Stronger association rules in algebra*

by **Gábor CZÉDLI** (Szeged)

to honour the 70th birthday of **Rudolf Wille**
at AAA75, Darmstadt, November 2–4

2007. november 2.

*http://www.math.u-szeged.hu/∼czedli/

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Hungarian–English Dictionary

Szegedi Tudományegyetem = University of Szeged

The rest of the labels are in Latin:

Universitatis Scientiarum Szegediensis

Vinum Universitatis

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B.

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B. Jónsson: CD, A. Day:CM, H.P. Gumm: CM, many others, Freese–McKenzie (Commutator Theory, Ch. XIII): ∞ many

Recently: —–, Horváth and Lipparini (AU, 2005): **all that imply modularity**.

All these above are based on the **Wille–Pixley** algorithm.

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Warning about this talk: mostly I will just look around !

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— A real application of $\mathcal{C}_{new}$ (in universal algebra)

Strong association rules versus Galois–Wille's $\mathcal{C}_{gw}$

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Context

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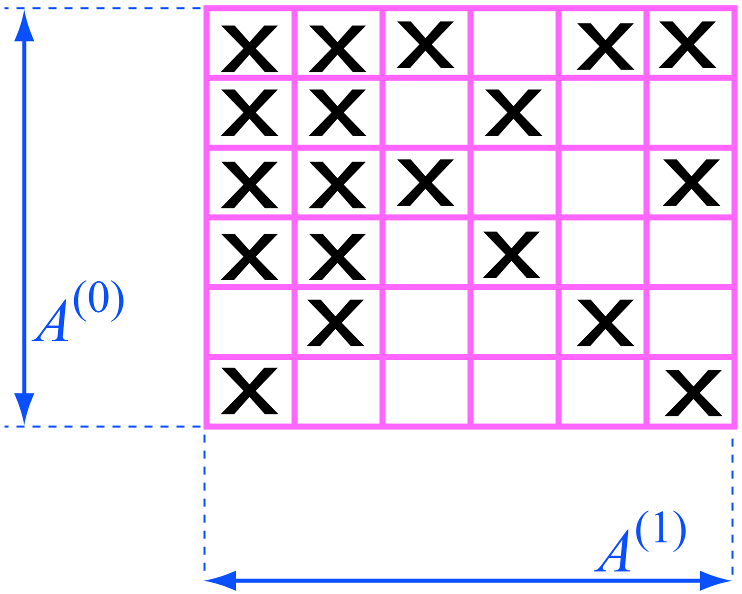
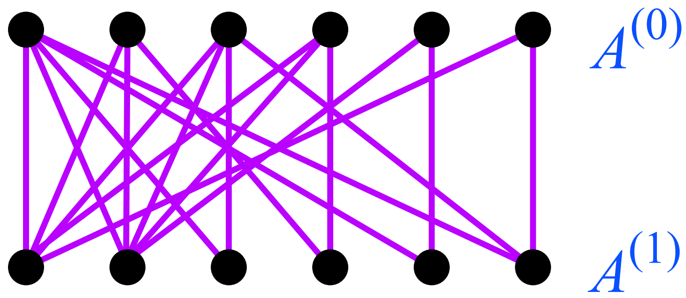


Another possibility
(the same context):



$\bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n^{(1-i)}(Y)} \{y\}\rho_{1-i}$ 5' (C) Czédli 40'

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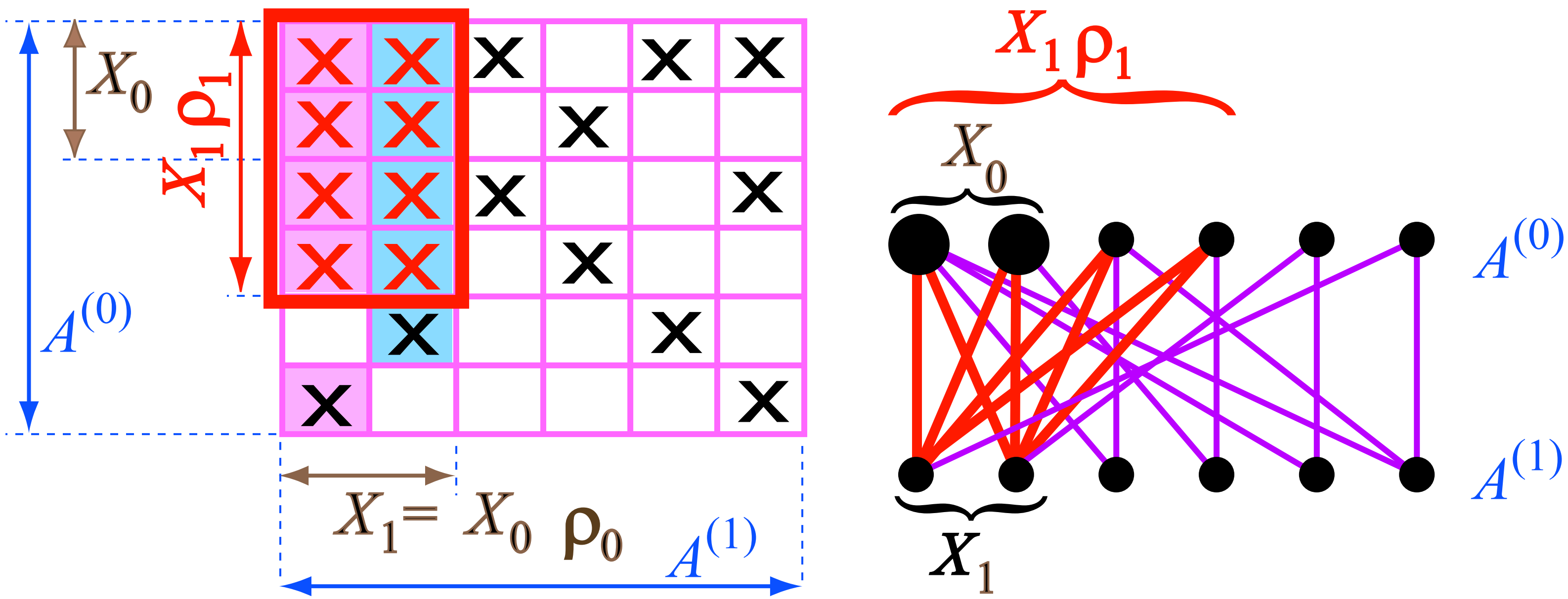

To get $\mathcal{C}_{gw}^{(0)}(X_0)$, take the intersection of sides of maximal triangles; only those sides that include X_0.

