PLANAR SEMIMODULAR LATTICES: STRUCTURE AND DIAGRAMS

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1. Foreword

Our conference talk is based on our joint chapter "Planar Semimodular Lattices: Structure and Diagrams" in the monograph "Lattice Theory: Special Topics and Applications", edited by George Grätzer and Fred Wehrung; we hope that it appears next year. This book chapter is ready; only some new references have to be added or updated.

Since the whole chapter would be too long for a 2-hour conference talk, and since I totally ran out of time, it is me (=G. Cz.) who extracts this lecture note from the chapter. The extracting procedure may create and the Foreword may contain some ungrammatical sentences; it is only me who is responsible for all errors.

From several aspects, planar semimodular lattices are quite easy mathematical objects. Thus one can ask the question why they are important. To give an answer, hopefully only a partial answer, we mention three recent results below; each of them is based on a proper understanding of (slim) planar semimodular lattices.

1. A sharp result on congruence lattice representation. For every result representing a finite distributive lattice D with n join-irreducible elements as the congruence lattice of a finite lattice L in some class **K** of lattices, the natural question arises: How small can we make L as a function of n and **K**?

There are only two results of this type in the literature. For the first result, **K** is the class of all lattices (no restriction on L). It was proved in G. Grätzer, I. Rival, and N. Zaguia [24] that $|L| = O(n^2)$ is best possible in this case.

For the second result, that is the result relevant here, **K** is the class of rectangular lattices, to be defined later. Note that these lattices are planar and semimodular. For this case, it was proved in G. Grätzer and E. Knapp [20] and [21] that $|L| = O(n^3)$ is best possible.

2. Jordan-Hölder theorem. Based on a proper understanding of planar semimodular lattices, we could strengthen the 140 year old Jordan-Hölder theorem for groups. Although this result is not as deep as the previous one, it sells (or should sell) well outside Lattice Theory. The details will be given later.

3. Dropping planarity. Now that planar semimodular lattices are more or less understood, there is a hope that we can drop planarity from our assumptions. See E. T. Schmidt's home page, `http://www.math.bme.hu/∼schmidt/`, his "unpublished papers" there, for a lot of ideas. Also, we can mention G. Czédli [9], where the lattices corresponding to antimatroids and dually corresponding to convex geometries are coordinatized. This result leads to coordinatizations of point-separating antimatroids and convex geometries, so it may sell well even in Combinatorics. The details will not be discussed.

Rediscoveries. A small part of the results presented here are rediscoveries of previously known things. However, the present approach is justified by the fact that the "proper understanding", which was necessary to achieve the above-mentioned results, was not available before.

Slim semimodular lattices are exactly the join-distributive lattices with convex dimension at most 2. Join-distributive lattices (their duals, to be precise) were introduced and studied by R. P. Dilworth [17] in 1940. There are quite many equivalent definitions for these lattices, see [9] for an overview; so it is not a surprise that they were discovered many times, see Monjardet [30].

The permutations on which our approach to the Jordan-Hölder theorem is based were discovered by R. P. Stanley [34], and these permutations are in connection with geometry and Coxeter groups, see Armstrong [4].

2. INTRODUCTION

While the study of *planar* lattices goes back to the 1970-s (K. A. Baker, P. C. Fishburn, and F. S. Roberts [5] and D. Kelly and I. Rival [28]), a systematic study of *planar semimodular* lattices began only in 2007 (G. Grätzer and E. Knapp [19]–[21] and G. Grätzer and T. Wares [25]). This was followed by G. Czédli and E. T. Schmidt [14], [15], [16], and G. Czédli [6]. This chapter presents some of the results in these papers.

Many properties of planar semimodular lattices are properties of their planar diagrams; we emphasize this point of view in this chapter.

In Section 3, we develop the basic concepts of planarity of lattices and diagrams. Slim lattices are introduced in Section 4. In Section 5, we introduce a construction of planar semimodular lattices from planar distributive lattices by inserting forks. The twin construction, using resections, is presented in Section 6. Rectangular lattices form an important subclass of planar semimodular lattices; we specialize the results of the previous two sections to this subclass in Section 7. Slim semimodular lattices can be described by 0-1-matrices, as described in Section 8. They can also be described by permutations, see Section 9. Finally, in Section 10, we present variants of the Jordan-Hölder Theorem.

Conventions. A planar lattice or a planar diagram is finite by definition. Hence, unless otherwise stated, all lattices and diagrams are assumed to be *finite*. Lattice *properties* are also used for diagrams in a self-explanatory way. For example, for "a planar diagram D of a semimodular lattice", we write "a planar semimodular diagram D". A diagram of a planar lattice is a *planar diagram*. If D is a planar diagram of a planar lattice L, then $x \in D$ and $x \in L$ have the same meaning. We use $x \in D$ to indicate that the context is D, for instance, for the left boundary.

3. Planarity and diagrams

Just as for planar geometry, our geometric intuition regards many statements about planar lattices as obvious. However, sometimes it is not so easy to provide proofs. In this section, we discuss several such statements; most of them can be found in D. Kelly and I. Rival [28].

Unless otherwise stated, semimodularity is not assumed in this section.

A finite lattice L is *planar* if it has a *planar diagram*, that is, a diagram in which edges can be incident only at their endpoints. For a lattice L, the set of *planar diagrams* of L will be denoted by $\mathrm{Dgr}(L)$; this set is nonempty iff L is a planar lattice.

To make the definition of a planar diagram more precise, let $\mathbb{R}$ be the field of real numbers, so $\mathbb{R}^2$ is the plane.

Here is a formal definition of planar diagrams from D. Kelly and I. Rival [28]:

Definition 3.1. A *planar diagram D* of a finite lattice L is a pair $D = (\varphi, E)$ with the following three properties:

  (i) φ is a one-to-one map of L into $\mathbb{R}^2$ such that if $a < b$ in L and $\varphi(a) = (a_1, a_2)$, $\varphi(b) = (b_1, b_2)$, then $a_2 < b_2$;
  (ii) E is the set of line segments between $\varphi(a)$ and $\varphi(b)$ for all $a \prec b$ in L;
  (iii) two distinct line segments of E are not incident except possibly at their endpoints.

The elements of E are called the *edges* of the diagram.

Next, we recall some basic concepts from D. Kelly and I. Rival [28]. Using Definition 3.1, it is easy to formally define them.

A planar diagram D of a lattice L has a *left boundary chain* $\mathrm{C_l}(D)$, a *right boundary chain* $\mathrm{C_r}(D)$, and a *boundary* $\mathrm{Bnd}(D) = \mathrm{C_l}(D) \cup \mathrm{C_r}(D)$.

If C is a maximal chain of L, then it has a *left side*, $\mathrm{LS}(C, D)$ ($\mathrm{LS}(C)$, for short), and a *right side*, $\mathrm{RS}(C, D)$ ($\mathrm{RS}(C)$, for short). Observe that

$$L = \mathrm{LS}(C) \cup \mathrm{RS}(C),$$
$$C = \mathrm{LS}(C) \cap \mathrm{RS}(C).$$

Assume that $a \le b$ in L and $D \in \mathrm{Dgr}(L)$. Let $D_{a,b}$ be the restriction of the diagram D to $[a, b]$. Let C_1 and C_2 be maximal chains of $[a, b]$ such that $C_1 \subseteq \mathrm{LS}(C_2, D_{a,b})$ and $C_2 \subseteq \mathrm{RS}(C_1, D_{a,b})$. Then

$$R = \mathrm{RS}(C_1) \cap \mathrm{LS}(C_2)$$

is a *region* of D. It is a convex sublattice, $\mathrm{C_l}(R, D_{a,b}) = C_1$, and $\mathrm{C_r}(R, D_{a,b}) = C_2$.

A minimal non-chain region is called a *cell*, a four-element cell is a *4-cell*; it is also a *covering square*, that is, cover-preserving four-element Boolean sublattice of L. A diagram of $\mathrm{M_3}$ has exactly two 4-cells and three covering squares. A 4-cell A of D consists of its bottom, 0_A, top, 1_A, *left corner*, $\mathrm{lc}(A)$, and *right corner*, $\mathrm{rc}(A)$. (Upper case acronyms define sets, lower case acronyms, elements.)

A planar lattice diagram is called a *4-cell diagram* if all of its cells are 4-cells. A planar lattice L is a *4-cell lattice* if it has a 4-cell diagram. Equivalently, if *all* planar diagrams of L are 4-cell diagrams. For example, $\mathrm{M_3}$ is a 4-cell lattice but $\mathrm{N_5}$ is not.

