Fractal lattices I. *

by **Gábor CZÉDLI** (Szeged)

to honour the 70th birthday of **Tibor Katriňák**
at the conference in Tale (September, 2007)

2007. szeptember 7.

*http://www.math.u-szeged.hu/∼czedli/

Happy
birthday !

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Open problems:

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Open problems: Are there more than countably many fractal generated varieties? Are they all modular? {all lattices} ?

Theorem 2. *There are continuously many lattice varieties such that each of them is not semifractal generated.*

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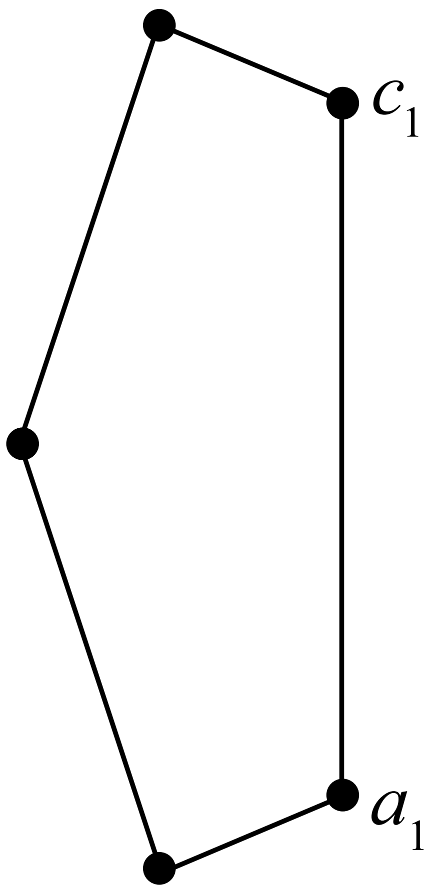
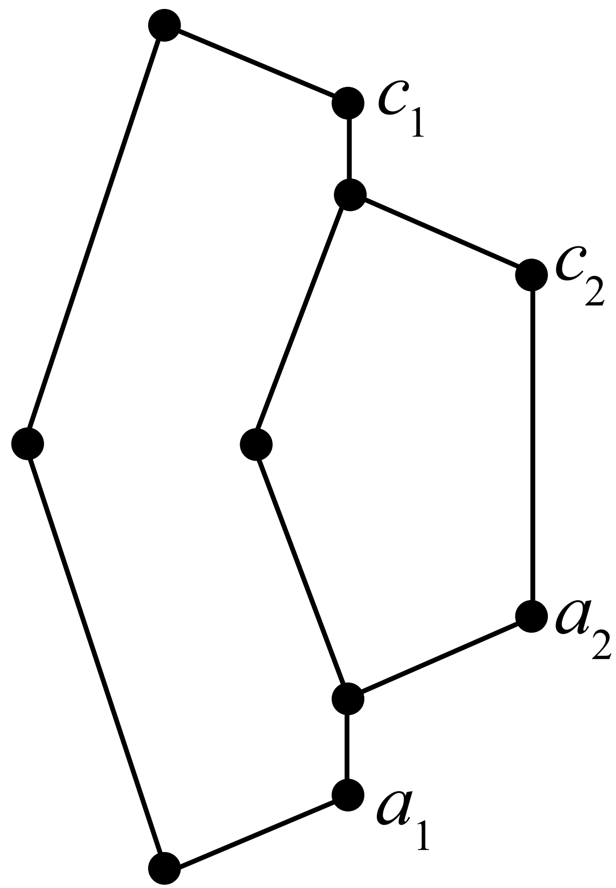
Key idea of the proof. Suppose L is a nondistributive semifractal and $\mathcal{V} = \mathbf{HSP}\{L\}$. Then N_5 of M_3 is a sublattice of L. Suppose $N_5 \leq L$. ($M_3 \leq L$ would be more complicated).



c_1
a_1



c_1
c_2
a_2
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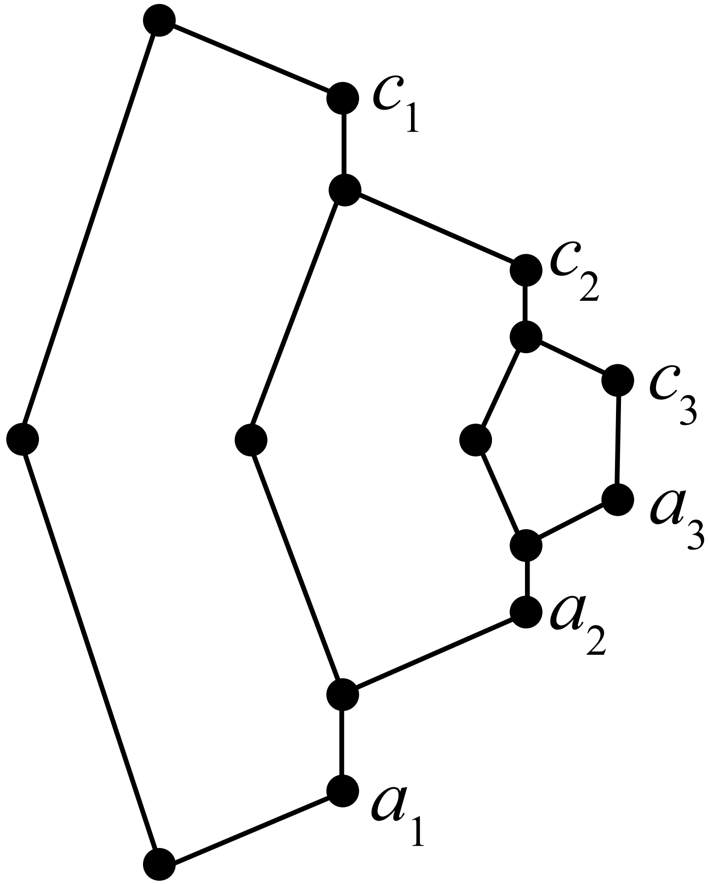