Our permanent notations

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$$\rho_0 = \rho, \quad \rho_1 := \rho^{-1},$$

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$$X\rho_i := \{y \in A^{(1-i)} : \forall x \in X, \ (x, y) \in \rho_i\} \quad (\text{set of } \rho\text{-neighbors}),$$

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$\mathcal{C}_{gw} = (\mathcal{C}_{gw}^{(0)}, \mathcal{C}_{gw}^{(1)})$: the pair of **Galois–Wille closure operators**.

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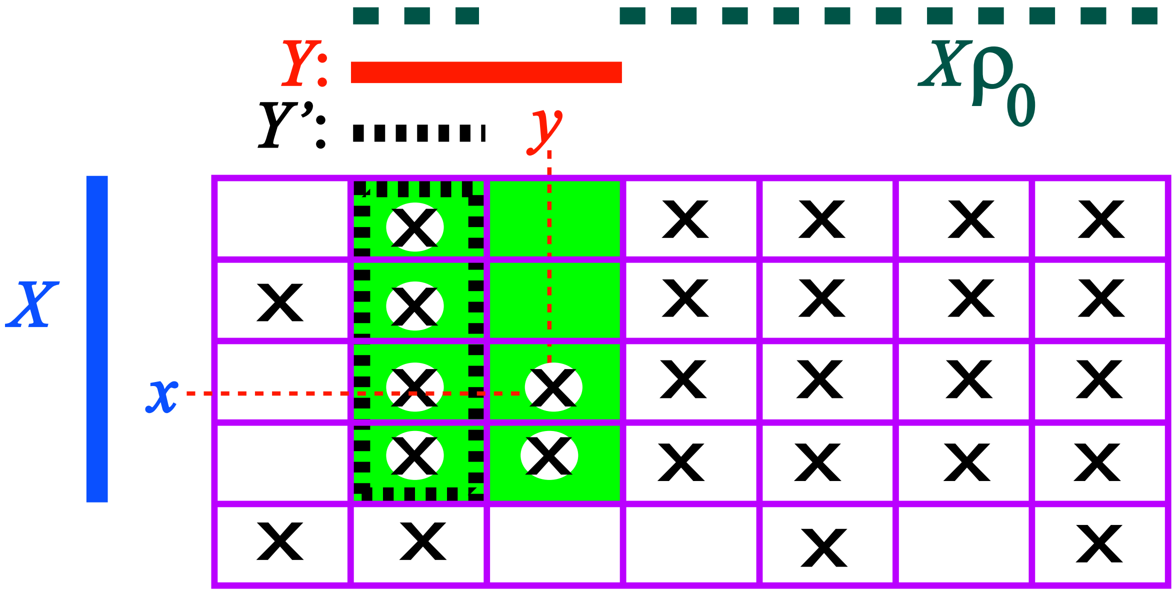

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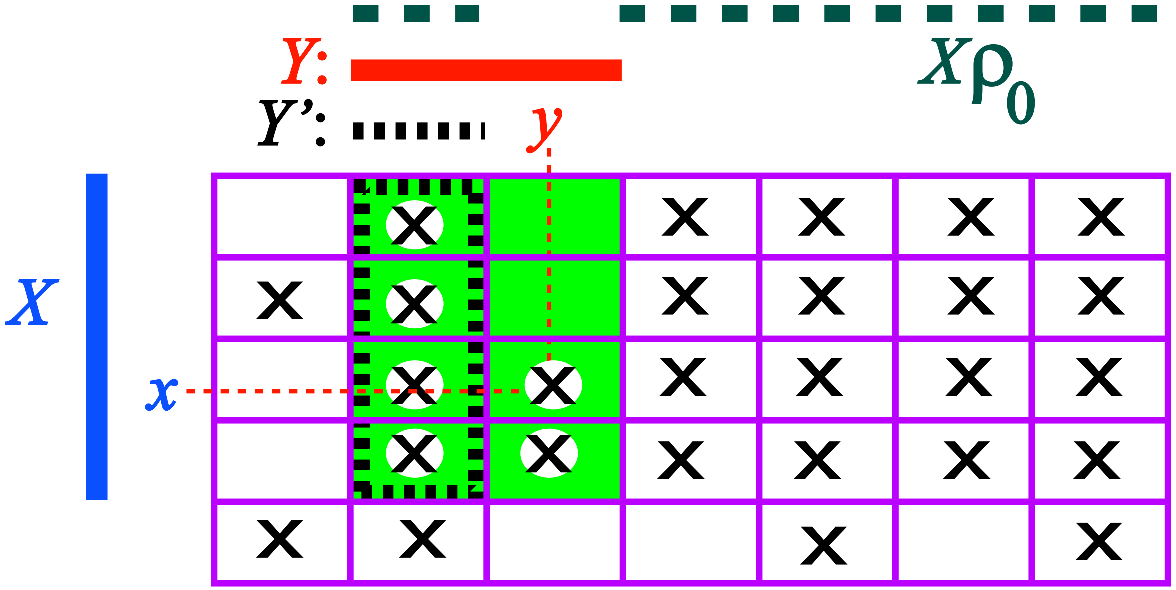