Recall that $\mathrm{Ji}\, L$ is the order of non-zero join-irreducible elements of L, and $\mathrm{Mi}\, L$ is defined dually. Finally, $\mathrm{Di}\, L$ is the order of doubly-irreducible elements of L

The following lemma is in D. Kelly and I. Rival [28, Lemmas 1.2 and 1.5 and Proposition 2.2].

◊**Lemma 3.2.** *For a planar lattice L, let $D \in \mathrm{Dgr}(L)$, and let C be a maximal chain of L.*

  (i) *If $x, y \in D$ are on different sides of C and $x \leq y$, then there is an element $z \in C$ with $x \leq z \leq y$. In particular, if $x \prec y$, then they cannot be on different sides of C outside of C.*
 (ii) *Every interval of L is a region of D.*
(iii) *$|L| \geq 3$, then there are doubly irreducible elements in $\mathrm{C_l}(D)$ and $\mathrm{C_r}(D)$.*

The following statement follows easily from Lemma 3.2.

◊**Lemma 3.3.** *Let R be a region of $D \in \mathrm{Dgr}(L)$.*

  (i) $\mathrm{int}(R) \subseteq \mathrm{int}(L)$.
 (ii) *If $u < v$ in L and $|R \cap \{u, v\}| = 1$, then $[u, v] \cap \mathrm{Bnd}(R)$ is nonempty.*
(iii) *If $x \in \mathrm{int}(R)$, then all upper and lower covers of x in L belong to R.*

For $i \in \{1, 2\}$, let L_i be a planar lattice and let $D_i \in \mathrm{Dgr}(L_i)$. A bijective map $\varphi \colon D_1 \to D_2$ is a *diagram isomorphism* if it is a lattice isomorphism $\varphi \colon L_1 \to L_2$. Equivalently, if $x \prec y$ iff $\varphi(x) \prec \varphi(y)$ for any pair of vertices $x, y \in D_1$. A diagram isomorphism $\varphi \colon D_1 \to D_2$ is called a *similarity map* if

(3.1)
$$\text{for all } x, y, z \in D_1 \text{ such that } x \prec y \text{ and } x \prec z,$$
$$y \text{ is to the left of } z \text{ iff } \varphi(y) \text{ is to the left of } \varphi(z),$$

and symmetrically. Following D. Kelly and I. Rival [28, p. 640], we say that D_1 and D_2 are *similar* lattice diagrams if there exists a similarity map $D_1 \to D_2$.

Similarity is an equivalence relation on $\mathrm{Dgr}(L)$. Since all the concepts we have defined so far are invariant under similarity, we consider lattice diagrams up to similarity.

In addition to similarity, there is left-right similarity. Two lattice diagrams, D_1 and D_2, are *left-right similar* if D_1 is similar to D_2 or D_1 is similar to the mirror image of D_2 over a vertical axis. We say that the diagrams of a planar lattice L are *unique up to left-right symmetry* if D_1 is left-right similar to D_2 for any $D_1, D_2 \in \mathrm{Dgr}(L)$.

Planar semimodular lattices can be characterized by properties of their diagrams, see G. Grätzer and E. Knapp [19, Lemmas 4 and 5].

Lemma 3.4. *Let L be a planar lattice.*

  (i) *If L is semimodular, then it is a 4-cell lattice. If $D \in \mathrm{Dgr}(L)$ and A, B are 4-cells of D with the same bottom, then these 4-cells have the same top.*
 (ii) *If L has a planar 4-cell diagram E in which no two 4-cells with the same bottom have distinct tops, then L is semimodular.*

We also need the following well-known concepts.

Definition 3.5. An element a of a lattice L is a *narrows* if a is comparable with all elements of L. The set of narrows of L is denoted by $\mathrm{Nar}(L)$. L is *(glued sum) indecomposable* if $|L| \geq 3$ and $\mathrm{Nar}(L) = \{0, 1\}$. For finite lattices L_1 and L_2, we obtain the *glued sum* of L_1 and L_2 (LTF, page 8) by putting L_2 atop L_1 and identifying 1_{L_1} with 0_{L_2}.

Some questions on lattices can be reduced to the indecomposable case.

4. Slim lattices, the basics

Slim semimodular lattices were defined in G. Grätzer and E. Knapp [19]. We use here the definition in G. Czédli and E. T. Schmidt [13], which does not require semimodularity.

A finite lattice L is called *slim* if $\mathrm{Ji}\,L$ contains no three-element antichain. It follows from R. P. Dilworth [18] that L is slim iff $\mathrm{Ji}\,L$ is the union of two chains.

◊**Lemma 4.1.** *Every slim lattice is planar.*

The following result is folklore.

Lemma 4.2. *A slim semimodular lattice can be uniquely decomposed into a glued sum of maximal chain intervals and indecomposable slim semimodular lattices.*

The next lemma is from G. Czédli and E. T. Schmidt [13] and G. Grätzer and E. Knapp [19].

Lemma 4.3. *For a finite lattice L, the following seven statements are equivalent.*
  (i) *L is a slim semimodular lattice.*
  (ii) *L is a slim semimodular lattice and a planar 4-cell lattice.*
  (iii) *L is a planar semimodular lattice with no cover-preserving diamond sublattice.*
  (iv) *L is a planar semimodular lattice and for all $D \in \mathrm{Dgr}(L)$, the 4-cells of D and the covering squares of L are the same.*
  (v) *L is a planar semimodular lattice and there exists a diagram $D \in \mathrm{Dgr}(L)$ such that the 4-cells of D and the covering squares of L are the same.*
  (vi) *L has a planar 4-cell diagram in which no two distinct 4-cells have the same bottom.*
  (vii) *All $D \in \mathrm{Dgr}(L)$ are 4-cell diagrams with no two distinct 4-cells having the same bottom.*

The following lemma is proved in G. Grätzer and E. Knapp [19, Lemma 6] and G. Czédli and E. T. Schmidt [14, Lemma 15]:

◊**Lemma 4.4.** *A slim, planar, semimodular lattice L is distributive iff N_7 (see Figure 1) is not a cover-preserving sublattice of L.*

Let L be a planar lattice and $D \in \mathrm{Dgr}(L)$. For $u \in L$, the *left support* of u the largest element of $C_l(D) \cap \downarrow u$; it is denoted by $\mathrm{lsp}(u, D)$, $\mathrm{lsp}(u)$, for short. We define the *right support* of u, $\mathrm{rsp}(u, D)$, $\mathrm{rsp}(u)$, symmetrically.

According to Lemma 4.3(iv), for a slim semimodular lattice L, we can consider the 4-cells of L. The set of 4-cells is denoted by $\mathrm{Cells}(L)$ or, for $D \in \mathrm{Dgr}(L)$, by $\mathrm{Cells}(D)$. By dropping the assumption of semimodularity, the following theorem generalizes some statements from G. Czédli and E. T. Schmidt [14, Lemma 6] and [15].

Theorem 4.5. *Let L be a slim lattice. Then the following statements hold:*
  (i) *$\mathrm{Bnd}(D) = \mathrm{Bnd}(E)$ for $D, E \in \mathrm{Dgr}(L)$ (that is, $\mathrm{Bnd}(L)$ does not depend on the diagram chosen).*
  (ii) *$\mathrm{Ji}\,L \subseteq \mathrm{Bnd}(L)$.*
  (iii) *If L is an indecomposable slim lattice, then its planar diagrams are unique up to left-right symmetry.*

From Theorem 4.5(iii), we obtain the following statement immediately.
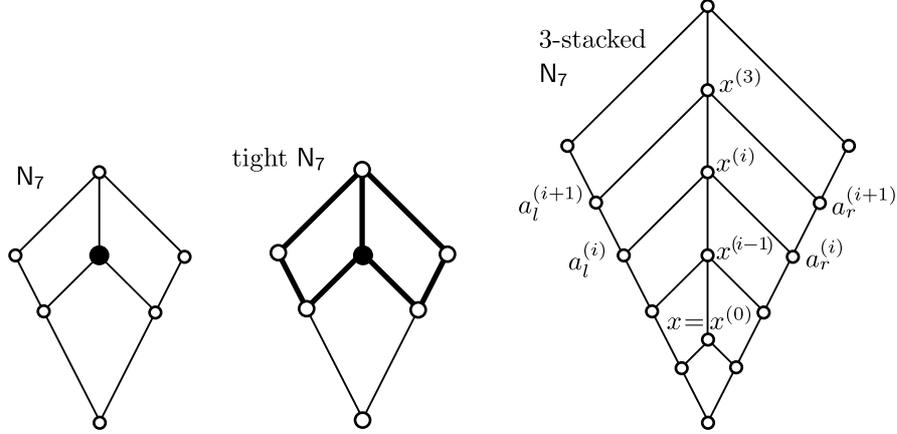
FIGURE 1. The lattice N_7 and its variants