c_1
c_2
c_3
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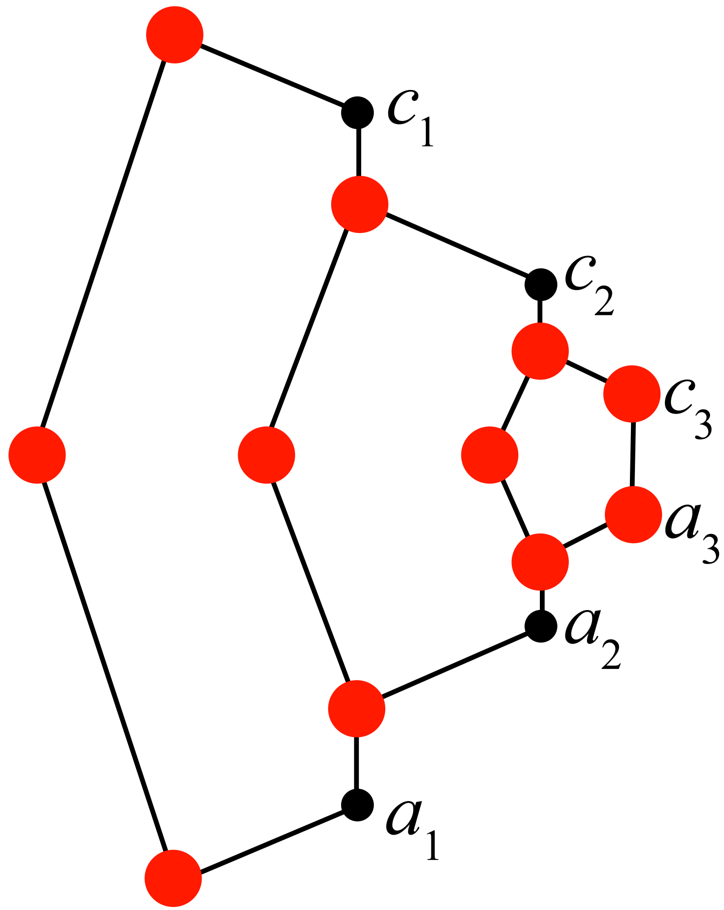
a_2
a_1



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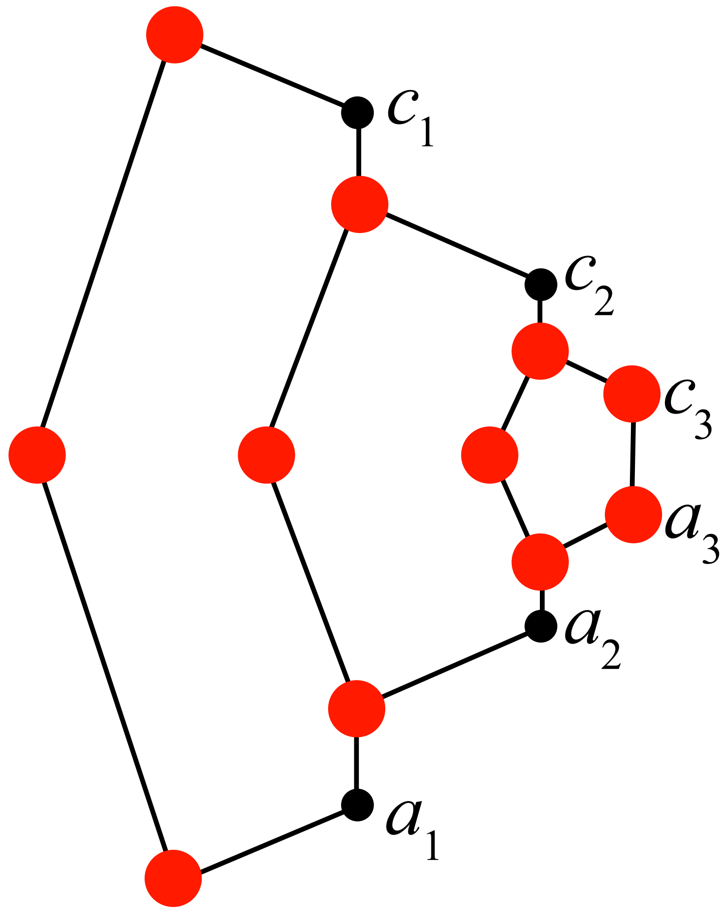

S.i. length not bounded. $2^{\aleph_0}$ many varieties fails this. Q.e.d.