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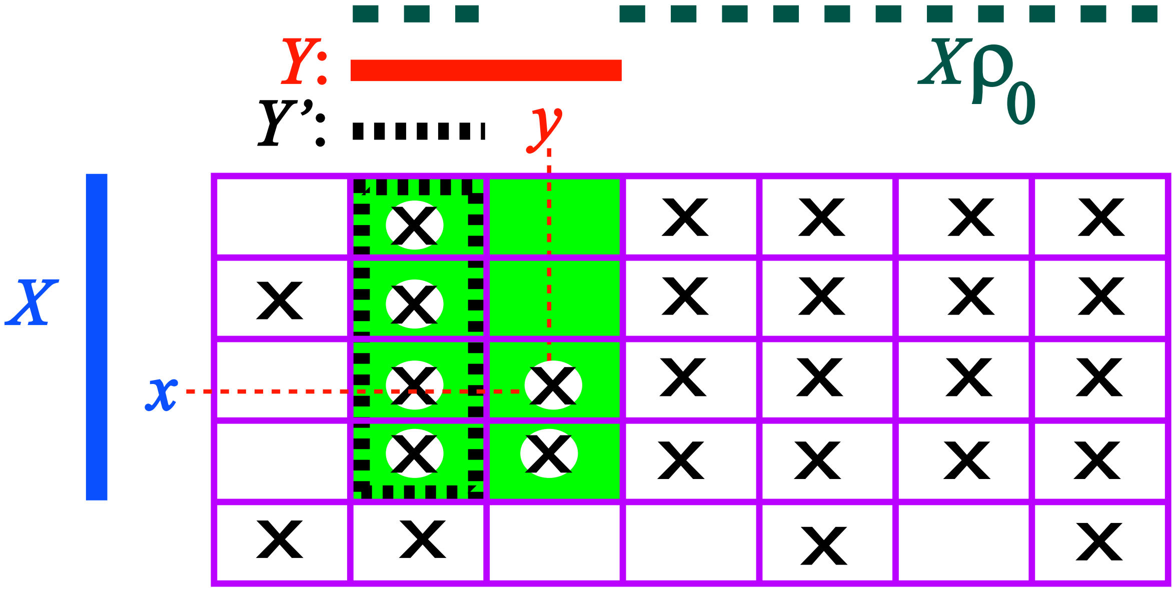

Let $X \in P(A^{(i)})$. (Here $i = 0$.) We say that Y **is a ρ-image of** X if there is a surjection $\varphi : X \to Y$ with $\varphi \subseteq \rho_i$. E.g.: Y', Y.