Theorem 4.6. *Let E_1 and E_2 be slim lattice diagrams, and let $\varphi \colon E_1 \to E_2$ be a diagram isomorphism. Then φ is a similarity map iff $\varphi\big(\mathrm{C_l}(E_1)\big) = \mathrm{C_l}(E_2)$ iff $\varphi\big(\mathrm{C_r}(E_1)\big) = \mathrm{C_r}(E_2)$.*

This theorem makes it possible to define quotient diagrams for the slim lattices. Let L be a slim lattice, $E \in \mathrm{Dgr}(L)$, and let $\boldsymbol{\alpha}$ be a *join-congruence* of E, that is, a congruence of $(L; \vee)$. Then the quotient join-semilattice $L/\boldsymbol{\alpha}$ is a lattice. If there is a diagram $E' \in \mathrm{Dgr}(L/\boldsymbol{\alpha})$ such that

$$\mathrm{C_l}(E') = \{x/\boldsymbol{\alpha} \mid x \in \mathrm{C_l}(E)\},$$
$$\mathrm{C_r}(E') = \{x/\boldsymbol{\alpha} \mid x \in \mathrm{C_r}(E)\},$$

then E' is called the *quotient diagram* of E modulo $\boldsymbol{\alpha}$, and it is denoted by $E/\boldsymbol{\alpha}$. By Theorem 4.6, $E/\boldsymbol{\alpha}$ is uniquely determined up to similarity. With some additional conditions, we next show that $E/\boldsymbol{\alpha}$ exists.

The kernels of cover-preserving join-homomorphisms are called *cover-preserving join-congruences*. The following theorem generalizes G. Czédli [6, Lemma 11].

Theorem 4.7. *If $\boldsymbol{\alpha}$ is a cover-preserving join-congruence of a slim semimodular diagram E, then $E/\boldsymbol{\alpha}$ exists, and it is a slim semimodular diagram.*

Let L be a planar semimodular lattice and let $D \in \mathrm{Dgr}(L)$. If we omit the interior elements of D in all intervals of length two, then we obtain a $\{0, 1\}$-sublattice, $\mathrm{Slim}\, L$. The elements of $D - \mathrm{Slim}\, L$ are called the *eyes* of D.

As illustrated by M_3, $\mathrm{Slim}\, L$, as a subset, depends on the diagram chosen. However, the following (straightforward) statement establishes that the isomorphism class of $\mathrm{Slim}\, L$ does not depend on D, so there is the *full slimming* (*lattice*) of L, up to isomorphism.

◊**Lemma 4.8.** *Let L_1 and L_2 be planar semimodular lattices. If L_1 is isomorphic to L_2, then $\mathrm{Slim}\, L_1$ is isomorphic to $\mathrm{Slim}\, L_2$.*

The slimming construction has a natural inverse. Let L' be a planar semimodular lattice and $D' \in \mathrm{Dgr}(L')$. Let S be a 4-cell of D'. Replace S by a copy of the diamond M_3 (with a fixed diagram). That is, we insert a new element, which is
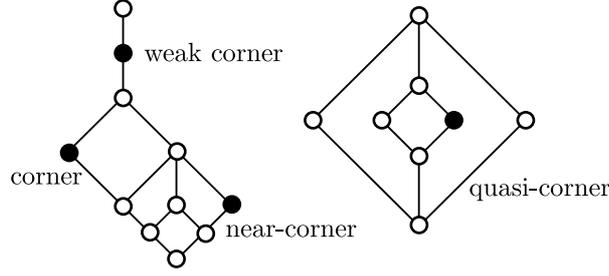
FIGURE 2. Corner variants

called an *eye*, into the interior of S. This way we obtain a new diagram, which determines a new lattice. If D is obtained from D' by inserting eyes one-by-one, then D and the corresponding L are called an *anti-slimming* of D', and of L', respectively. Clearly, L is an anti-slimming of $\mathrm{Slim}\,L$.

◊**Proposition 4.9.** *A planar lattice is semimodular iff some (equivalently, all) of its full slimming sublattices is slim and semimodular.*

◊**Corollary 4.10.** *Planar semimodular lattices are characterized as anti-slimmings of slim semimodular lattices.*

5. CONSTRUCTION WITH FORKS

Our goal is to present a construction of all planar semimodular lattices from planar distributive lattices. In view of Corollary 4.10, it suffices to deal with *slim* semimodular lattices. This section is based on G. Czédli and E. T. Schmidt [14].

Corners. G. Grätzer and E. Knapp [19] introduced *corners* (corner elements). Several variants of this concept appeared in G. Czédli and E. T. Schmidt [14] and [15], and in G. Czédli [8].

Definition 5.1. Let d be an element of a planar lattice L, and let $D \in \mathrm{Dgr}(L)$; see Figure 2 for an illustration where d is one of the black-filled elements.

(i) If $d \in \mathrm{Di}\,L \cap \mathrm{Bnd}(D)$, then d is called a *weak corner* of D. The elements of $d \in \mathrm{Di}\,L \cap \mathrm{C}_l(D)$ are *left weak corners*. *Right weak corners* are defined similarly.

(ii) A *near corner* is a weak corner d such that d_* has exactly two covers and d^* has at least two lower covers.

(iii) A *corner* is a near corner d such that d_* has exactly two covers and d^* has exactly two lower covers. As in (i), corners and near corners of D are *left* or *right*.

Assume that L is a *slim* semimodular lattice. Then, since $\mathrm{Bnd}(D)$ does not depend on $D \in \mathrm{Dgr}(L)$ by Theorem 4.5(i), we can define weak corners of L. A near corner or a corner of L can be removed to form a cover-preserving sublattice.

Consider the reverse procedure. If $a \prec b \prec c$ is a subchain of $\mathrm{C}_l(D)$ and $a \in \mathrm{Mi}\,L$, then we can *add a near corner d* to D by stipulating that $a \prec d \prec c$ and d be to the left to b. This way, we obtain a new diagram D' with $\mathrm{C}_l(D') = (\mathrm{C}_l(D) - \{b\}) \cup \{d\}$; if $c \in \mathrm{Ji}\,L$, we *add a corner*; see Figure 3 for examples. Of course, we can add a near corner or a corner to the right boundary chain analogously. If $D \in \mathrm{Dgr}(L)$ is
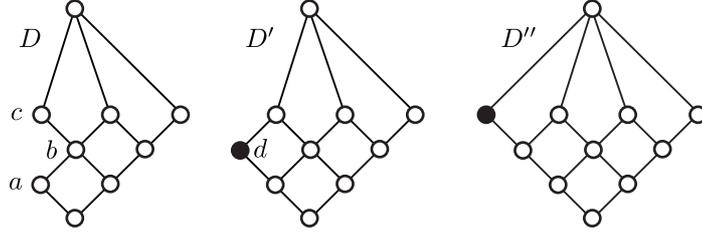
FIGURE 3. Adding a corner to D and a near corner to D'
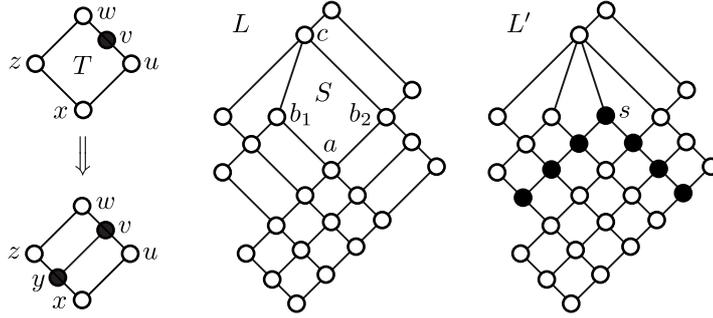
FIGURE 4. Adding a fork to L

understood, we say that $L' = L \cup \{d\}$, the lattice determined by D', is obtained from L by adding a near corner or a corner.

Proposition 5.2.

  (i) *Let L' be obtained from a planar lattice L by adding a near corner. Equivalently, let L be obtained from a planar lattice L' by removing a near corner. Then L is semimodular iff L' is semimodular. Similarly, L is slim and semimodular iff L' is slim and semimodular.*

  (ii) *Each slim semimodular lattice can be obtained from a chain by adding near corners, one-by-one.*

◊ **Forks.** Proposition 5.2(ii) is our first constructive description of slim semimodular lattices. However, as N_7 of Figure 1 illustrates, we cannot replace near corners by corners in the proposition. We obtain a deeper result using the following construction.

Definition 5.3.

  (i) Let $S = \{a = b_1 \wedge b_2,\ b_1,\ b_2,\ c = b_1 \vee b_2\}$ be a 4-cell of a slim semimodular diagram D. We change D to a new diagram D' as follows.

   Firstly, we replace S by a copy of N_7. We get three 4-cells replacing S.

