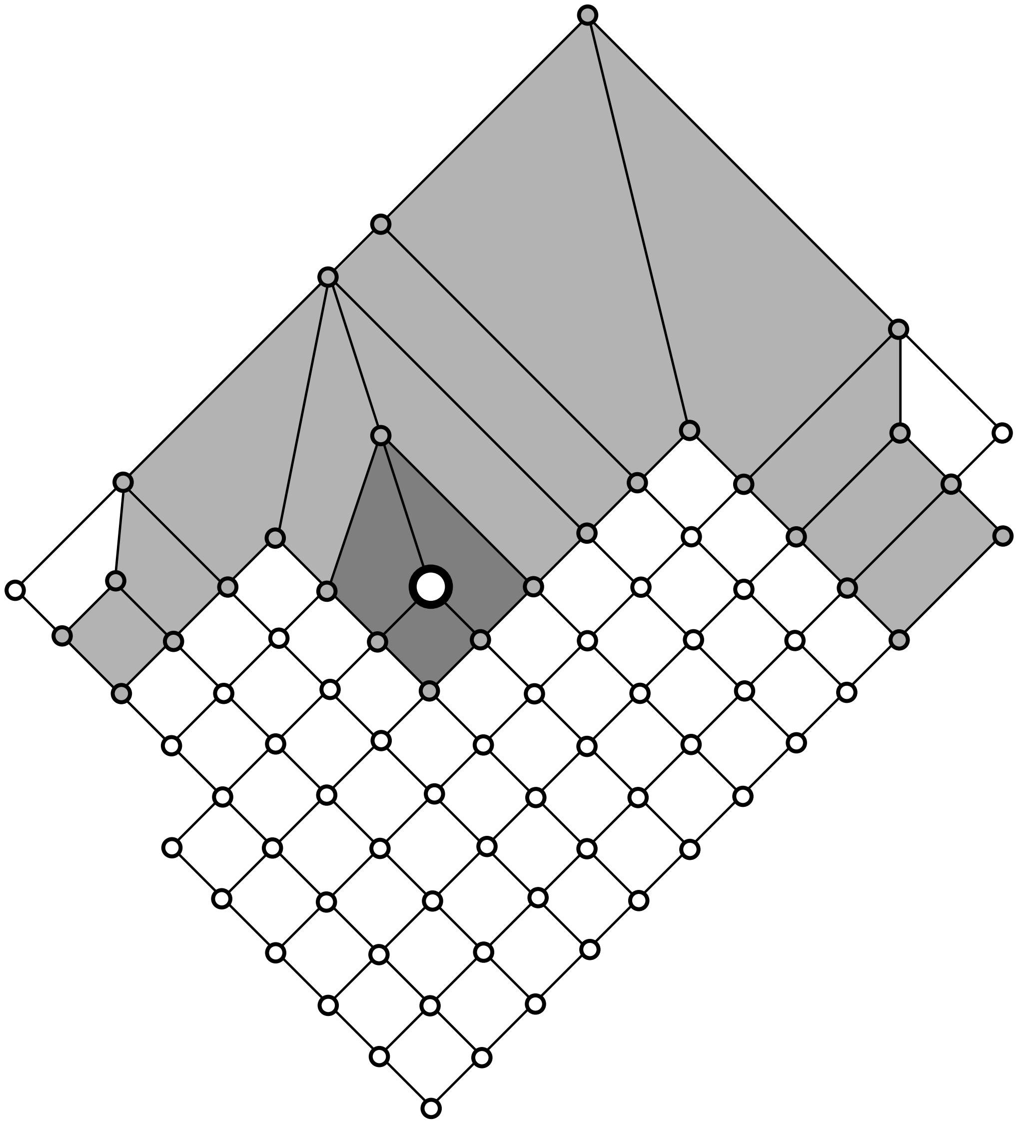


The left and right ends of a C_2-trajectory are on the boundary.

The *elements* of a C_i-trajectory are the elements of the C_i-chains forming it. Let A be a cover-preserving C_i-chain in D. By planarity, there is a unique C_i-trajectory through A. The C_i-chains of this trajectory to the left of A and including A form the **left wing** W_l of A. The **right wing** W_r of A is defined analogously.



Resections
(Czédli and Grätzer, 2012).
Let L be a slim and semimodular.
We start with a cover-preserving
$A = \mathsf{C}_3^2$ (dark gray). Assume that
its wings, W_l and W_r, terminate
on the boundary of D.
Delete the two black-filled elements
of A to get an
N_7. Then delete all the
black-filled elements, going up
and down to the left and
to the right, to preserve
semimodularity for the result
of the **resection**.



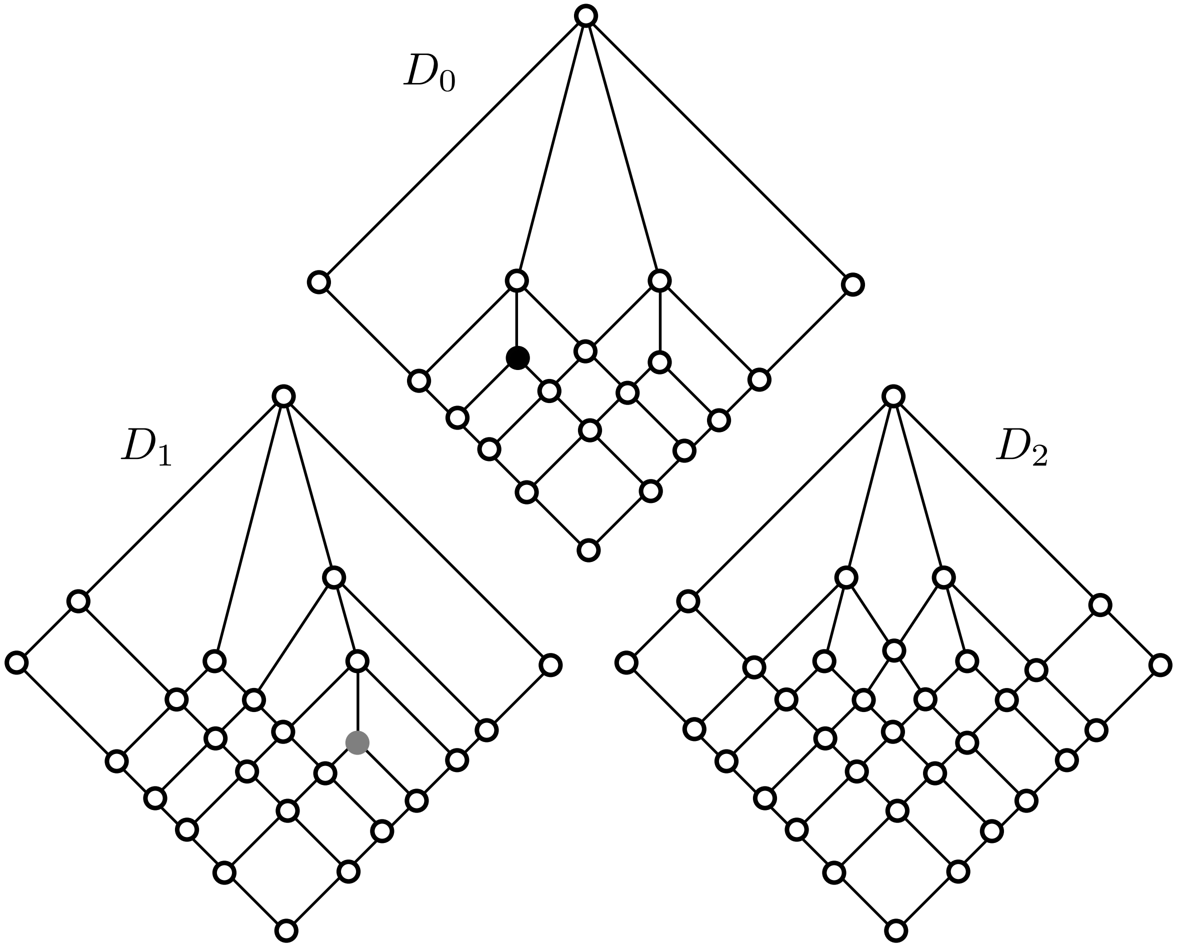
Here is the
result of the resection.

Theorem (GCz-GG). Slim semimodular lattice diagrams are characterized as diagrams obtained from slim distributive lattice diagrams by a sequence of resections.