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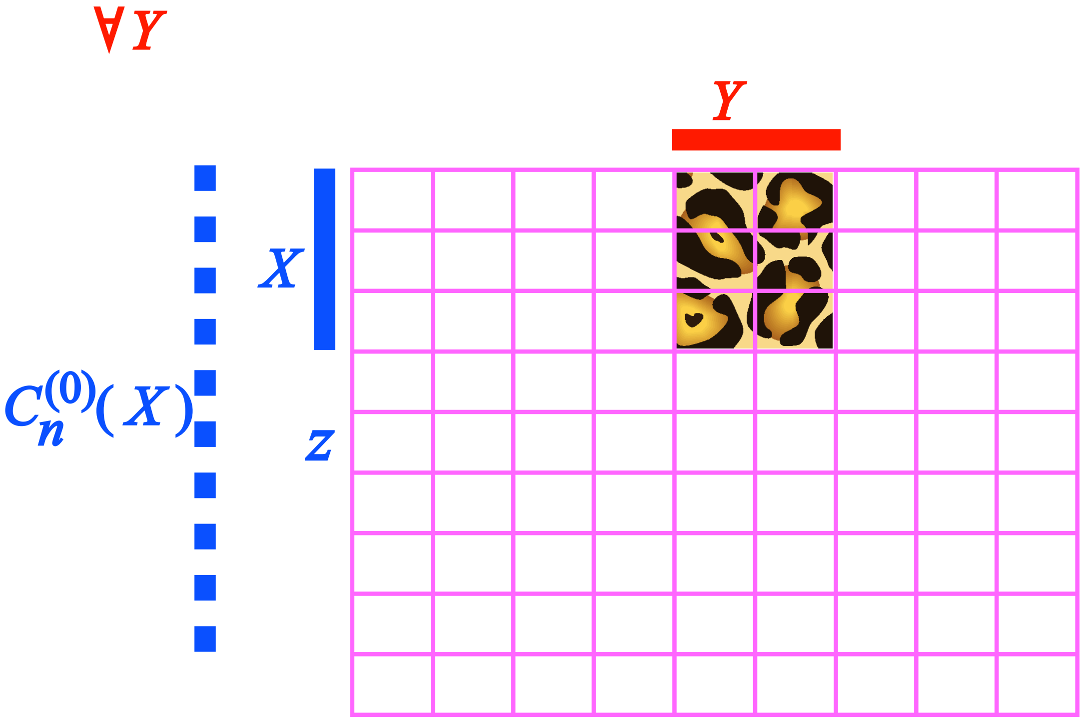
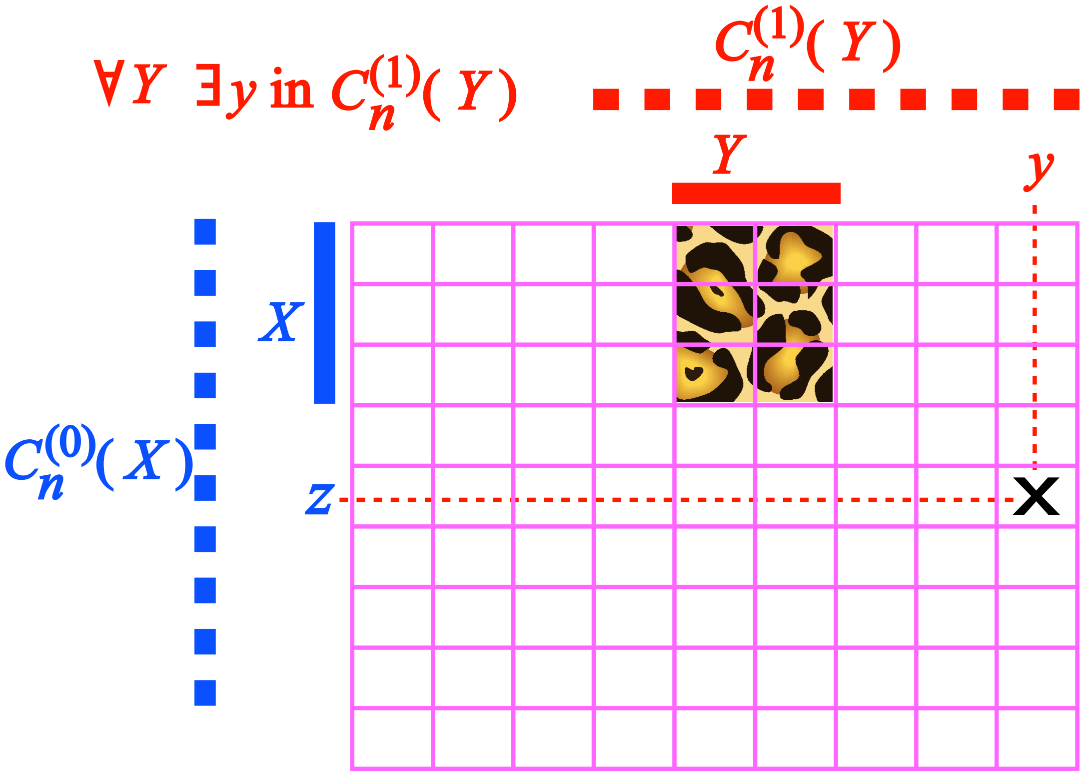
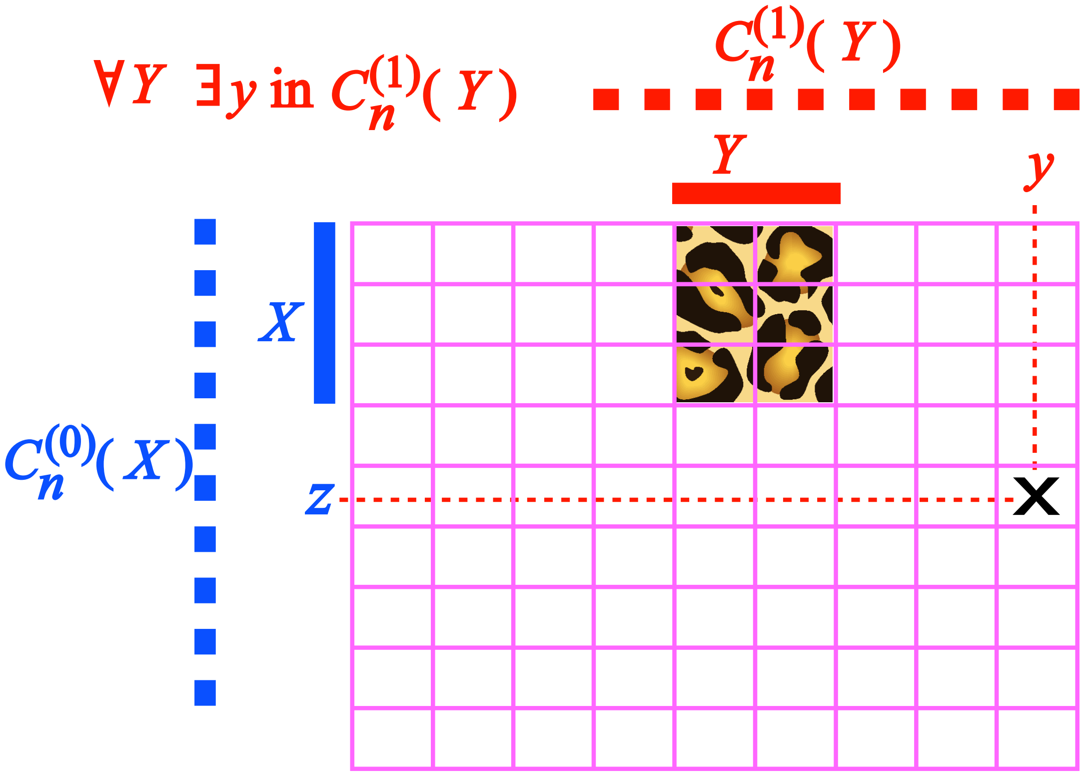


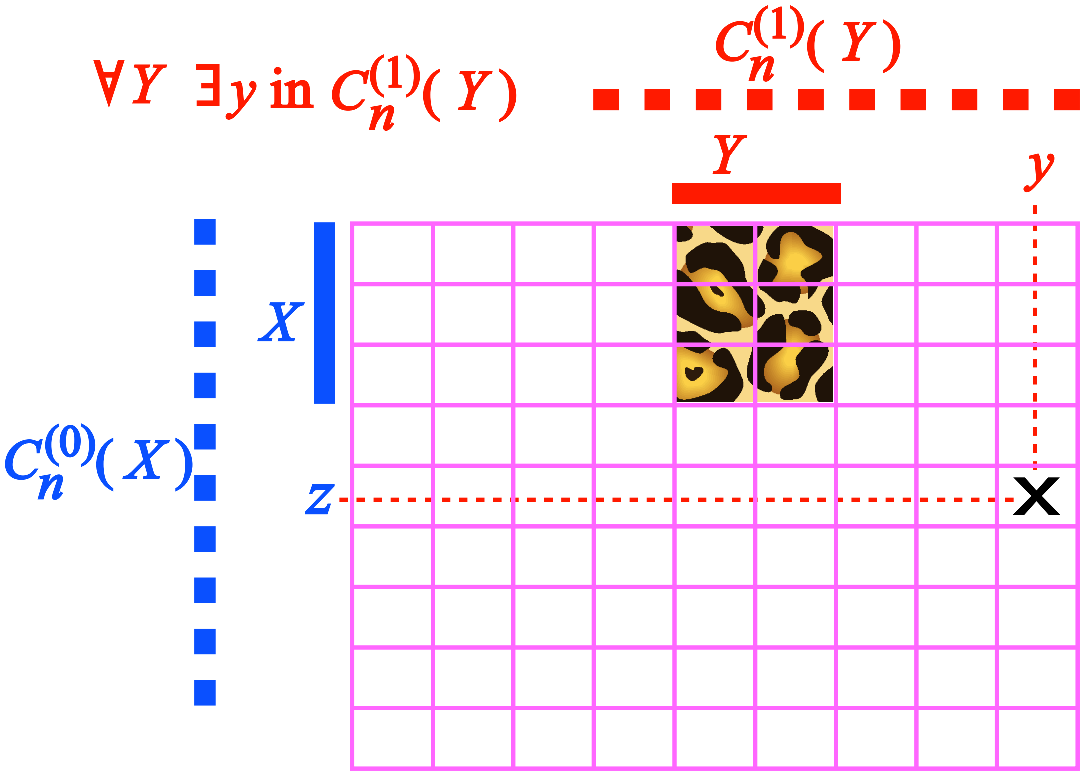
Let $z \in \mathcal{C}_n{}^{(0)}(X)$. Then $z \in \mathcal{C}_{n+1}^{(0)}(X)$ iff for **each** ρ-image Y of X iff