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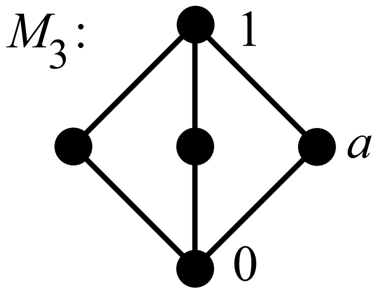

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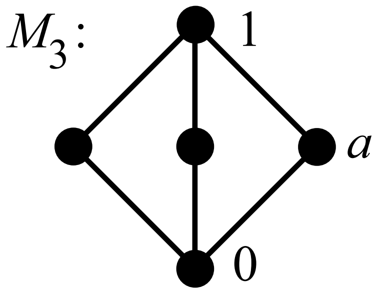

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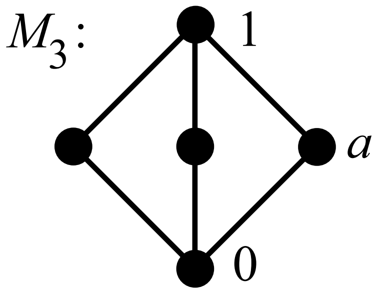

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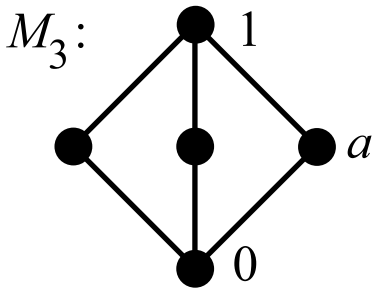

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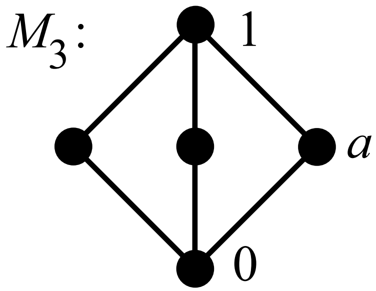

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Easy: Insert L to each interval, then M_3 to each three-element chain, $\aleph_0$ steps; union. Q.e.d. **Problem:** only „quasi"?

An application of quasifractals

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Now, if $\mathcal{V}$ is a *convexity* and some $L \in \mathcal{V}$ has a nontrivial distributive interval (e.g., $a \prec b$) then $\mathbf{HCP}\{\mathbf{2}\}$ is included in $\mathcal{V}$. J. Lihová: $\mathbf{HCP}\{\mathbf{2}\} \subseteq \mathcal{V}$ in many other cases.

Theorem 4. *(Partial answer to Jakubík's problem.) If L is an M_3-simple quasifractal then the convexity $\mathcal{V} := \mathbf{HCP}\{L\}$ includes no minimal subconvexity.*

Note that such an L exists by Thm. 3.

Proof.

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Finally, if $L' \in \mathbf{HCP}\{M\}$ then Lemma (with M in place of L) $\Rightarrow$ $|L'| \geq |M|$, a contradiction. Hence $\mathbf{HCP}\{M\} \subset \mathcal{W}$

Lemma. *Each subd.irr.* $L' \in \mathcal{V} = \mathbf{HCP}\{L\}$ *is again an* M_3*-simple quasifractal and* $|L'| \geq |L|$.

Proof of the Lemma. $L' \in \mathbf{HCP}\{L\} =^{\text{Lihová}} \mathbf{P}_s\mathbf{HCP}_u\{L\}$. So $L' \in \mathbf{HCP}_u\{L\}$. Now $\mathbf{P}_u$ preserves the red properties (by Łoś' thm.), so does $\mathbf{C}$ (by def's), and $\mathbf{H}$ does nothing. Q.e.d.

Proof. Suppose $\mathcal{W} \subseteq \mathcal{V}$ is minimal. Choose a subd.irr. $L' \in \mathcal{W}$ and an ultrapower M of L' with $|M| > |L'|$. Lemma $\Rightarrow$ L' has the red properties, the ultrapower preserves the red properties, so M is an M_3-simple quasifractal, $|M| > |L'|$ and $M \in \mathcal{W}$.

Finally, if $L' \in \mathbf{HCP}\{M\}$ then Lemma (with M in place of L) $\Rightarrow$ $|L'| \geq |M|$, a contradiction. Hence $\mathbf{HCP}\{M\} \subset \mathcal{W}$ shows that $\mathcal{W}$ is not minimal. Q.e.d.

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Thank you, Miroslav, for organizing

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