The inverse procedure is called:
insertion

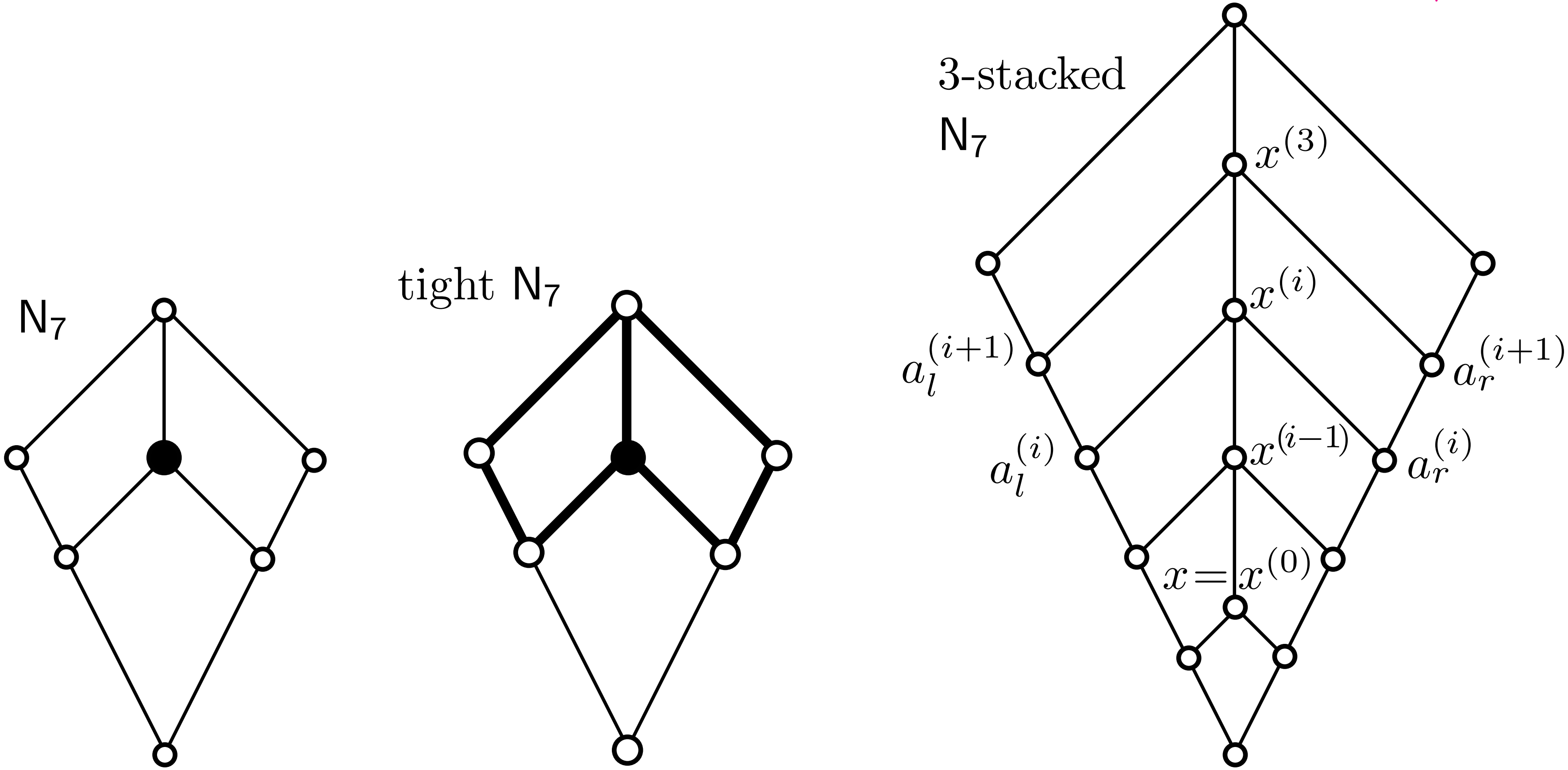
Proof $\Rightarrow$: resection preserves semimodularity (4-cell, same bottom $\Rightarrow$ same top). $\surd$

Conversely: Take a covering N_7, see the previous figure. Perform an insertion at this N_7 to get the earlier figure. Then we have fewer covering N_7-s. Proceed this way until a diagram is obtained without covering N_7-s. Finally, when no covering N_7 remains, we obtain a planar distributive K. Clearly, L is obtained from K by a sequence of resections.

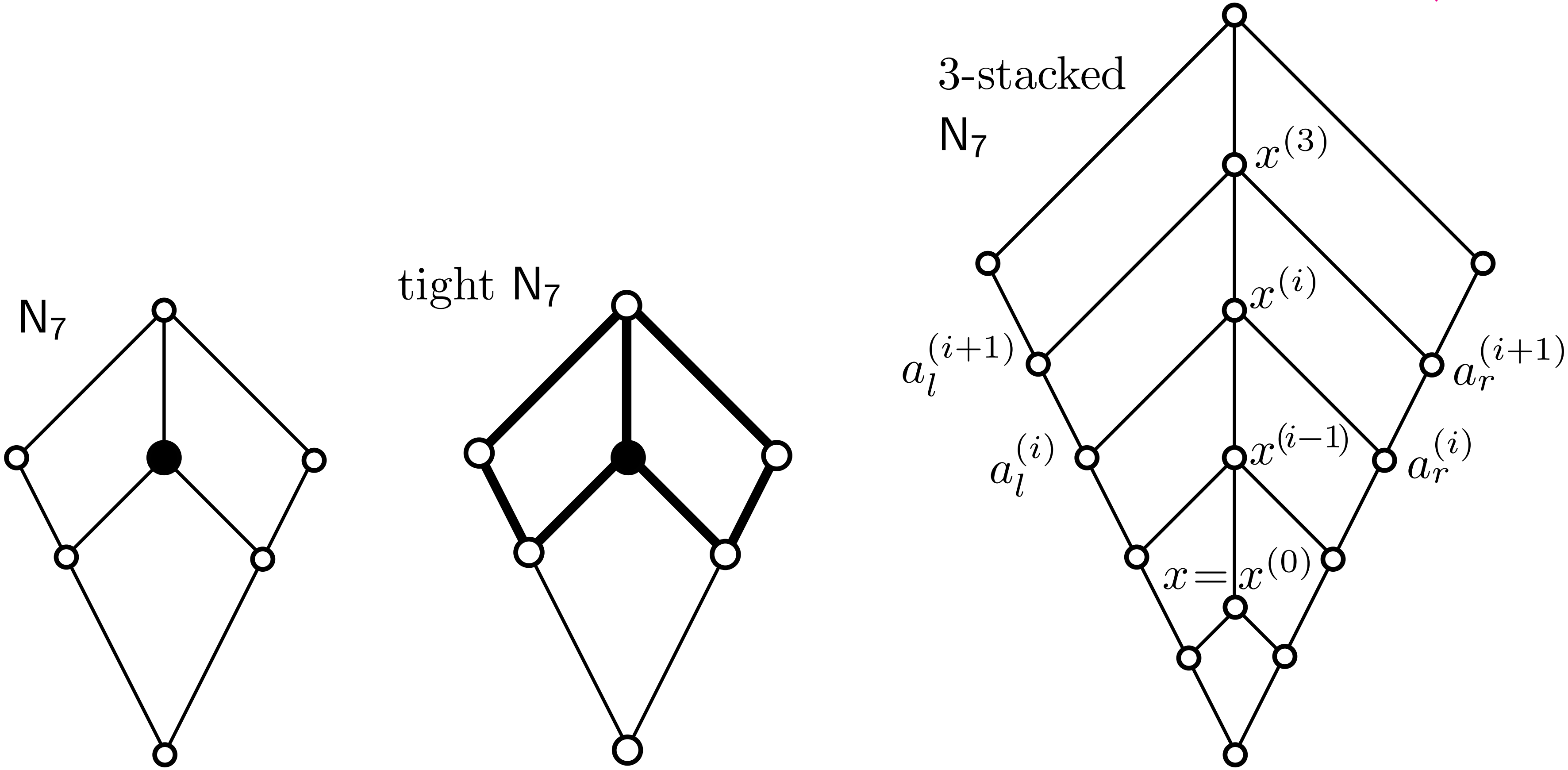


Two covering N_7-s in D_0. After two insertions, there are still two covering N_7-s in D_2. And so on. The number of covering N_7-s is never zero; the illusion of a proof fails!

Sketch of the real proof



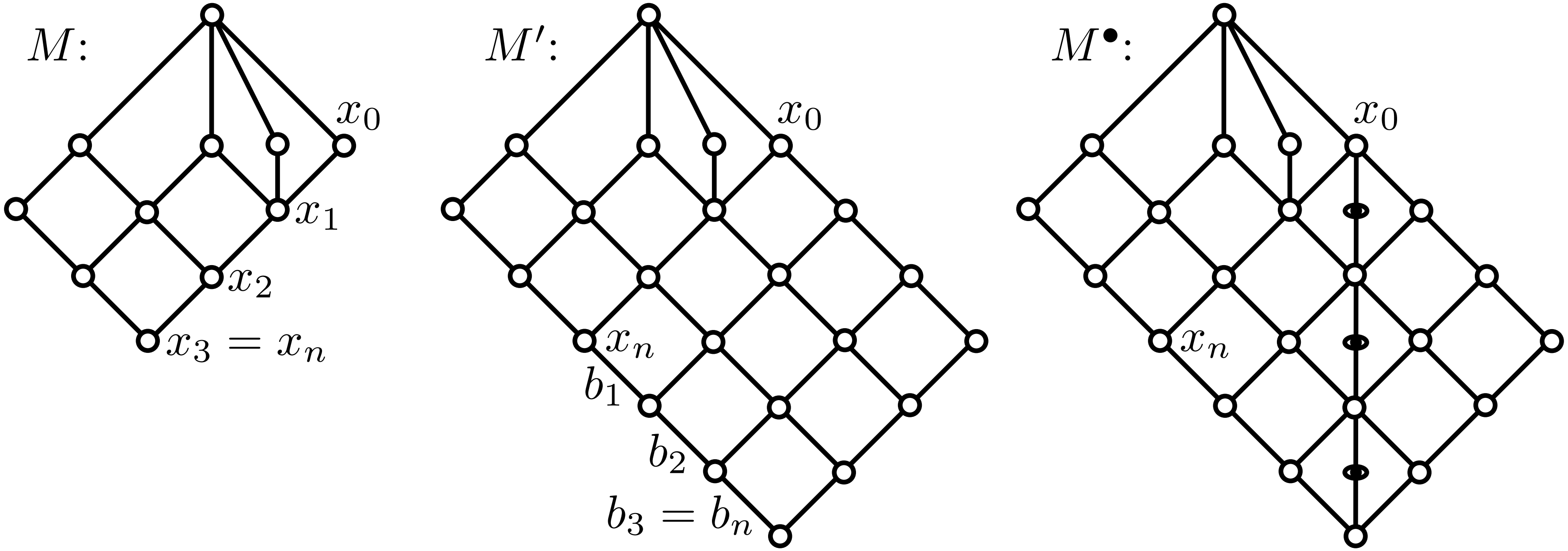
Anchor: the interior element of a covering N$_7$. The **rank** of an anchor x is the largest number t such that there is a tight t-stacked N$_7$ with least inner element x. (In the figure, rank=3.)
L



Anchor: the interior element of a covering N_7. The **rank** of an anchor x is the largest number t such that there is a tight t-stacked N_7 with least inner element x. (In the figure, rank=3.)