there is a $y \in \mathcal{C}_n{}^{(1)}(Y)$ with the cross indicated (i.e., $(z, y) \in \rho$).



there is a $y \in \mathcal{C}_n{}^{(1)}(Y)$ with the cross indicated (i.e., $(z, y) \in \rho$). The same is described with the following formula (whose „hard part'' is always in the running head above):



$$\mathcal{C}_{n+1}^{(i)}(X) := \mathcal{C}_n{}^{(i)}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n{}^{(1-i)}(Y)} \{y\}\rho_{1-i}\,.$$

$\bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n{}^{(1-i)}(Y)} \{y\}\rho_{1-i}$ 13'

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$$\mathcal{C}_{new} = (\mathcal{C}_{new}^{(0)}, \mathcal{C}_{new}^{(1)}) := (\bigcap_{n=0}^{\infty} \mathcal{C}_n{}^{(0)}, \bigcap_{n=0}^{\infty} \mathcal{C}_n{}^{(1)}),$$

which means that, for all $X \in P(A^{(i)})$,

$$\mathcal{C}_{new}^{(i)}(X) = \bigcap_{n=0}^{\infty} \mathcal{C}_n{}^{(i)}(X).$$

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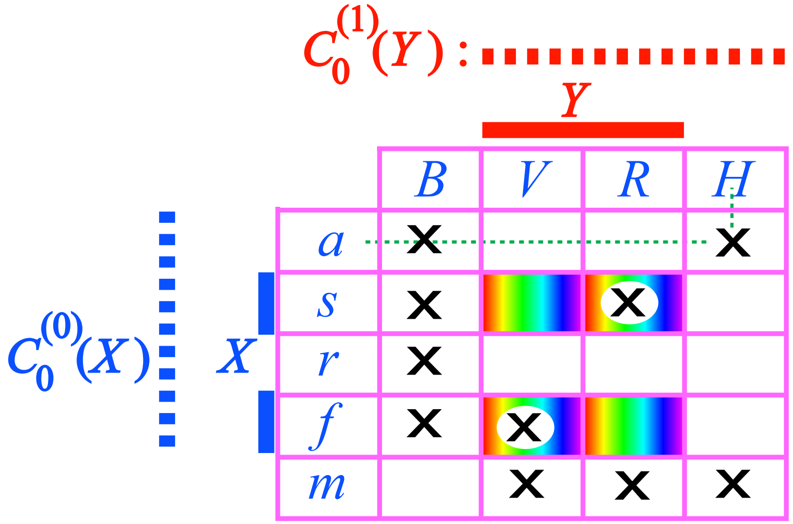
$$\mathcal{C}_{gw} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots \supseteq \mathcal{C}_{new}$$

(understood componentwise).

Proof: later.



The smallest interesting example: $A^{(0)} = \{a, s, r, f, m\}$, $X = X_0 = \{s, f\}$,

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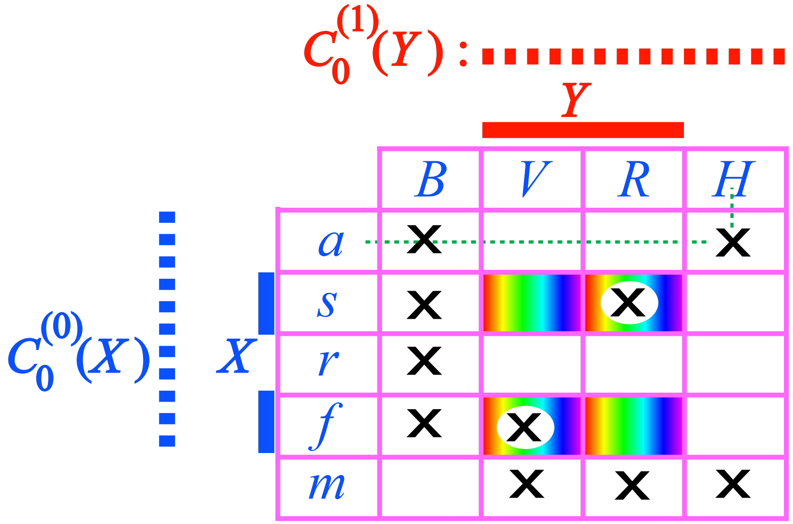

