Fractal lattices and what they generate*

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Talk: Budapest (March19), Novi Sad(March23,2007)

in memoriam of András P. Huhn (1947–1985)

2007. március 20.

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In colloquial usage, a *fractal* is a geometric shape that is self-similar (at least approximately) to its arbitrarily small parts. Nature has many objects that approximate fractals. These objects include river networks, systems of blood vessels and pulmonary vessels, cauliflower or broccoli, snow flakes, mountains and lightning bolts. Fractals are frequently used tools to make mathematics popular for wider audience. Therefore it is quite natural to investigate the related notion in algebra and, first of all, in lattice theory.

Recall: if $L \in \mathbf{HSP}\mathcal{V}$ and L is a subdirect irreducible lattice then $L \in \mathbf{HSP}_u\mathcal{V}$ (Jónsson Lemma).

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Answer: not yet. But:

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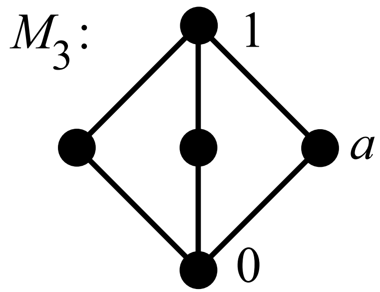

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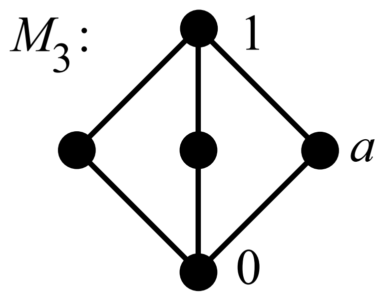

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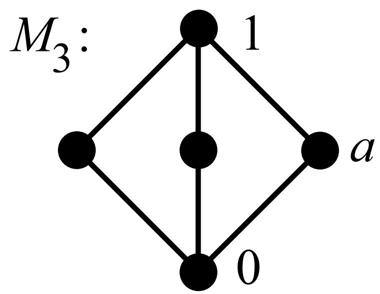

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The main theorem comes. (Asserts more than used for convexities.)

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(4) $|L| \ge 3 \Rightarrow L$ is 0–1-embedded in an M_3-simple quasifractal. $\exists$ $2^{\aleph_0}$ many countable quasifractals generating {all lattices}.

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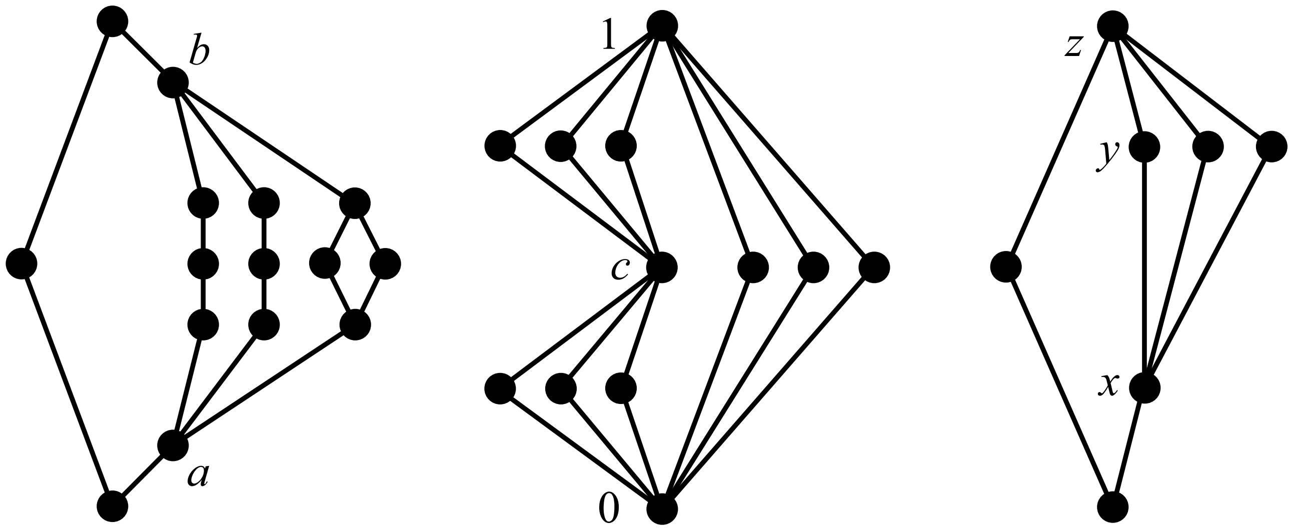

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How to get $2^{\aleph_0}$ many? Put $FL(\omega)$ and, for $T \subseteq \{$prime numbers$\}$, the Sub($\mathbf{Z}_p^3$) into a single lattice $L(T)$, and use one of Herrmann and Huhn's result (on Huhn's 3-diamonds) to show that the $L(T)$ are embedded in distinct quasifractals.

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$(\mathbf{N}, \leq)$ is the directed union of the chains $\mathbf{n}$, $n \in \mathbf{N}$. The same holds for $(\mathbf{N}, \leq)^d$! Too bad. Remedy:

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Lemma: (A) if $A_i \leq_{hu} A_{i+1}$ then $\bigcup_{i\in\mathbf{N}} A_i$ is unique. (B) If $A_i \leq_{hu} A_{i+1}$, $B_i \leq_{hu} B_{i+1}$ and $A_i \leq_{hu} B_i$ then $\bigcup_{i\in\mathbf{N}} A_i \leq \bigcup_{i\in\mathbf{N}} B_i$.

Proof: (A) exercise or the uniqueness of direct limit (in category theory). (B) exercise.

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Sub$(F^p) \leq L_H$. Hence L_H is not a quasifractal. Now Sub$(F^p) \leq_{01} L_H$ iff $p \in H$, proving that the L_H are distinct.♠

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(3) What is the cardinality of the set of β-fractal generated varieties?

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