Lemma. Perform an insertion at an anchor x, then the rank of x decreases by 1, and no new anchor enters (but the ranks of other anchors may increase.) In particular, if rank$(x) = 0$, then the number of covering N_7-s decreases.

(Remember Grätzer and Knapp's result $|L| = O(n^3)$)



Following G. Grätzer and E. Knapp [2009], a semimodular lattice diagram D is *rectangular* if $\mathrm{C_l}(D)$ has exactly one weak corner, lc(D) and $\mathrm{C_r}(D)$ has exactly one weak corner, rc(D), and these two weak corners are complementary, that is, their meet is 0 and their join is 1. (D

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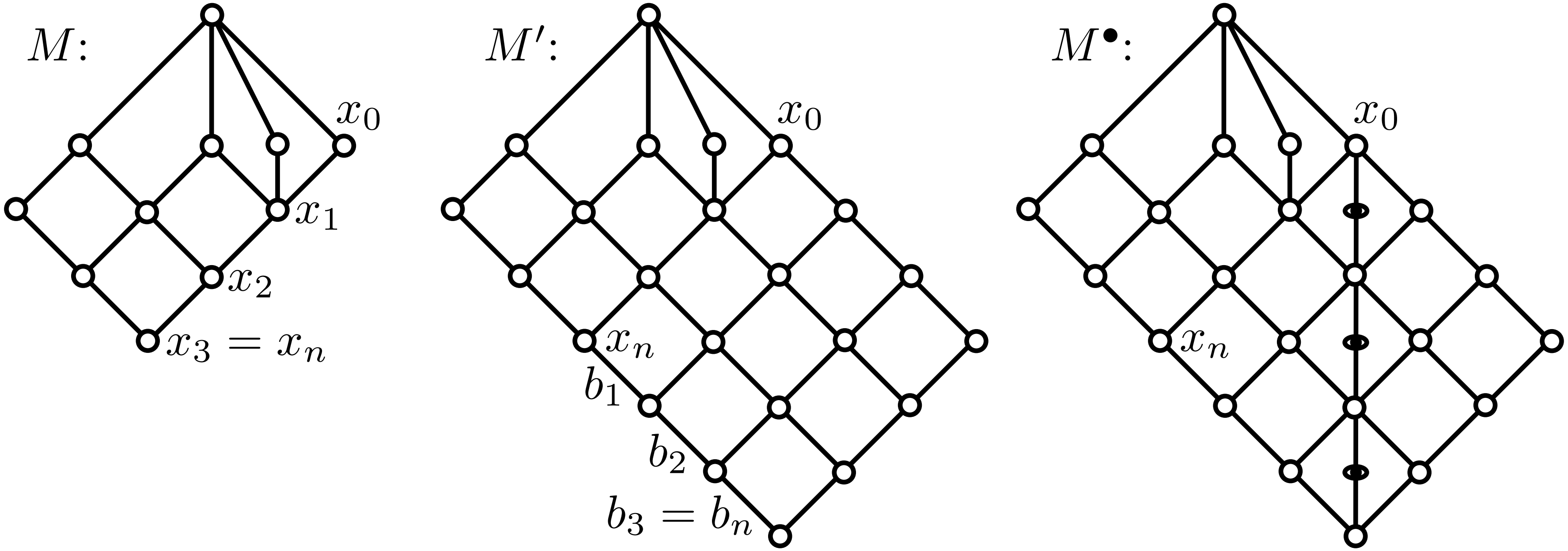

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Rectangular lattices

Czédli-Grätzer, 2012 $66'/34'$

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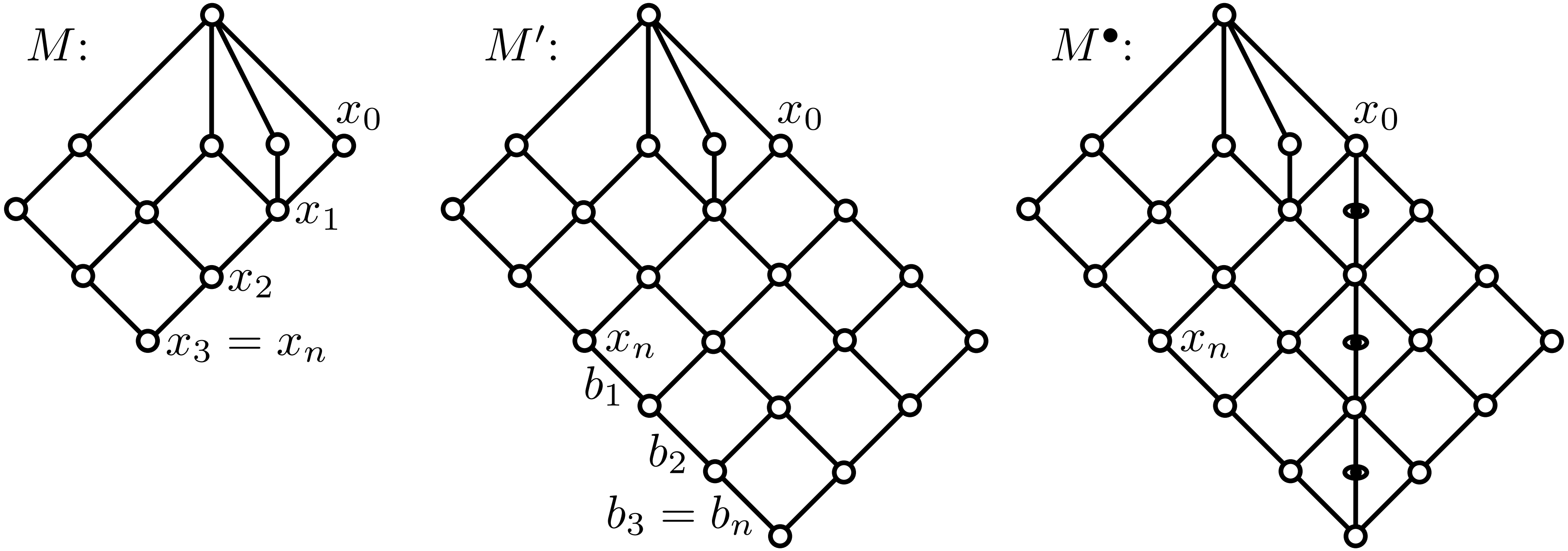

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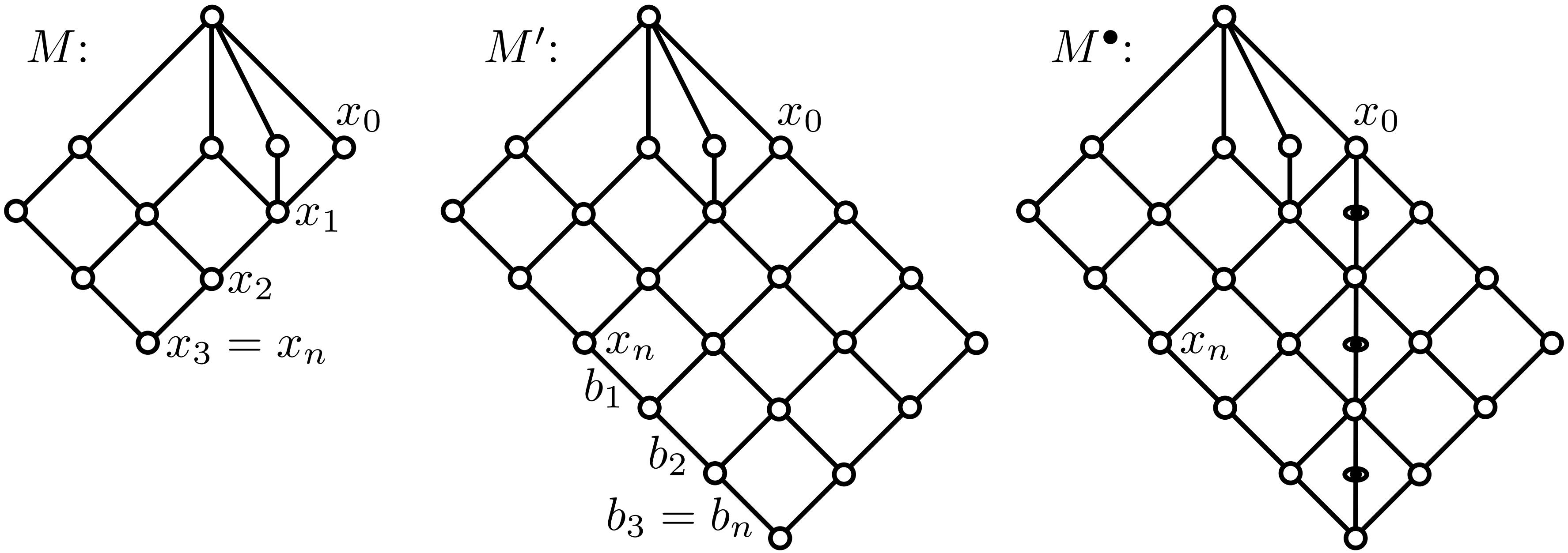
Lemma. The rest (that is, non-corner elements) of the boundary of a rectangular lattice is unique.