   Secondly, we do a series of steps: if there is a chain $u \prec v \prec w$ such that v is a new element and $T = \{x = u \wedge z,\ z,\ u,\ w = z \vee u\}$ is a 4-cell in the original diagram D but $x \prec z$ at the present stage of the construction, see Figure 4, we insert a new element y such that $x \prec y \prec z$ and $y \prec v$. We get two 4-cells to replace the 4-cell T.

Let D' denote the diagram we obtain when the procedure terminates. (The collection of all new elements, which is an order, is called a *fork*.) We say that D' is obtained from D by *adding a fork to D* at the 4-cell S.

(ii) Let S be a covering square of a slim semimodular lattice L. Choose a diagram $D \in \mathrm{Dgr}(L)$. By Lemma 4.3(iv), S is a 4-cell of D. By adding a fork to D at S we obtain a diagram D', which determines a lattice L'. We say that L' is obtained from L by *adding a fork* at S.

(iii) "Adding forks" means adding forks one-by-one.

One can prove that L' does not depend on the choice of $D \in \mathrm{Dgr}(L)$, and it is a slim semimodular lattice.

$\Diamond$**Theorem 5.4.** *A slim semimodular diagram can be obtained from a chain by adding forks and corners.*

A chain with more than one element is a *nontrivial chain*. The direct product of two nontrivial chains is a *grid*. The diagram of a grid is a *grid diagram*.

Now we can state the main result of G. Czédli and E. T. Schmidt [14].

$\Diamond$**Theorem 5.5.** *A slim semimodular diagram* (*or lattice*) *with at least three elements can be obtained from a grid by*

(i) *first, adding forks,*

(ii) *then removing corners.*

6. Construction with resections

Based on G. Czédli and G. Grätzer [11], we now present a twin of the construction of adding forks, presented in the last section. First, we introduce the concept of trajectories.

Trajectories. Let D be a slim semimodular diagram. Two prime intervals of D are *consecutive* if they are opposite sides of a 4-cell (see Section 3). As in G. Czédli and E. T. Schmidt [13], maximal sequences of consecutive prime intervals form a C_2-*trajectory*. So a C_2-trajectory is an equivalence class of the transitive reflexive closure of the "consecutive" relation.

Similarly, let A and B be two cover-preserving C_3-chains of D. If they are opposite sides of a cover-preserving $C_3 \times C_2$, then A and B are called *consecutive*. An equivalence class of the transitive reflexive closure of this "consecutive" relation is called a C_3-*trajectory*.

We recall the basic properties of C_2-trajectories from [13] and [15]; they also hold for C_3-trajectories. For $i \in \{2, 3\}$, a C_i-trajectory goes from left to right (unless otherwise stated); they do not branch out. A C_i-trajectory is of two types: an *up-trajectory,* which goes up (possibly, in zero steps) and a *hat-trajectory,* which goes up (possibly in zero steps), then turns to the lower right, and finally it goes down (possibly, in zero steps).

Note that the left and right ends of a C_2-trajectory are on the boundary of L; this may fail for a C_3-trajectory.

The *elements* of a C_i-trajectory are the elements of the C_i-chains forming it. Let A be a cover-preserving C_i-chain in D. By planarity, there is a unique C_i-trajectory through A. The C_i-chains of this trajectory to the left of A and including A form the *left wing of A*. The *right wing of A* is defined analogously.

Resections. We construct slim (planar) semimodular lattices from planar distributive lattices by a series of *resections*. A resection starts with a cover-preserving C_3^2 (the dark gray square of the three-element chain in Figure 5), and it deletes two elements to get an N_7 (see Figure 1), and then deletes some more elements (all the black-filled ones), going up and down to the left and to the right, to preserve semimodularity; see Figure 6 for the result of the resection.

Let B be a cover-preserving $C_3^2 = C_3 \times C_3$ of the diagram D. Let W_l be the left wing of the upper left boundary of B and let W_r be the right wing of the upper right boundary of B. Assume that W_l and W_r terminate on the boundary of D (that is, the last C_3-chains are on the boundary of D). In this case, the collection of elements of $S = B \cup W_l \cup W_r$ is called a C_3-*scheme* of D, see Figure 5 for an example. The elements of W_l and W_r form the *left wing* and the *right wing* of this C_3-scheme, respectively, while B is the *base*. The middle element of S is the *anchor* of the scheme. A C_3-scheme is uniquely determined by its anchor. Of course, D may have cover-preserving C_3^2's that cannot be extended to C_3-schemes. For example, the slim semimodular diagrams in Figure 7 have cover-preserving C_3^2 sublattices but no C_3-schemes.

The concept of a C_2-scheme and the related terminology are analogous, see Figures 6 for an example. The base of a C_2-scheme is a cover-preserving N_7, and its wings are in C_2-trajectories. The middle element of the base is again called the anchor, and it determines the C_2-scheme. Since C_2-trajectories always reach the boundary of D, each cover-preserving N_7 sublattice is the base of a unique C_2-scheme.

For $i \in \{2, 3\}$ and a C_i-scheme S, we define the *upper boundary*, the *lower boundary*, and the *interior* of S as expected.

Let S be a C_3-scheme of a slim semimodular diagram D. By removing all the interior elements of S but its anchor, we obtain a new slim semimodular diagram, D', and S turns into a C_2-scheme of D'. We say that D' is obtained from D by a *resection*; this process is illustrated in Figures 5 and 6. The reverse procedure, transforming a C_2-scheme to a C_3-scheme by adding new interior elements, is called an *insertion*.

A *grid* is a planar diagram of the form $C_m \times C_n$ for $m, n \geq 2$. We obtain a slim distributive diagram from a grid by a sequence of steps; each step omits a doubly irreducible element from a boundary chain. Our main result generalizes this to slim semimodular lattice diagrams.

◊**Theorem 6.1.** *Slim semimodular lattice diagrams are characterized as diagrams obtained from slim distributive lattice diagrams by a sequence of resections.*

The proof of this theorem appears clear. Let D be a slim semimodular lattice diagram. Find in it a covering N_7 as in Figure 6. Perform an insertion to obtain the diagram of Figure 5. The diagram of Figure 5 has one fewer covering N_7-s. Proceed this way until a diagram is obtained without covering N_7-s.

However, this argument does not necessarily work. Start with the first diagram in Figure 8. Apply an insertion at the black-filled element, to obtain the second diagram. Apply an insertion at the gray-filled element of the second diagram, to obtain the third diagram. And so on. It is clear that the number of covering N_7-s is not diminishing.

The proof proceeds by defining the rank of an anchor. The *rank* of an anchor x is the largest number t such that there is a tight t-stacked N_7 with least inner
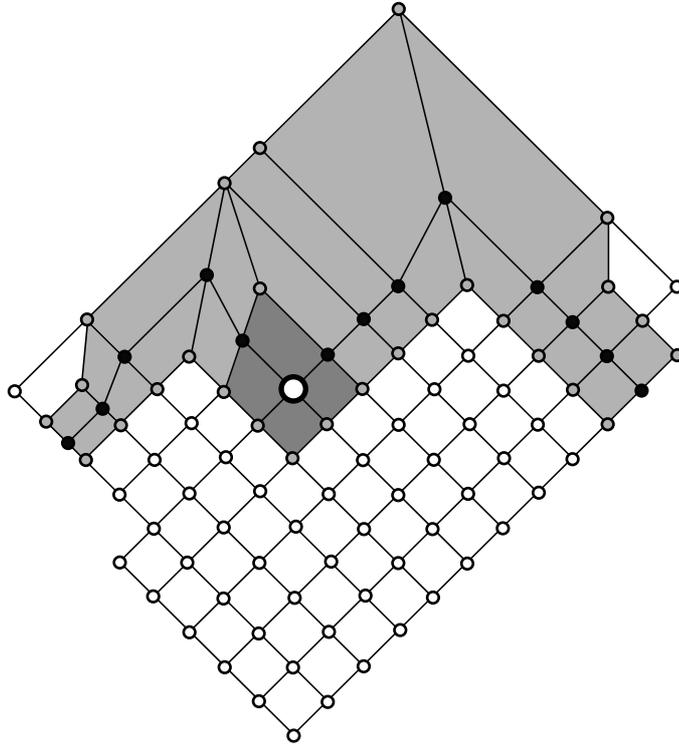
FIGURE 5. Resect this diagram at the element marked by the big circle by deleting the black-filled elements
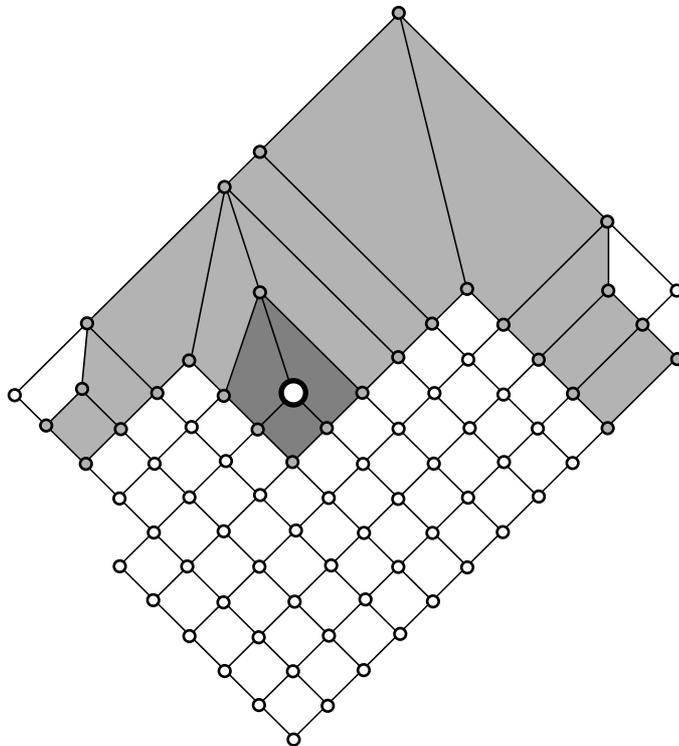