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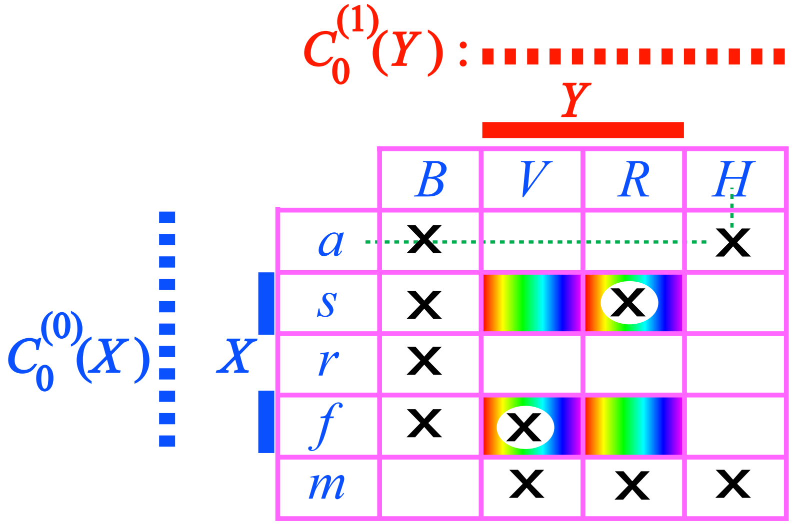

The smallest interesting example: $A^{(0)} = \{a, s, r, f, m\}$, $X = X_0 = \{s, f\}$, $\mathcal{C}_{gw}^{(0)}(\{s, f\}) = \{s, f, a, r\}$. However, $\mathcal{C}_1^{(0)}(\{s, f\}) = \mathcal{C}_{new}^{(0)}(\{s, f\}) = \{s, f, a\}$. So $\mathcal{C}_{new}$ is distinct from $\mathcal{C}_{gw}$.

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$\bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n{}^{(1-i)}(Y)} \{y\}\rho_{1-i}$ 16' (C) Czédli 29'

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X: members of a political/scientific body. Which member from $A^{(0)}$ should we adopt next?

X: chemical compounds (and the attributes are pharmaceutical effects they produce), which chemical compound should we add next (and spend a lot of money on testing the new medicine)?

Notice that $\mathcal{C}_{new}$ does not always give new insight into things, for we do not now wich new object is better: one in $\mathcal{C}_{new}^{(0)}(X)$ or one in $\mathcal{C}_{gw}^{(0)}(X) \setminus \mathcal{C}_{new}^{(0)}(X)$?

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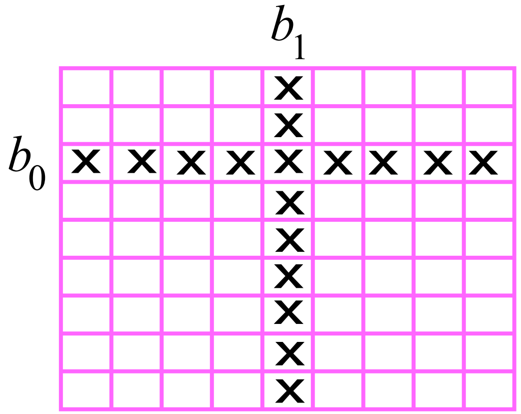
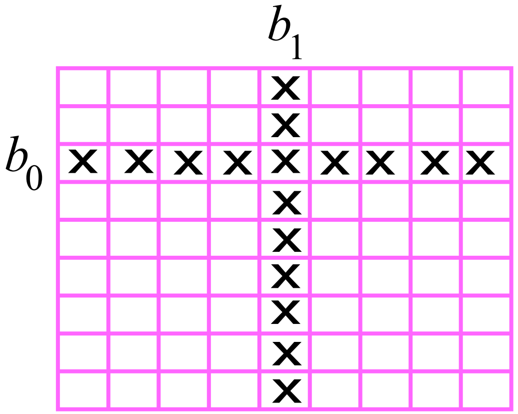
There is a generalization of $\mathcal{C}_{new}$ for the situation **where only a part of the attributes are positive**, cf. my website for details.



$\mathcal{C}_{new}$ in algebra

Relations, and the Galois-Wille's $\mathcal{C}_{gw}$ associated with them occur everywhere in mathematics. They often give us something useful.



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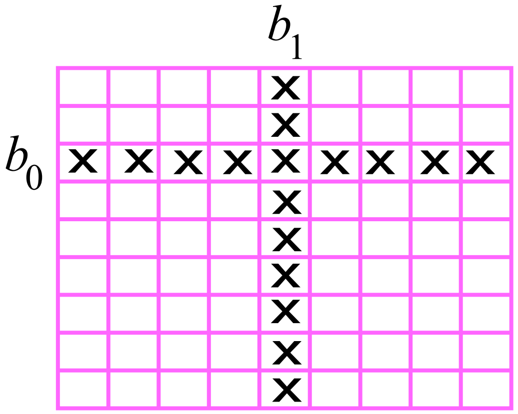

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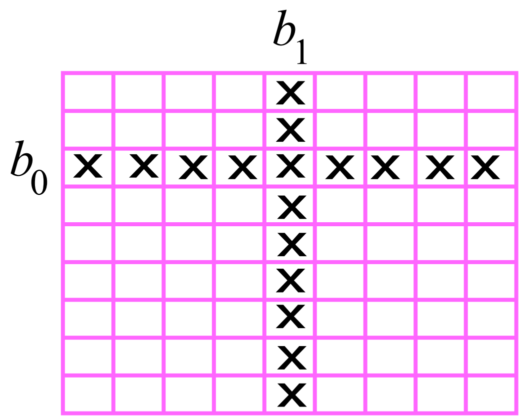

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Relations, and the Galois-Wille's $\mathcal{C}_{gw}$ associated with them occur everywhere in mathematics. They often give us something useful.

A possible project: Check $\mathcal{C}_{new}$? for the relations in our favorite field! But $0 \in A^{(0)}$ and $1 \in A^{(1)}$ should be removed, for otherwise $\mathcal{C}_{new} = \mathcal{C}_{gw}$, and we do not obtain anything new.

Before further steps, it is reasonable to ask if $\mathcal{C}_{new} \neq \mathcal{C}_{gw}$, and how often.

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For lattices we have Wille's classical example: the concept lattice of the context $(J(L), M(L), \leq)$.