Lemma (Grätzer and Knapp [2010]). Let D be a rectangular diagram. Then the intervals $[0, \mathrm{lc(D)}]$, $[\mathrm{lc(D)}, 1]$, $[0, \mathrm{rc(D)}]$, and $[\mathrm{rc(D)}, 1]$ are chains.

So the chains $\mathrm{C_l}(D)$ and $\mathrm{C_r}(D)$ are split into two parts, a lower and an upper part: $\mathrm{C_{ll}}(D) = [0, \mathrm{lc(D)}]$, $\mathrm{C_{ul}}(D) = [\mathrm{lc(D)}, 1]$, $\mathrm{C_{lr}}(D) = [0, \mathrm{rc(D)}]$, and $\mathrm{C_{ur}}(D) = [\mathrm{rc(D)}, 1]$.



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Because they are the building stones of planar semimodular lattices.

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- L is an anti-slimming of a lattice obtained from the four-element Boolean lattice by adding forks.

So, instead of forks and corners, we only need forks and gluings.

[gluing indecomposable $\iff$ antislimming of (C_2^2 + forks)]

Sketch of proof by G. Grätzer. The property (= gluing indecomposable) is invariant with respect to slimming and antislimming. Replacing L by Slim L if necessary, we can assume that L is slim. Not too difficult to show: the property is invariant with respect to adding/deleting corners. Add corners as long as possible. One can show that if no more corner can be added, then L is rectangular. Finally, for a rectangular L, apply this:

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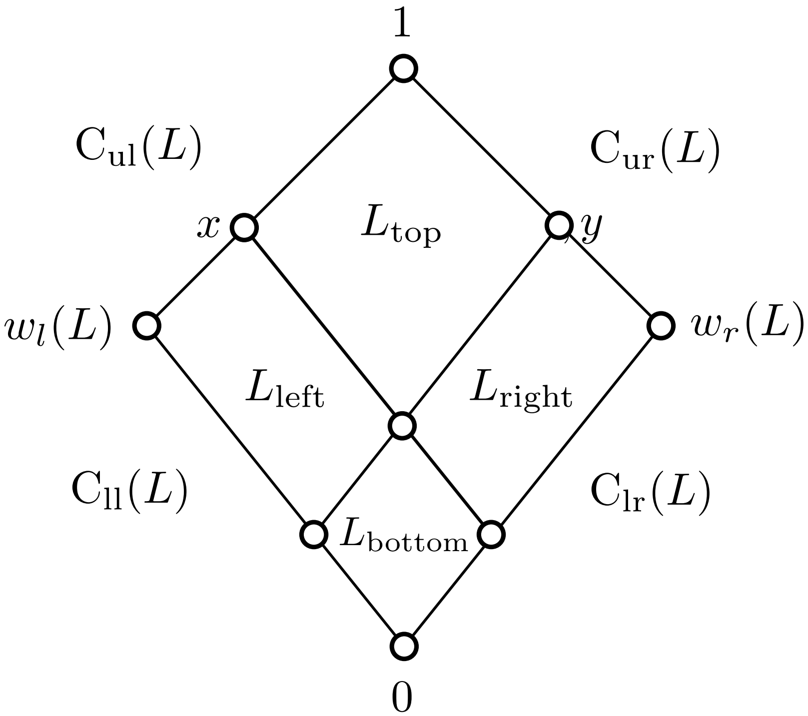


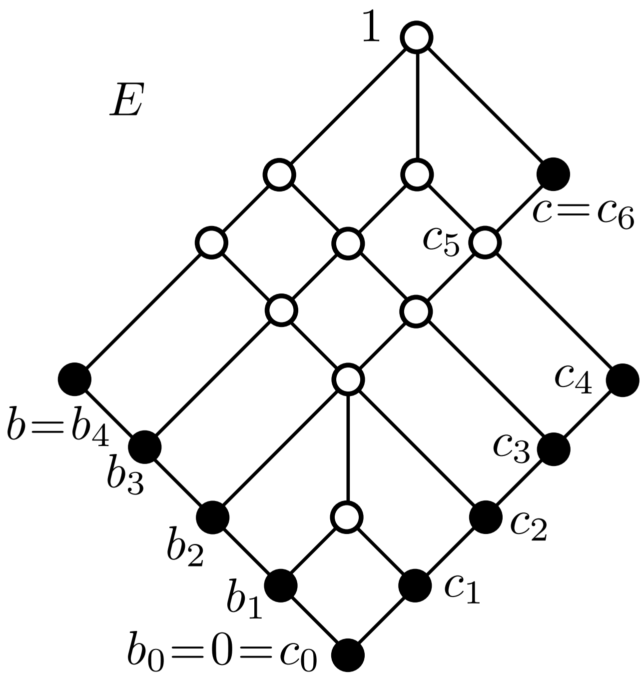
Decomposition Theorem
(Grätzer and Knapp 2010).
If L is rectangular, x is in the open upper-left chain, y is in the open upper-right chain, $x \wedge y$ as indicated, then the four intervals indicated are rectangular, and L can be reconstructed from them by repeated gluing.

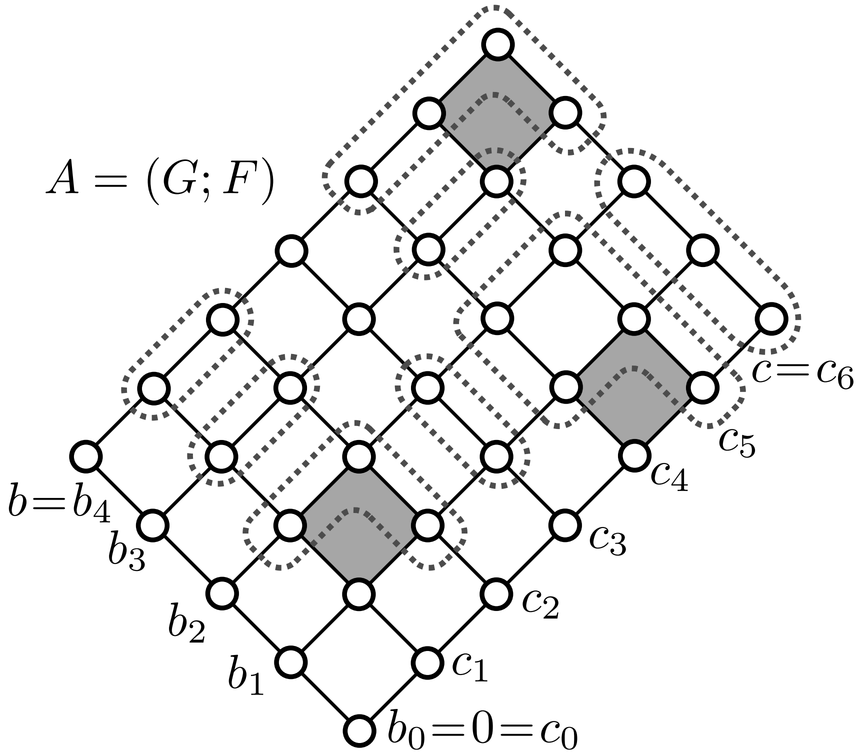
The matrix of a diagram

Czédli-Grätzer, 2012

$80'/20'$

M. Stern (1999?): slim sm lattice = cover-preserving join-homomorphic image of a grid. Minimal grid=?





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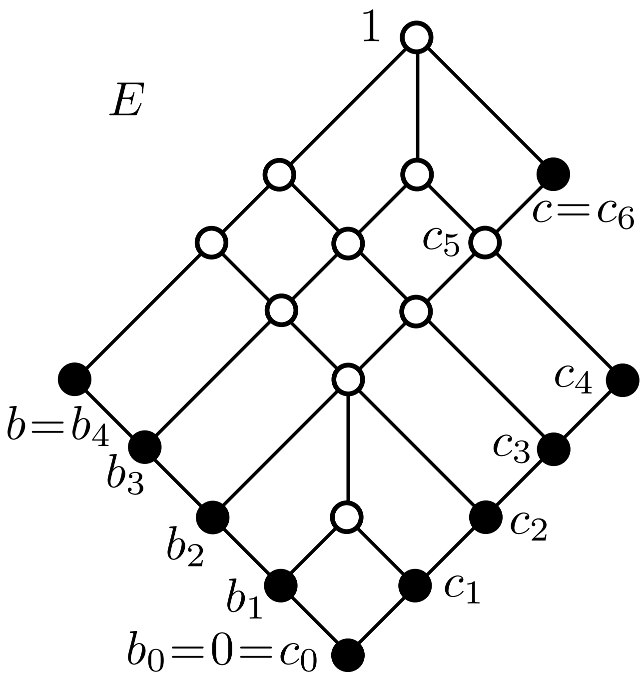