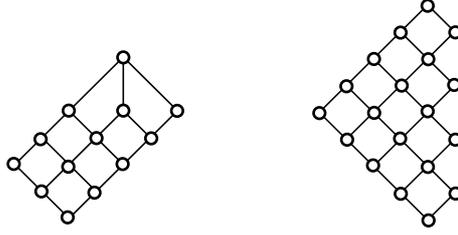FIGURE 6. to obtain this diagram
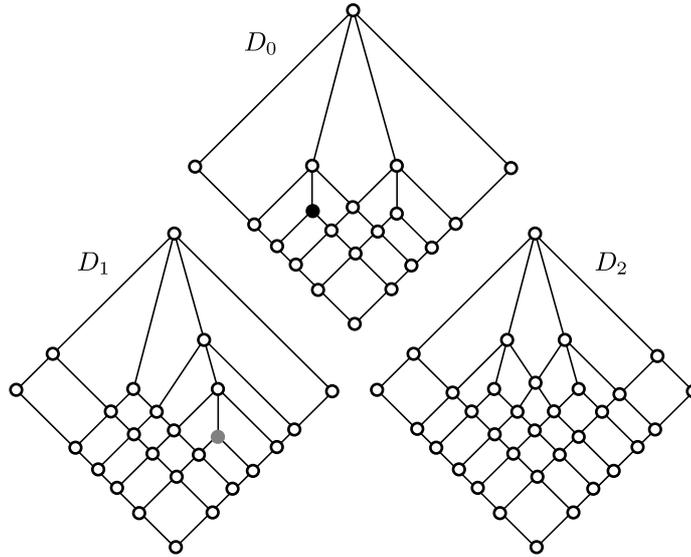
FIGURE 7. Two slim semimodular diagrams

FIGURE 8. The process does not stop

element x; see Figure 1 for $t = 3$. One can show that if we perform a resection at an anchor x, then the rank of x decreases by 1 and no new anchor enters (but the ranks of other anchors may increase.) In particular, if x is of rank 0, the x will not remain an anchor after the resection, and the number of anchors decreases by 1. Therefore, if we keep performing a resection at an anchor of minimal rank, then the procedure terminates.

7. ◊ RECTANGULAR LATTICES

◊ **From the basics to structure theorems.** Following G. Grätzer and E. Knapp [20], a semimodular lattice diagram D is *rectangular* if $C_l(D)$ has exactly one weak corner, $\mathrm{lc}(D)$ and $C_r(D)$ has exactly one weak corner, $\mathrm{rc}(D)$, and these two weak corners are complementary, that is,

$$\mathrm{lc(D)} \vee \mathrm{rc(D)} = 0,$$
$$\mathrm{lc(D)} \wedge \mathrm{rc(D)} = 1.$$

A semimodular lattice L is called a *rectangular lattice* if some $D \in \mathrm{Dgr}(L)$ is rectangular; equivalently, if all $D \in \mathrm{Dgr}(L)$ are rectangular. Rectangular lattices have nice rectangle-shaped diagrams.

Given a rectangular lattice, for instance, the diamond M_3, its weak corners are not unique. But the rest of the boundary is unique.

◊**Lemma 7.1** (G. Grätzer and E. Knapp [21]). *Let D be a rectangular diagram. Then the intervals $[0, \mathrm{lc}(D)]$, $[\mathrm{lc}(D), 1]$, $[0, \mathrm{rc}(D)]$, and $[\mathrm{rc}(D), 1]$ are chains.*

So the chains $C_l(D)$ and $C_r(D)$ are split into two, a lower and an upper part: $C_{ll}(D) = [0, \mathrm{lc}(D)]$, $C_{ul}(D) = [\mathrm{lc}(D), 1]$, $C_{lr}(D) = [0, \mathrm{rc}(D)]$, and $C_{ur}(D) = [\mathrm{rc}(D), 1]$ (C_{ll}, C_{ul}, C_{lr}, and C_{ur}, for short).

The structure of rectangular lattices is described in the following two statements.

◊**Theorem 7.2** (G. Czédli and E. T. Schmidt [14]). *L is a rectangular lattice iff it is an anti-slimming of a lattice that can be obtained from a grid by adding forks.*

◊**Theorem 7.3** (G. Czédli and G. Grätzer [11]). *Every slim rectangular lattice L can be constructed from a grid by a sequence of resections.*

◊ **Gluings of rectangular lattices.** Let L be a nontrivial lattice. If L cannot be obtained as a gluing of two lattices, we call L *gluing indecomposable*.

◊**Theorem 7.4** (G. Czédli and E. T. Schmidt [15]). *Let L be a planar semimodular lattice with at least four elements. Then the following six conditions are equivalent.*

 (i) *L is gluing indecomposable;*
 (ii) *L is gluing indecomposable over chains;*
 (iii) *L is a rectangular lattice whose weak corners, $\mathrm{lc}(D)$ and $\mathrm{rc}(D)$, are dual atoms for some rectangular diagram D of L.*
 (iv) *L has a planar diagram such that the intersection of the leftmost dual atom and the rightmost dual atom is 0;*
 (v) *for any planar diagram of L, the intersection of the leftmost dual atom and the rightmost dual atom is 0;*
 (vi) *L is an anti-slimming of a lattice obtained from the four-element Boolean lattice by adding forks.*

For a slim rectangular lattice L, let $x \in C_{ul}(L) - \{1, \mathrm{lc}(L)\}$ and let $y \in C_{ur}(L) - \{1, \mathrm{rc}(L)\}$. We introduce some notation (see Figure 10):

$$L_{\mathrm{top}}(x, y) = [x \wedge y, 1],$$
$$L_{\mathrm{left}}(x, y) = [\mathrm{lc}(L) \wedge y, x],$$
$$L_{\mathrm{right}}(x, y) = [x \wedge \mathrm{rc}(L), y],$$
$$L_{\mathrm{bottom}}(x, y) = [0, (\mathrm{lc}(L) \wedge y) \vee (x \wedge \mathrm{rc}(L))].$$

The following result is from G. Grätzer and E. Knapp [21].

Theorem 7.5 (**Decomposition Theorem**). *Let L be a slim rectangular lattice, and let $x \in C_{ul}(L) - \{1, u_l\}$, $y \in C_{ur}(L) - \{1, u_r\}$. Then L can be decomposed into four slim rectangular lattices $L_{\mathrm{top}}(x, y)$, $L_{\mathrm{left}}(x, y)$, $L_{\mathrm{right}}(x, y)$, $L_{\mathrm{bottom}}(x, y)$, and the lattice L can be reconstructed from these by repeated gluing.*