1. Proposition. *If* L *is a finite* **modular** *lattice and we consider the context* $(J(L), M(L), \leq)$ *then* $\mathcal{C}_{new}$ *equals* $\mathcal{C}_{gw}$.

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Proof: easy (half a page), resembles the proof of KORP.

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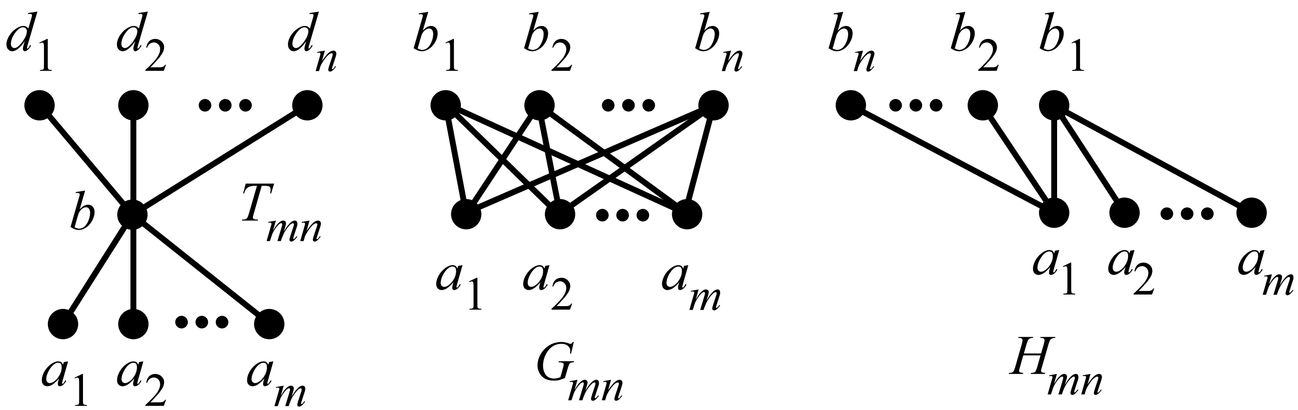
The converse is false.

After lattices, posets are also our favorites. Let $Q = (Q, \leq)$ denote a poset.



1. Theorem. *$\mathcal{C}_{new}(Q, Q, <) = \mathcal{C}_{gw}(Q, Q, <)$ iff $U(Q \setminus \max(Q)) \neq \emptyset$ and $L(Q \setminus \min(Q)) \neq \emptyset$.*

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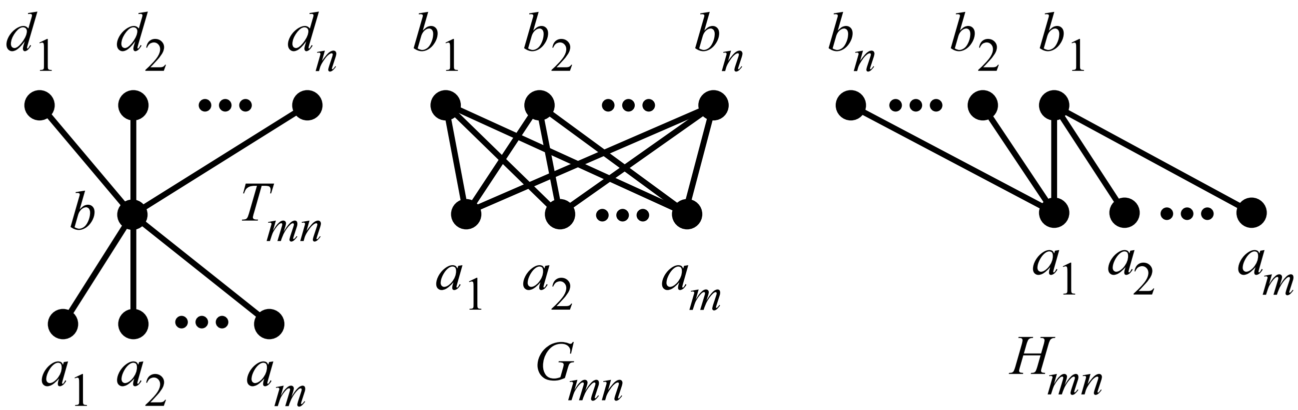

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Proof: additional 6 lines

3. Theorem. $\mathcal{C}_{new}(Q, Q, \leq) = \mathcal{C}_{gw}(Q, Q, \leq)$ *if and only if either*

$$|\max(Q)| = |\min(Q)| = 1,$$

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$$|\max(Q)| \geq 2, \ |\min(Q) \geq 2,$$
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Proof 0.8 page.