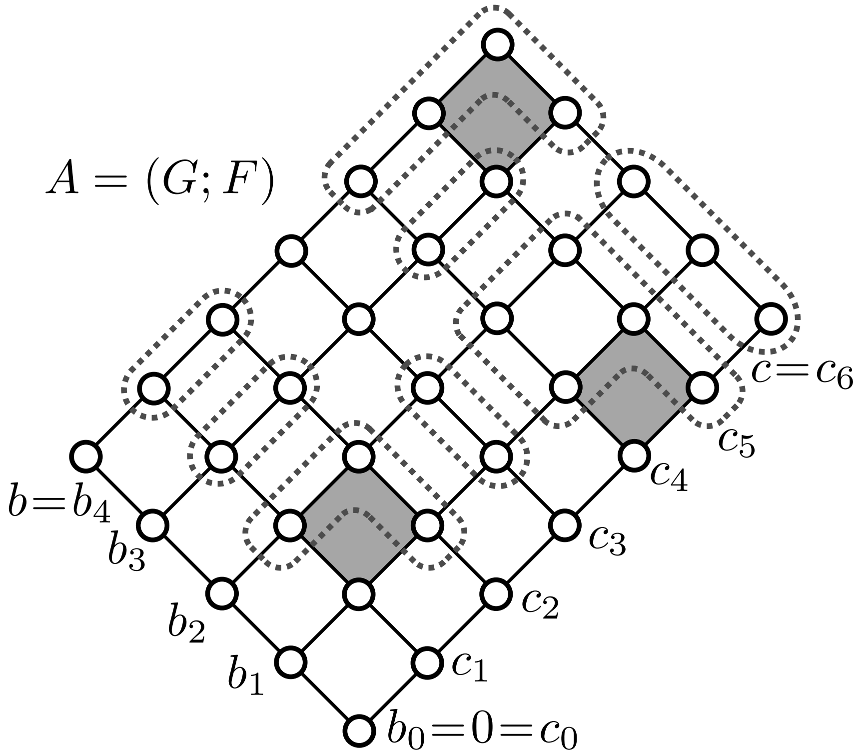

The matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ of a slim sm diagram E.

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An m-by-n 0-1-matrix is **regular** if

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An m-by-n 0-1-matrix is **regular** if

- every row and every column contains at most one unit(=1);

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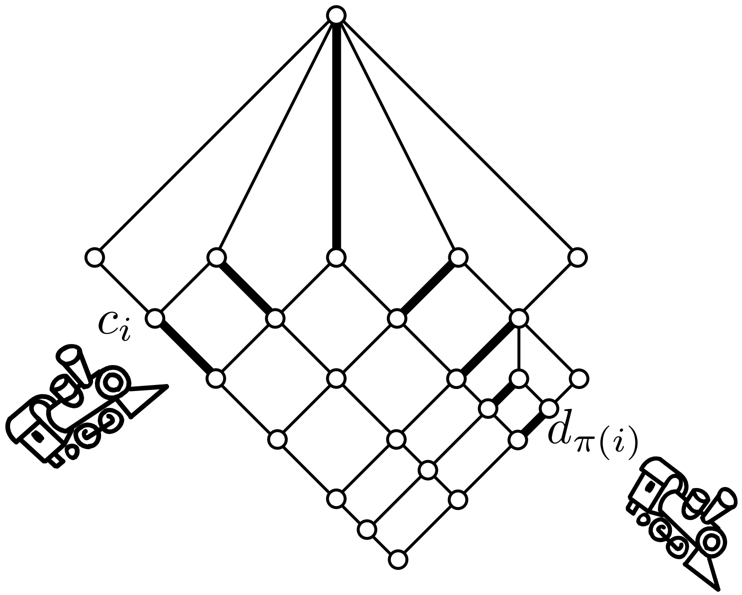
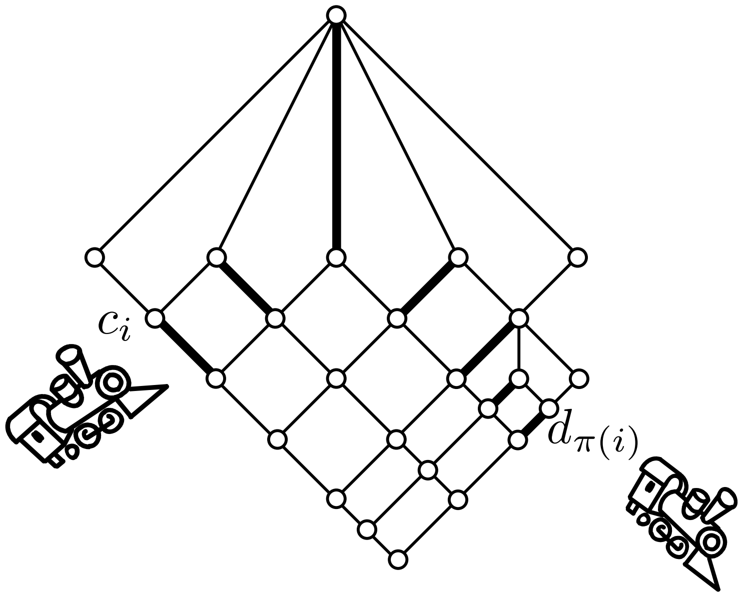
Theorem (Czédli 2012). This gives a bijective correspondence between slim semimodular diagrams E and the so-called **regular matrices**, which are exactly the minimal matrices.



Definition 1.



Definition 1. Czédli and Schmidt (2011): length$(D) = n$; if the i-th prime interval of $\mathrm{C_l}(D)$ and the j-th prime interval of $\mathrm{C_r}(D)$ belong to the same trajectory, then $j = \pi(i)$.

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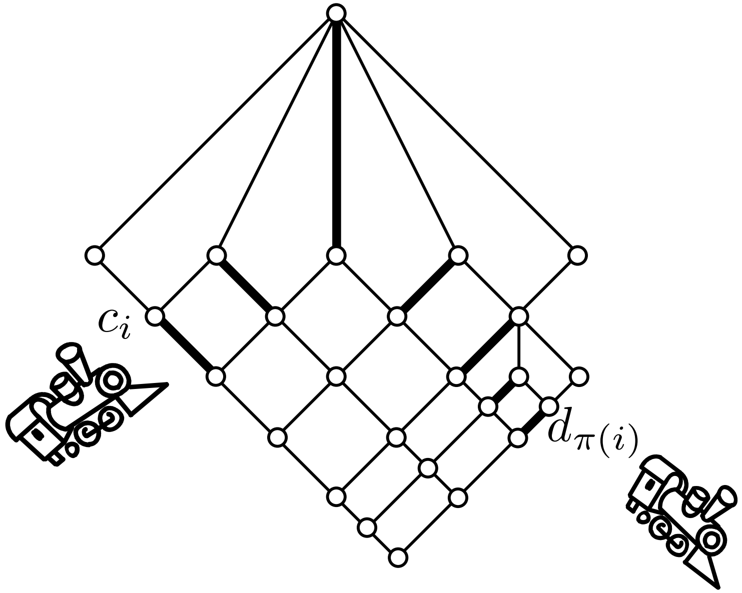
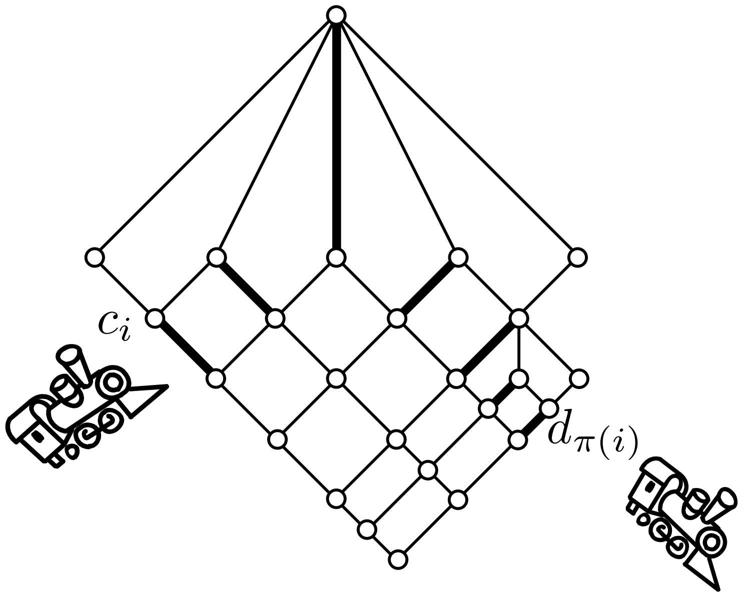
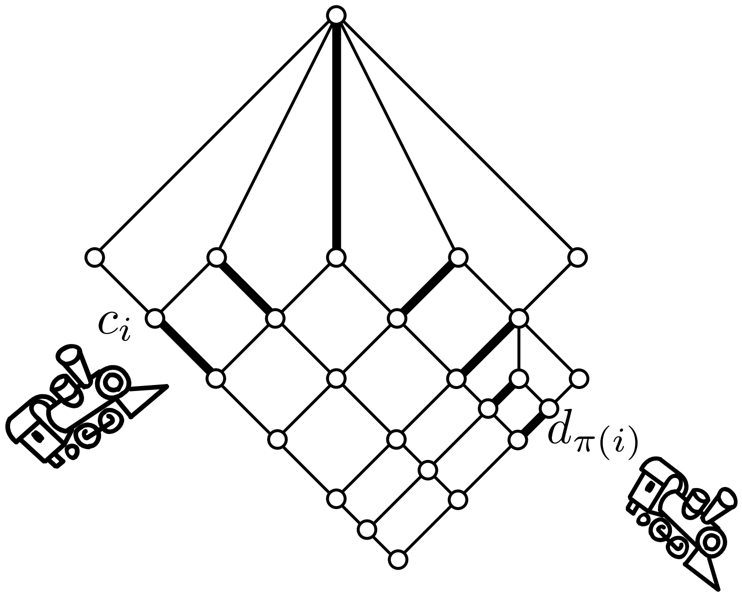