Now, Theorem 7.4 can be derived from Theorem 7.5 in the following way, see G. Grätzer [22]. The trick is the following: first, the validity of Theorem 7.4 is
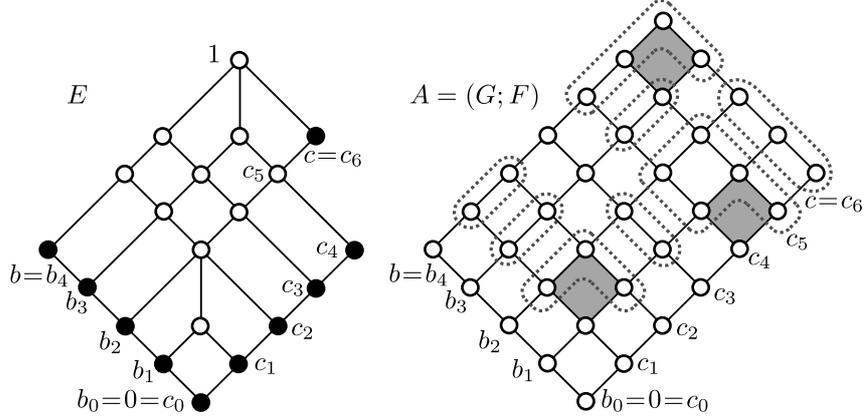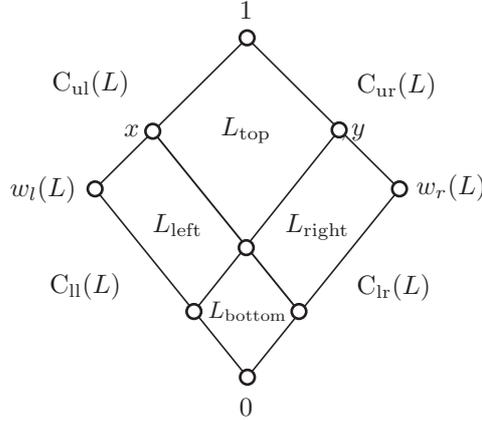
FIGURE 9. E and $A = \mathrm{Mtx}(E)$

FIGURE 10. Decomposing a slim rectangular lattice

invariant under slimming, so we can assume that L is slim. Next, the validity of Theorem 7.4 is invariant under removing corners; thus Theorem 5.5(ii) allows us to assume that L is a rectangular lattice. Once L is rectangular, Theorem 7.5 applies.

8. ◊ A DESCRIPTION BY MATRICES

The main result. The quotient join-semilattice of a grid modulo a cover-preserving join-congruence is a slim semimodular lattice, and every slim semimodular lattice can be represented this way. We are going to deal with minimal representations of this kind. Since (finite) chains are trivial as slim semimodular lattices, we only deal with *non-chains*. Remember that, by Definition 3.5, an indecomposable lattice is not a chain.

This section is based on G. Czédli [6]. We consider diagrams up to similarity. Except for symmetric diagrams, we distinguish between a diagram and its vertical mirror image.

Let K be a grid, and let $G \in \mathrm{Dgr}(K)$ be a grid diagram. If $F \subseteq \mathrm{Cells}(G)$, then $A = (G; F)$ is called a *matrix diagram* and the elements of F are called *F-cells*. See Figure 9 for an illustration, where the F-cells are gray-filled. For an m-by-n matrix diagram $A = (G; F)$, we use the notation

$$\mathrm{C_{ll}}(G) = \{0 = b_0 \prec \cdots \prec b_m\},$$
$$\mathrm{C_{lr}}(G) = \{0 = c_0 \prec \cdots \prec c_n\}.$$

The unique 4-cell of G with top $b_i \vee c_j$ is denoted by $\mathrm{cell}(i, j)$. This notation will be used even where we use different symbols for the elements of $\mathrm{C_{ll}}(G)$ and $\mathrm{C_{lr}}(G)$.

The i-th *row* of $A = (G; F)$ is

$$\{\mathrm{cell}(i, j) \in \mathrm{Cells}(G) \mid 0 \leq j \leq n\},$$

and the j-th column is defined analogously.

Matrix diagrams are in a bijective correspondence with 0-1-matrices as follows; the m-by-n 0-1-matrix $P = (p_{ij})_{m \times n}$ corresponding to $A = (G; F)$ is defined by the rule $p_{ij} = 1$ if the $\mathrm{cell}(i, j) \in F$ and $p_{ij} = 0$, otherwise. For example, the 0-1-matrix corresponding to $A = (G; F)$ in Figure 9 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 8.1. For a matrix diagram $A = (G; F)$, we define a slim semimodular diagram $\mathrm{QDgr}(A)$ as follows. For $U \in \mathrm{Cells}(G)$, let $\mathrm{con}_\vee(U)$ denote the smallest join-congruence of G that collapses $\{\mathrm{lc(U)}, \mathrm{rc(U)}, 1_U\}$. Let $\boldsymbol{\beta} = \bigvee \{\mathrm{con}_\vee(U) \mid U \in F\}$. If $\boldsymbol{\beta}$ is a cover-preserving join-congruence, then $\mathrm{QDgr}(A)$ is the *quotient diagram* $G/\boldsymbol{\beta}$, see Theorem 4.7. Otherwise, $\mathrm{QDgr}(A)$ is undefined.

We also need a construction in the opposite direction.

Definition 8.2. With an indecomposable, slim, semimodular diagram E, we associate a matrix diagram as follows. Let b and c be the largest elements of $\mathrm{Ji}\, E \cap \mathrm{C_l}(E)$ and $\mathrm{Ji}\, E \cap \mathrm{C_r}(E)$, respectively. Let

$$(8.1) \qquad \begin{aligned} B &= \{0 = b_0 \prec \cdots \prec b_m = b\} = \mathrm{C_l}(E) \cap \downarrow b, \\ C &= \{0 = c_0 \prec \cdots \prec c_n = c\} = \mathrm{C_r}(E) \cap \downarrow c, \end{aligned}$$

see Theorem 4.5(ii) and Figure 9. Let G be the diagram of $B \times C$ such that

$$(8.2) \qquad \begin{aligned} \mathrm{C_{ll}}(G) &= \{(b_i, 0) \mid 0 \leq i \leq m\}, \\ \mathrm{C_{lr}}(G) &= \{(0, c_j) \mid 0 \leq i \leq n\}. \end{aligned}$$

On G, we define an equivalence $\boldsymbol{\alpha}$ by $((b_i, c_j), (b_h, c_k)) \in \boldsymbol{\alpha}$ iff $b_i \vee c_j = b_h \vee c_k$. In Figure 9, the blocks of $\boldsymbol{\alpha}$ are represented by dotted lines.

A 4-cell $U \in \mathrm{Cells}(G)$ is called a *source cell* of $\boldsymbol{\alpha}$ if $0_U \notin 1_U/\boldsymbol{\alpha}$ and $\mathrm{lc(U)}, \mathrm{rc(U)} \in 1_U/\boldsymbol{\alpha}$. The set of these source cells is denoted by $\mathrm{SCells}(\boldsymbol{\alpha})$. In Figure 9, $\mathrm{SCells}(\boldsymbol{\alpha})$ consists of the three gray-filled 4-cells. The matrix diagram we associate with E is $\mathrm{Mtx}(E) = (G, \mathrm{SCells}(\boldsymbol{\alpha}))$.

◊**Proposition 8.3.** *Let $A = (G; F)$ be a matrix diagram. Then the following two conditions are equivalent*

(i) $\mathrm{QDgr}(A)$ *is defined, it is an indecomposable, slim, semimodular diagram and for any matrix diagram $A' = (G'; F')$ such that $\mathrm{QDgr}(A')$ is similar to $\mathrm{QDgr}(A)$, we have $|\mathrm{C}_{\mathrm{ll}}(G)| \leq |\mathrm{C}_{\mathrm{ll}}(G')|$ and $|\mathrm{C}_{\mathrm{lr}}(G)| \leq |\mathrm{C}_{\mathrm{lr}}(G')|$.*

(ii) *A satisfies the following five conditions:*

(mr1) *every row and every column of A contains at most one F-cell;*

(mr2) $|F| < \min\{m, n\}$;

(mr3) $|F \cap \mathrm{Cells}(\downarrow (c_k \vee d_k))| < k$ *for $k = 1, \ldots, \min\{m, n\} - 1$;*

(mr4) *if $\mathrm{cell}(i, n) \in F$, then there is an i' such that $1 \leq i' < i$ and there is no F-cell in the i'-th row;*

(mr5) *if $\mathrm{cell}(m, j) \in F$, then there is a j' such that $1 \leq j' < j$ and there is no F-cell in the j'-th column.*

Matrix diagrams satisfying (mr1), ..., (mr5) are called *regular matrix diagrams*. By Proposition 8.3, they are the minimal matrix diagrams to characterize indecomposable slim semimodular diagrams in the following theorem. Although this theorem was stated in G. Czédli [6] for lattices rather diagrams in [6], the proof is similar.

◊**Theorem 8.4.** *Let E be an indecomposable, slim, semimodular diagram, and let A be a regular matrix diagram. Then $\mathrm{Mtx}(E)$ is a regular matrix diagram, $\mathrm{QDgr}(A)$ is an indecomposable slim semimodular diagram, $\mathrm{QDgr}(\mathrm{Mtx}(E)) = E$, and $\mathrm{Mtx}(\mathrm{QDgr}(A)) = A$.*

9. Description by permutations

The description of slim semimodular lattices by permutations can be extracted from Abels [1, Remark 2.9 and the next sentence]. However, it will be more convenient to follow G. Czédli and E. T. Schmidt [16]. We start here with a variant of the definition in G. Czédli, L. Ozsvárt, and B. Udvari [12].