Definition 1. Czédli and Schmidt (2011): length$(D) = n$; if the i-th prime interval of $\mathsf{C}_{\mathsf{l}}(D)$ and the j-th prime interval of $\mathsf{C}_{\mathsf{r}}(D)$ belong to the same trajectory, then $j = \pi(i)$.

This efinition is the most useful one for us. The concept of these permutations was already known, but defined differently, by R. Stanley (1972) and H. Abels (1991).



Definition 2 and a Lemma. (Abels 1991, Czédli and Schmidt 2012, Czédli, Ozsvárt and Udvari 2012). Denote
$\mathsf{C}_{\mathsf{l}}(D) = \{0 = b_0 \prec b_1 \prec \cdots \prec b_n = 1\}$,
$\mathsf{C}_{\mathsf{r}}(D) = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$.
For $i, j \in \{1, \ldots, n\}$, let
$\pi(i) = \min\{j \in \{1, \ldots, n\} \mid b_{i-1} \vee c_j = b_i \vee c_j\}$ and
$\sigma(j) = \min\{i \in \{1, \ldots, n\} \mid b_i \vee c_{j-1} = b_i \vee c_j\}$.



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Lemma. $\pi, \sigma \in S_n$, and $\sigma = \pi^{-1}$.

Definition 3 (Czédli and Schmidt 2012). The elements of $\mathsf{C}_\mathsf{l}(D)$ and $\mathsf{C}_\mathsf{r}(D)$ are denoted as before. For $u \in \mathrm{Mi}\, D$, let b_i be the smallest element of $\mathsf{C}_\mathsf{l}(D)$ such that $b_i \nleq u$,

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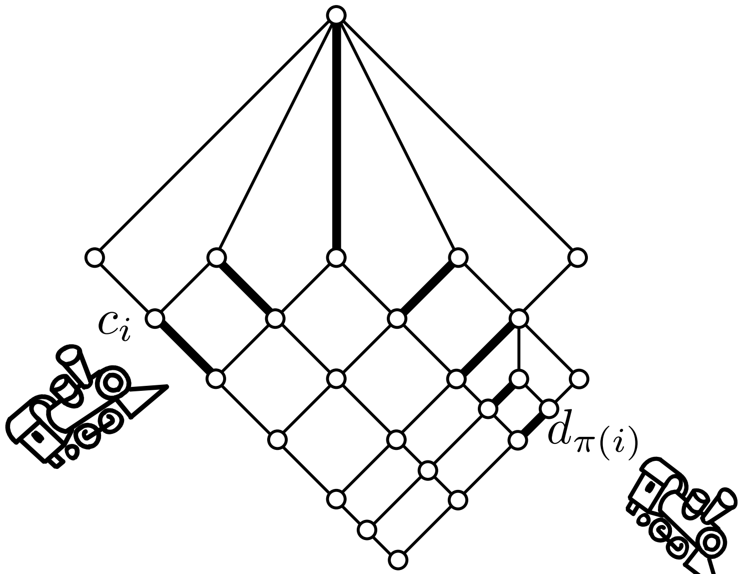
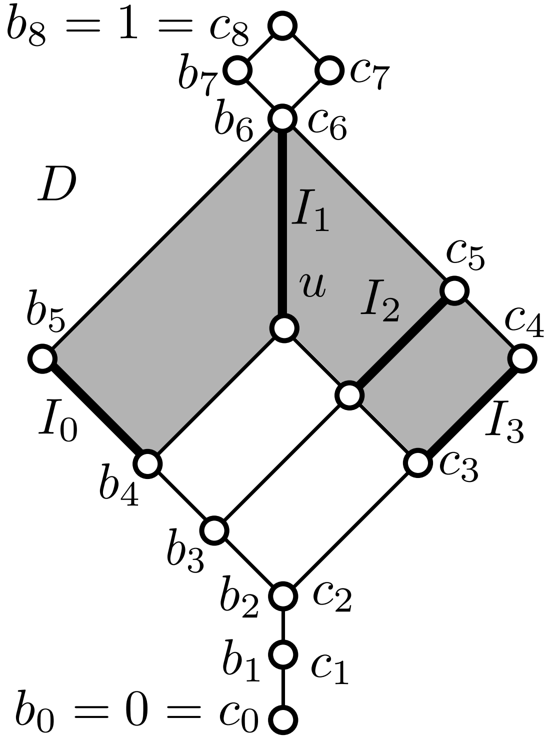
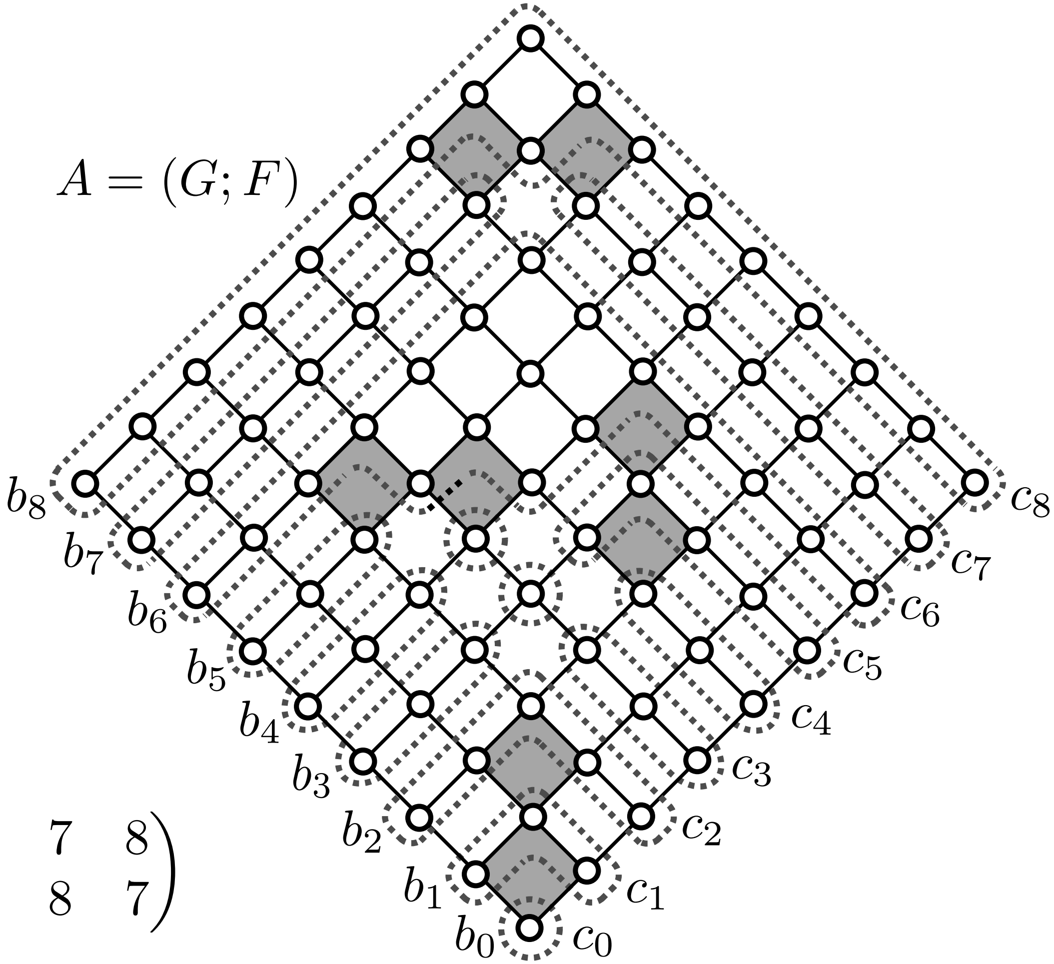


Lemma π is a permutation.

Corollary: $\text{length}(L) = |\operatorname{Mi} L|$, provided L is slim and semimodular.

Lemma. The three definitions give the same permutation.





$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 4 & 3 & 8 & 7 \end{pmatrix}$$

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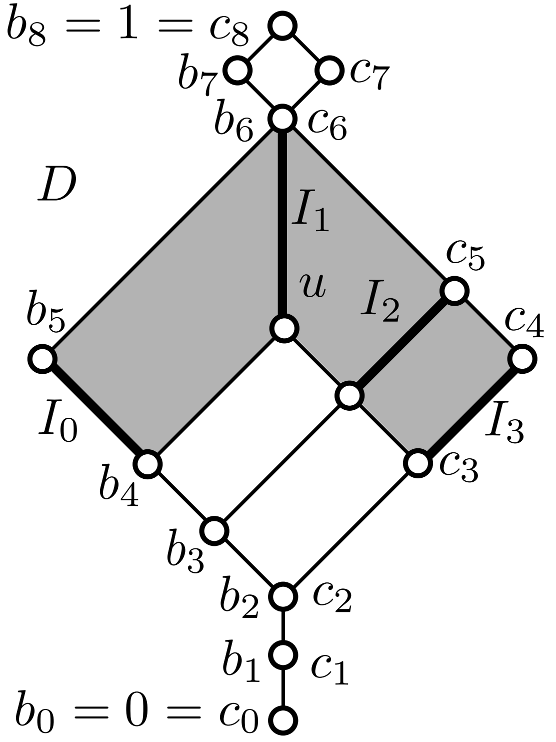
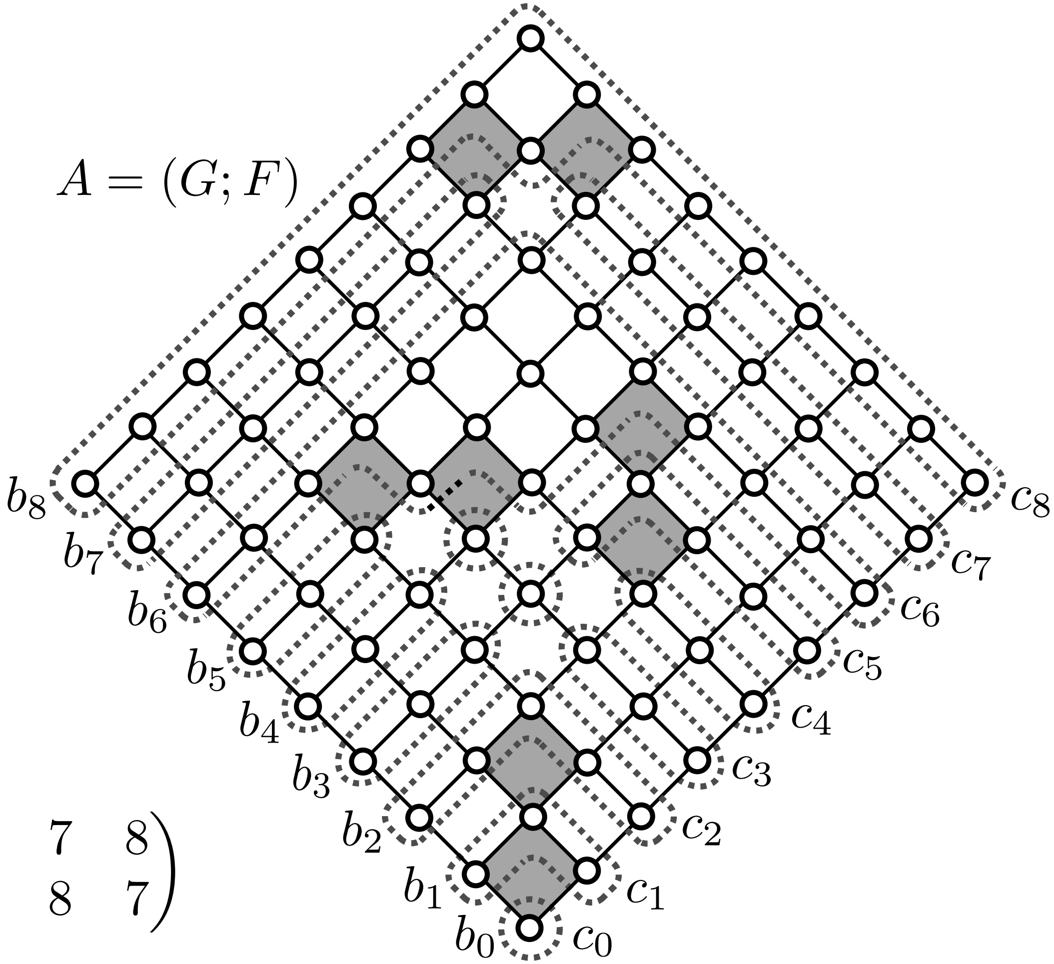
permutation $\mapsto$ diagram





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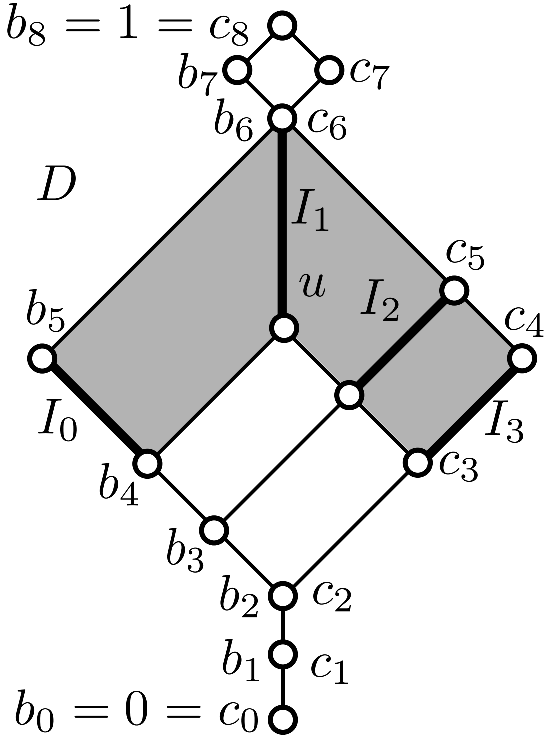

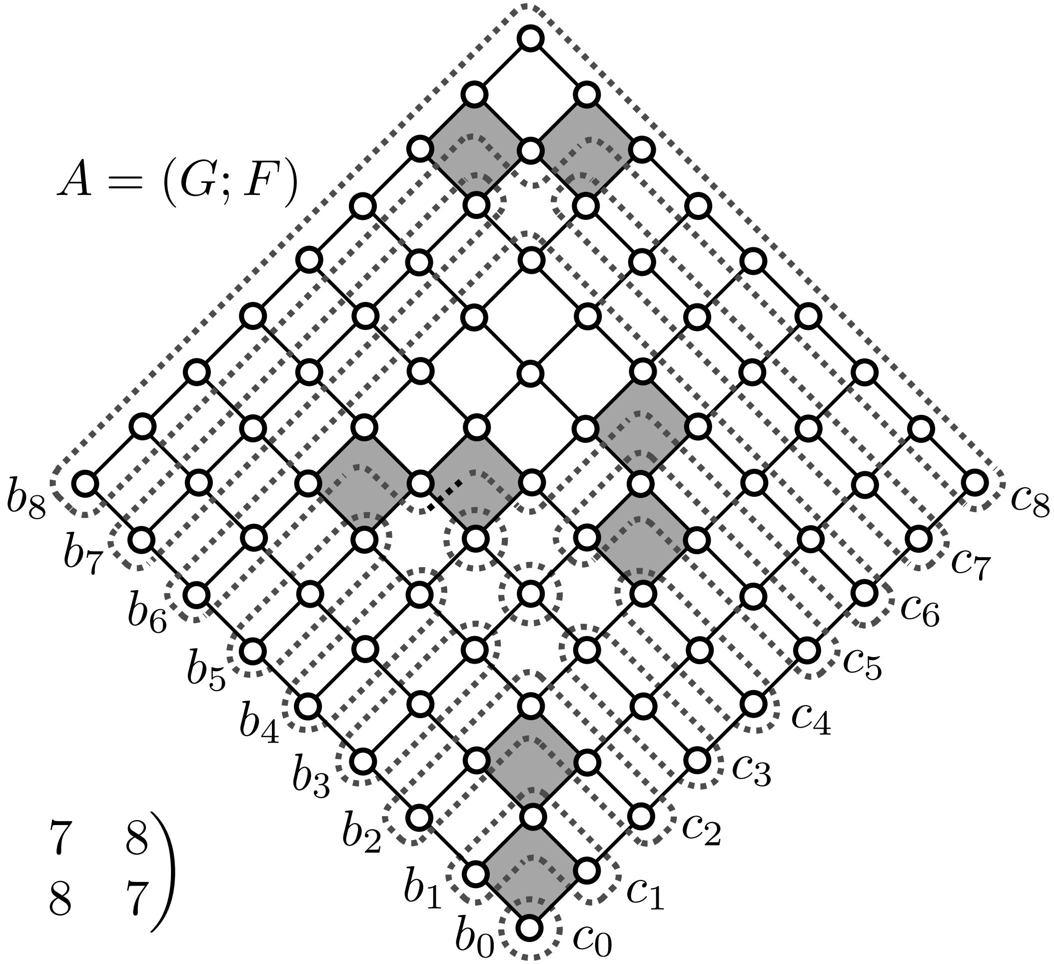

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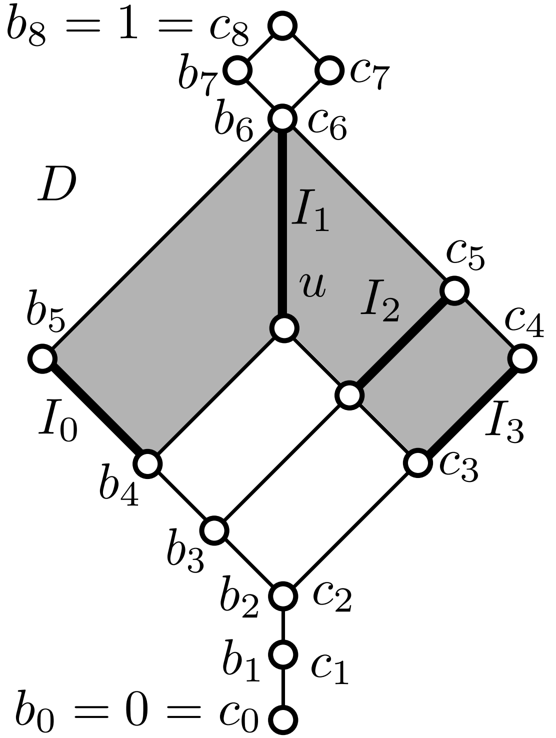