Definition 9.1. Assume that D is a slim semimodular diagram. Let

(9.1)
$$\mathrm{C}_{\mathrm{l}}(D) = B = \{0 = b_0 \prec b_1 \prec \cdots \prec b_h = 1\},$$
$$\mathrm{C}_{\mathrm{r}}(D) = C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_h = 1\}.$$

We define two maps, $\pi = \pi(D)$ and $\sigma = \sigma(D)$, as follows. For $i, j \in \{1, \ldots, h\}$, let

$$I(i) = \{j \in \{1, \ldots, h\} \mid b_{i-1} \vee c_j = b_i \vee c_j\},$$
$$\pi(i) = \text{the smallest element of } I(i),$$
$$J(j) = \{i \in \{1, \ldots, h\} \mid b_i \vee c_{j-1} = b_i \vee c_j\},$$
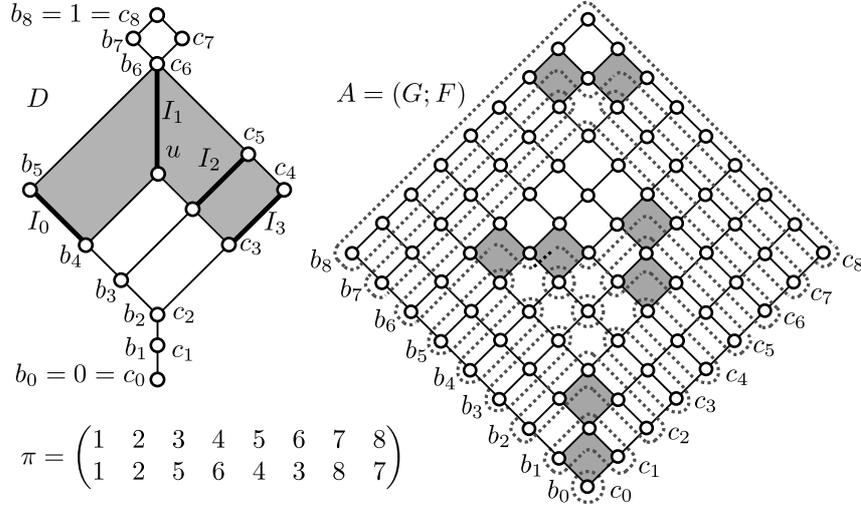$$\sigma(j) = \text{the smallest element of } J(j).$$

Then $\pi(D)$ is the *permutation associated with D*.

Of course, we have to prove that $\pi(D)$ is a permutation. The set of permutations acting on $\{1, \ldots, h\}$ is denoted by S_h.

Lemma 9.2 (H. Abels [2], G. Czédli and E. T. Schmidt [16] and G. Czédli, L. Ozsvárt, and B. Udvari [12]). *If D is a slim semimodular diagram, then $\pi(D)$ and $\sigma(D)$ are permutations, and $\pi(D)^{-1} = \sigma(D)$.*

Proof. Let $\pi = \pi(D)$ and $\sigma = \sigma(D)$. Clearly, $0 \notin I(i) \cup J(j)$ and $h \in I(i) \cap J(j)$. Assume that j belongs to $I(i)$ and $j < h$. Then

$$b_{i-1} \vee c_{j+1} = b_{i-1} \vee c_j \vee c_{j+1} = b_i \vee c_j \vee c_{j+1} = b_i \vee c_{j+1}$$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 4 & 3 & 8 & 7 \end{pmatrix}$$

FIGURE 11. A diagram D and the corresponding permutation

shows that $j + 1 \in I(i)$. Since the same argument works for $J(j)$, we conclude that, for $i, j \in \{1, \ldots, h\}$, both $I(i)$ and $J(j)$ are (order) filters of $\{1, \ldots, h\}$. For $i \in \{1, \ldots, h\}$, let $j = \pi(i)$. Since $j - 1 \notin I(i)$ and $j \in I(i)$, we obtain that

(9.2) $b_{i-1} \vee c_{j-1} < b_i \vee c_{j-1} \le b_i \vee c_j = b_{i-1} \vee c_j.$

Semimodularity implies that $b_{i-1} \vee c_{j-1} \preceq b_{i-1} \vee c_j$. This and (9.2) yield that $b_i \vee c_{j-1} = b_i \vee c_j$. Hence $i \in J(j)$, and we obtain that $\sigma(j) \le i$. If $\sigma(j) < i$, then $i - 1 \in J(j)$ implies that $b_{i-1} \vee c_{j-1} = b_{i-1} \vee c_j$, contradicting (9.2). Hence $i = \sigma(j) = \sigma(\pi(i))$, that is, $\sigma \circ \pi$ is the identity map on $\{1, \ldots, h\}$. By symmetry, so is $\pi \circ \sigma$. $\square$

The definition of $\pi(D)$ from G. Czédli and E. T. Schmidt [15] is easy to visualize. Namely, with notation (9.1), we let $\pi(i) = j$ iff the prime intervals $[c_{i-1}, c_i]$ and $[d_{j-1}, d_j]$ belong to the same trajectory.

Lemma 9.3 (G. Czédli and E. T. Schmidt [13] and [15]). *Let D be a slim semimodular diagram. Then the following statements hold:*

  (i) *Going from left to right, the trajectories depart from the left boundary chain, do not branch out, and arrive at the right boundary chain.*
 (ii) *While going from left to right, a trajectory first goes up, possibly in zero steps, then it may take a turn to the lower right, and finally it keeps going down, possibly in zero steps. In particular, once it is going down, there is no further turn.*
(iii) *$\pi(D)$ is a permutation.*
(iv) *$\pi(D)$ is the same as the permutation given in Definition 9.1*

Definition 9.4. Given $\pi \in S_h$, we define a matrix diagram $D(\pi)$ as follows. Let $B = \{b_0 \prec \cdots \prec b_h\}$ and $C = \{c_0 \prec \cdots \prec d_h\}$, and let G be the square grid diagram satisfying (8.2) with $m = n = h$. Let $F = \{\text{cell}(i, \pi(i)) \mid 1 \le i \le h\}$. This

way we obtain a square matrix diagram $A = (G; F)$. Let

$$\boldsymbol{\beta} = \boldsymbol{\beta}_\pi = \bigvee_{i=1}^{h} \text{con}_\vee(\text{cell}(i, \pi(i))) = \bigvee_{U \in F} \text{con}_\vee(U),$$

and define $D(\pi)$ to be the quotient diagram $G/\boldsymbol{\beta}$.

One can prove that $\boldsymbol{\beta}$ is cover-preserving. Hence $D(\pi)$ exists and it is a slim semimodular diagram by Theorem 4.7. For example, if π is the permutation in Figure 11, then $A = (G; F)$ and, with dotted lines, the $\boldsymbol{\beta}$-blocks are depicted in the figure. In this case, $D(\pi)$ equals D on the left of the figure.

Let $\text{SSD}(h)^\sim$ be the set of slim semimodular lattice diagrams of length h, where similar diagrams are considered equal.

◊**Theorem 9.5** (G. Czédli and E. T. Schmidt [16])**.** *For $h \in \mathbb{N}$, the maps*

$$S_h \to \text{SSD}(h)^\sim, \quad \pi \mapsto D(\pi), \qquad and \qquad \text{SSD}(h)^\sim \to S_h, \quad D \mapsto \pi(D),$$

are inverse bijections.

Assume that $1 \leq u \leq v \leq h$ and $\pi \in S_h$. Let $I = [u, v] = \{i \in \mathbb{N} \mid u \leq i \leq v\}$ be nonempty and let $[1, u-1]$, I, and $[v+1, h]$ be closed with respect to π. Then I is called a *section* of π. Sections minimal with respect to set inclusion are called *segments*.

Let $\text{Seg}(\pi)$ denote the set of all segments of π. For example, for the permutation π in Figure 11, we have $\text{Seg}(\pi) = \{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8\}\}$. For $\pi, \mu \in S_h$, we say that π and μ are *sectionally equal or inverted* if $\text{Seg}(\pi) = \text{Seg}(\mu)$ and $\mu|_I \in \{\pi|_I, (\pi|_I)^{-1}\}$ for all $I \in \text{Seg}(\pi)$.

We can derive the following statement from Theorem 9.5.

◊**Corollary 9.6** (G. Czédli and E. T. Schmidt [16])**.** *Let L_1 and L_2 be slim semimodular lattices of the same length, and let $D_i \in \text{Dgr}(L_i)$ for $i = 1, 2$. Then $L_1 \cong L_2$ iff the permutations $\pi(D_1)$ and $\pi(D_2)$ are sectionally equal or inverted.*

10. Variants of the Jordan-Hölder Theorem

Strengthening the Jordan-Hölder Theorem. The classical Jordan-Hölder Theorem goes back to C. Jordan [27] and O. Hölder [26]. Firstly, we deal with its lattice theoretical counterpart. It states that whenever L is a semimodular lattice of finite length, then any two maximal chains of L are of the same length and, in addition, if

(10.1) $B = \{b_0 \prec \cdots \prec b_h\}, \quad C = \{c_0 \prec \cdots \prec c_h\}$

are maximal chains of L, then there is a permutation $\pi \in S_h$ such that, for all $i \in \{1, \ldots, h\}$, the prime interval $[b_{i-1}, b_i]$ is projective to the prime interval $[c_{\pi(i)-1}, c_{\pi(i)}]$.

For intervals $[a_1, b_1]$ and $[a_2, b_2]$ of a lattice, $[a_1, b_1]$ is *up-perspective* to $[a_2, b_2]$, in notation, $[a_1, b_1] \nearrow [a_2, b_2]$ if $a_2 \vee b_1 = b_2$ and $a_2 \wedge b_1 = a_1$. Dually, $[a_2, b_2] \searrow [a_1, b_1]$ means that $[a_1, b_1] \nearrow [a_2, b_2]$. We say that $[a_1, b_1]$ is *up-and-down projective* to $[a_2, b_2]$, in notation $[a_1, b_1] \bigwedge\searrow [a_2, b_2]$, if there is an interval $[x, y]$ such that $[a_1, b_1] \nearrow [x, y]$ and $[x, y] \searrow [a_2, b_2]$. This concept was used in the first step of extending the Jordan-Hölder theorem, as follows.
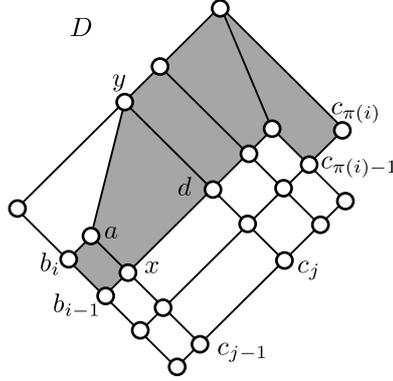
FIGURE 12. D with a trajectory

Theorem 10.1 (G. Grätzer and J. B. Nation [23])**.** *Assume that B and C in (10.1) are maximal chains of a semimodular lattice L. Then there is a permutation $\pi \in S_h$ such that $[b_{i-1}, b_i] \,\diagdown\!\!\!\nearrow\, [c_{\pi(i)-1}, c_{\pi(i)}]$ for all $i \in \{1, \ldots, h\}$.*

Although planarity is not assumed in this theorem, we need the theory of planar semimodular lattices to strengthen it with a uniqueness statement.

Theorem 10.2 (G. Czédli and E. T. Schmidt [13])**.** *The permutation π in Theorem 10.1 is uniquely determined. Furthermore, for all $i, j \in \{1, \ldots, h\}$, the up-and-down projectivity $[b_{i-1}, b_i] \,\diagdown\!\!\!\nearrow\, [c_{j-1}, c_j]$ implies that $j \leq \pi(i)$.*

Now, we convert Theorems 10.1 and 10.2 to group theoretic results. As usual, the relation *subnormal subgroup* is the transitive closure of the relation *normal subgroup*. For subnormal subgroups $A \lhd B$ and $C \lhd D$ of a given group G, the quotient B/A will be called *subnormally down-and-up projective* to D/C if there are subnormal subgroups $X \lhd Y$ of G such that $AY = B$, $A \cap Y = X$, $CY = D$ and $C \cap Y = X$. Clearly, $B/A \cong D/C$ in this case, because both are isomorphic to the group Y/X by the Second Isomorphism Theorem.

The well-known concept of a composition series in a group goes back to É. Galois (1831), see J. J. Rotman [32, Thm. 5.9]. The Jordan-Hölder theorem, stating that any two composition series of a finite group have the same length, was also proved in the nineteenth century, see C. Jordan [27] and O. Hölder [26]. The group does not have to be finite; it suffices to assume that there exists a finite composition series.

The first statement of the following theorem is in G. Grätzer and J. B. Nation [23], while the second statement is in G. Czédli and E. T. Schmidt [13].

Theorem 10.3. *Let*

$$(10.2) \qquad \begin{aligned} \vec{H} &: \{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G, \\ \vec{K} &: \{1\} = K_0 \lhd K_1 \lhd \cdots \lhd K_m = G \end{aligned}$$

be composition series of a group G. Then

(i) *$n = m$, and there exists a permutation π of the set $\{1, \ldots, n\}$ such that H_i/H_{i-1} is subnormally down-and-up projective to $K_{\pi(i)}/K_{\pi(i)-1}$ for all i;*

(ii) *this permutation π is* uniquely determined *and it has the following property: if $i, j \in \{1, \ldots, n\}$ and H_i/H_{i-1} is subnormally down-and-up projective to K_j/K_{j-1}, then $j \geq \pi(i)$.*

Proof. Let G be a group with a finite composition series. We know from a classical result of H. Wielandt [36] (see also R. Schmidt [33, Theorem 1.1.5] and the remark after its proof or see M. Stern [35, p. 302]) that the subnormal subgroups form a sublattice $\mathrm{SnSub}\, G$ of the lattice $\mathrm{Sub}\, G$ of all subgroups of G. It is not hard to strengthen this result to the following one:

$$(10.3) \qquad \mathrm{SnSub}\, G \text{ is a dually semimodular lattice;}$$

see [33, Theorem 2.1.8], or the proof of [35, Theorem 8.3.3], or the proof of J. B. Nation [31, Theorem 9.8]. Therefore, the theorem follows from Theorems 10.1 and 10.2 by the Duality Principle. $\qquad\square$

How many ways can two composition series intersect? Assume that $\vec{H}$ and $\vec{K}$ in (10.2) are composition series of a group G and $h = m = n$. Let

$$\mathrm{CSL}_h(\vec{H}, \vec{K}) = \{H_i \cap K_j \mid i, j \in \{0, \ldots, h\}\}.$$

Then $\mathrm{CSL}_h(\vec{H}, \vec{K}) = \big(\mathrm{CSL}_h(\vec{H}, \vec{K}); \subseteq\big)$ is an order. Since it has a largest element and it is closed with respect to intersection, $\mathrm{CSL}_h(\vec{H}, \vec{K})$ is a finite lattice; we call it a *composition series lattice*. We are going to determine which lattices are (isomorphic to) composition series lattices and how many there are. The following theorem strengthens G. Czédli and E. T. Schmidt [16, Corollary 3.5] and G. Czédli, L. Ozsvárt, and B. Udvari [12].

Theorem 10.4. *Composition series lattices are the duals of slim semimodular lattices. Furthermore, if G is the direct product of h nontrivial simple cyclic groups, then for each slim semimodular lattice L of length h, there exist composition series $\vec{H}$ and $\vec{K}$ of G such that L is isomorphic to the dual of $\mathrm{CSL}_h(\vec{H}, \vec{K})$.*

Let $N(h)$ denote the number of isomorphism classes of all composition series lattices $\mathrm{CSL}_h(\vec{H}, \vec{K})$. This number counts how many ways two composition series of length h can intersect. By Theorem 10.4, $N(h)$ is also the number of isomorphism classes of slim semimodular lattice of length h. Using matrices and Theorem 8.4, G. Czédli, L. Ozsvárt, and B. Udvari [12] gave a recursive method of computing $N(h)$. On a personal computer, it can be used up to $h = 100$.

The following tables, in which $N_0(h)$ and $N(h)$ denote the number of isomorphism classes of indecomposable slim semimodular lattices of length h and of slim semimodular lattices of length h, respectively, was computed in a fraction of a second.

h	0	1	2	3	4	5	6	7	8
$N_0(h)$	0	0	1	2	8	39	242	1,759	14,674
$N(h)$	1	1	2	5	17	73	397	2,623	20,414

h	9	10	11	12
$N_0(h)$	137,127	1,416,430	16,006,403	196,400,810
$N(h)$	181,607	1,809,104	19,886,032	238,723,606

G. Czédli, L. Ozsvárt, and B. Udvari [12] also contains the following result:

◊**Theorem 10.5.** *The asymptotic value of $N(h)$ is $h!/2$.*

There are also results on the number of slim semimodular diagrams as a function of their size, rather than length, see G. Czédli, T. Dékány, L. Ozsvárt, N. Szakács, and B. Udvari [10] and G. Czédli [7].

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