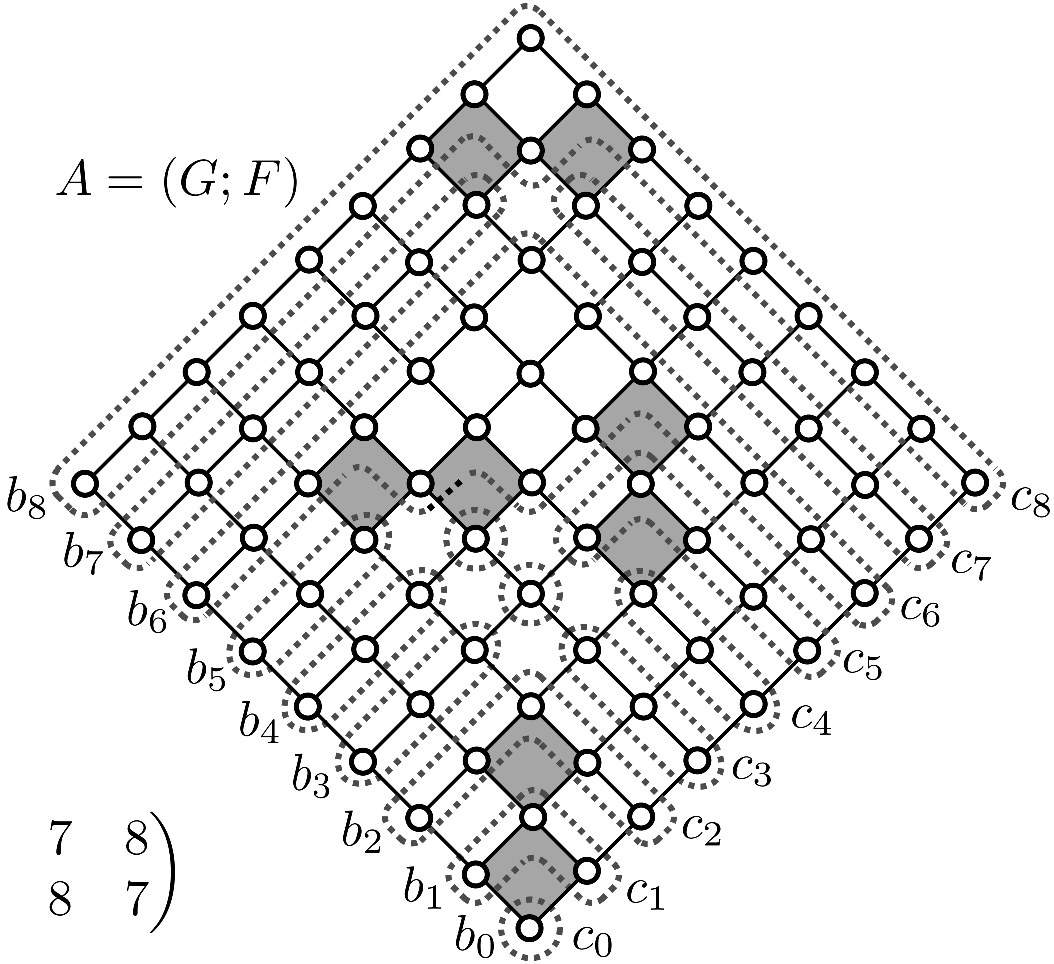

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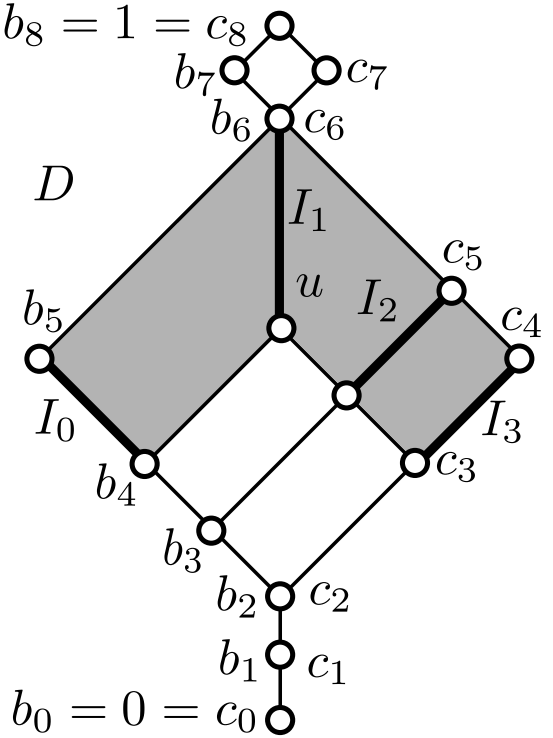
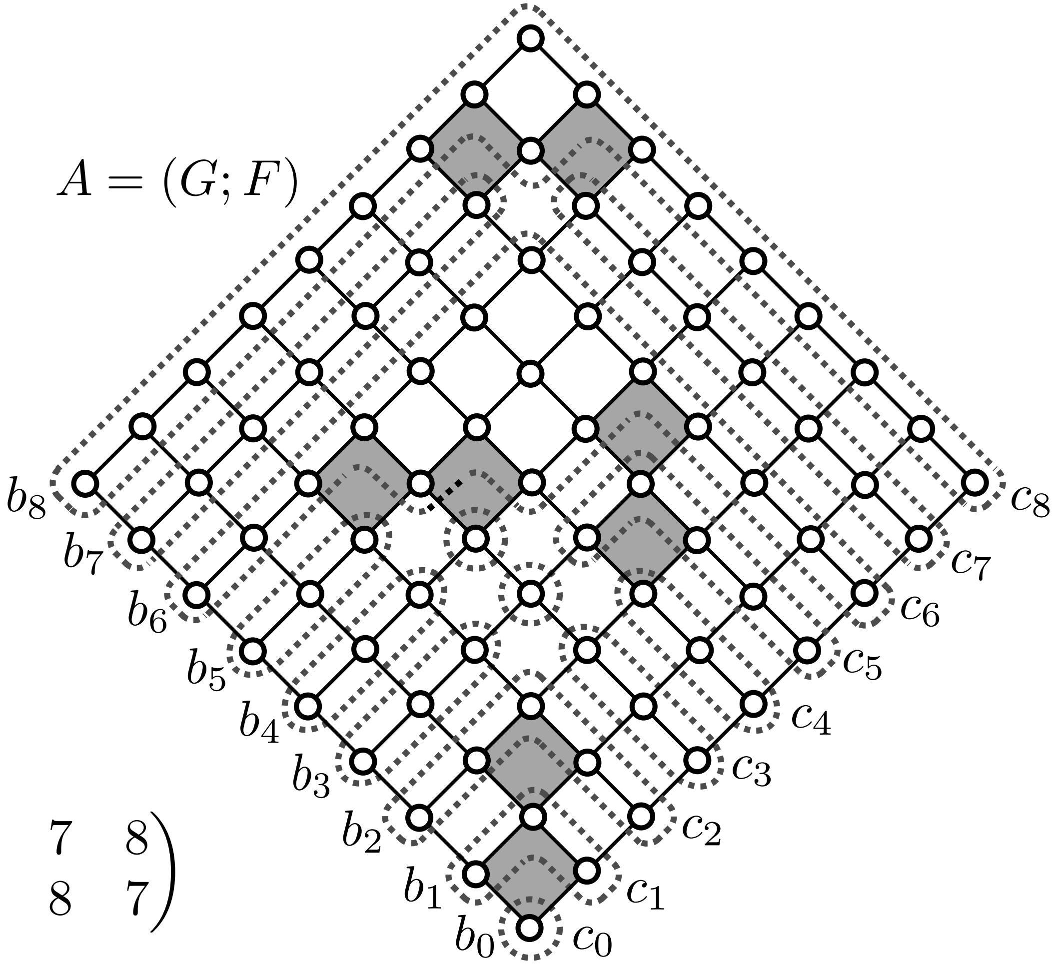
Theorem (A





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Theorem (Abels 1991, Czédli and Schmidt 2012). The described relation between slim, semimodular, planar diagrams of length n and S_n is a **bijection**.

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http://www.math.u-szeged.hu/~czedli/
http://server.math.umanitoba.ca/homepages/gratzer/