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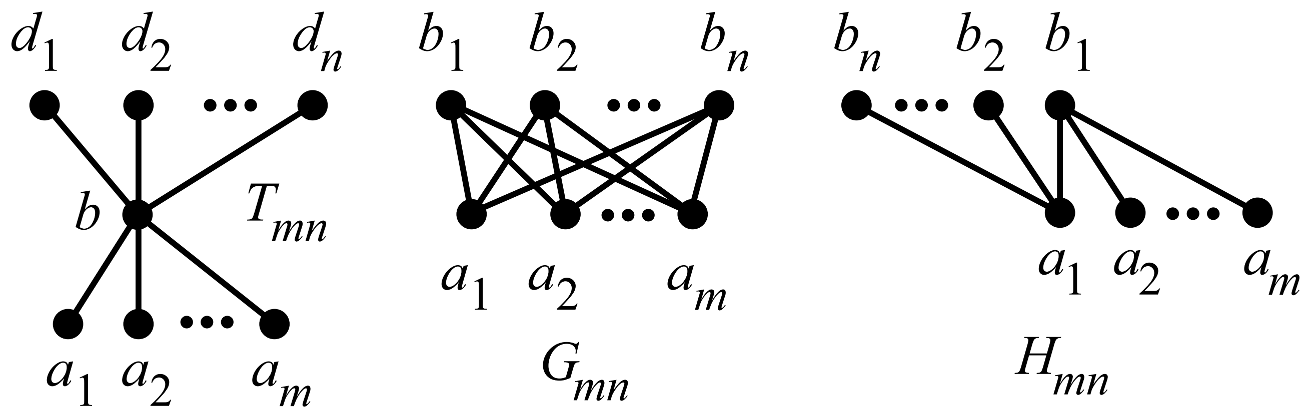
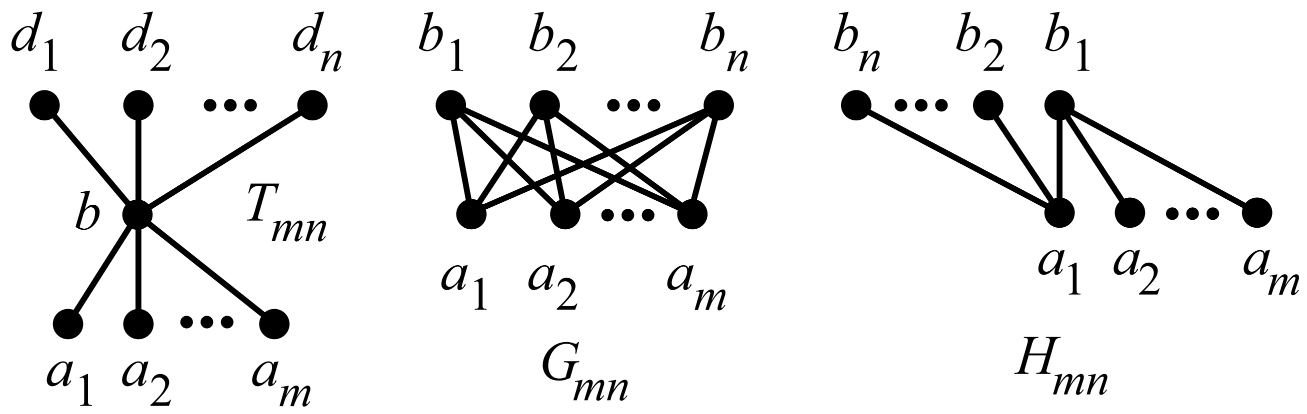


4. Theorem. *$\mathcal{C}_{new}(Q, Q, \preceq) = \mathcal{C}_{gw}(Q, Q, \preceq)$ if and only if either Q is (isomorphic to) T_{mn} for some $m, n \geq 1$, or Q is H_{mn} or G_{mn} for some $m, n \geq 2$, or Q is a disjoint union of at most two-element chains.*



Proof: 2 pages.

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And in the outer world?

$\lvert A^{(0)}\rvert = \lvert A^{(1)}\rvert =$	4	5	6	8	10	12	14	20	30	40	48
$\lvert\{\text{tests}\}\rvert$	1000	1000	1000	100	100	100	100	100	100	100	100
$\mathcal{C}_{new} \neq \mathcal{C}_{gw}$	549	757	889	98	100						
$\neq$, $\lvert X\rvert = 2$	282	517	736	94	100	100	100	100	100	100	100

Papers of this topic, the computer program, and (from the next week) this presentation:

www.math.u-szeged.hu//∼czedli/

$\lvert\mathcal{C}_{new}\rvert = \lvert\mathcal{C}_{gw}\rvert =$	3	4	5	6	7	8	9
$\lvert\{\text{tests}\}\rvert$	1000	1000	1000	1000	1000	1000	1000
$\mathcal{C}_{new}^{(i)}(X) \neq \mathcal{C}_{gw}^{(i)}(X)$	17	39	82	185	306	402	571

If both the context (of a given size) and X are chosen randomly.

$\bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n^{(1-i)}(Y)} \{y\}\rho_{1-i}$ 24' (C) Czédli 21'

Here is a statistics for those context from **Ganter and Wille's book** : *Formal Concept Analysis* whose size fits into the program:

	1.1	1.5a	1.5b	1.13	1.16	1.21	1.23	1.24	2.4	2.13	2.15
$\|A^{(0)}\|$	8	8	5	5	14	6	6	8	7	12	14
$\|A^{(1)}\|$	9	5	4	25	16	12	8	8	7	9	9
distinct	yes	no?	no?	yes		yes	yes	no?	yes	no?	yes

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Conclusion: $\mathcal{C}_{new}$ and $\mathcal{C}_{gw}$ are **often** distinct.

A real application of $\mathcal{C}_{new}$.

A real application of $\mathcal{C}_{new}$. Let $k \in \mathbf{N}$. A congruence is called k-**uniform**, if each of its classes has exaxtly k elements. A congruence is called **uniform** if it is k-uniform for some $k \in \mathbf{N}$.

K. Kaarli, $\sim$ 2005: If all congruences of a finite lattice L are uniform, then L is congruence permutable. (This answers Grätzer, Quackenbush and Schmidt.)

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$$\mathbf{1} \notin \mathbf{typ}\, V \iff \exists \text{ idempotent Mal'cev condition}$$

then any two 2-uniform congruences of A permute.

Remark

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Remark This gives something new even for lattices: there can be 2-uniform congruences even if other congruences are not uniform! **The proof heavily uses $\mathcal{C}_{new}$!** Some proofs come now:

2. Lemma. *$\mathcal{C}_{new}$ and $\mathcal{C}_n$, $n = 0, 1, \ldots$, are indeed pairs of closure operators. Further*

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$$\mathcal{C}_{gw} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots \supseteq \mathcal{C}_{new}$$

(understood componentwise).

$\bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n{}^{(1-i)}(Y)} \{y\}\rho_{1-i}$ 27' (C) Czédli 18'

The heart of the easy proof. $\mathcal{C}_0 = \mathcal{C}_{gw}$ is OK (Galois). Suppose $\mathcal{C}_n$ is already a pair of closure operators and we already know that the $(n+1)$-th closure map candidates $\mathcal{C}_{n+1}^{(0)}$ and $\mathcal{C}_{n+1}^{(1)}$ are **monotone and extensive** mappings.

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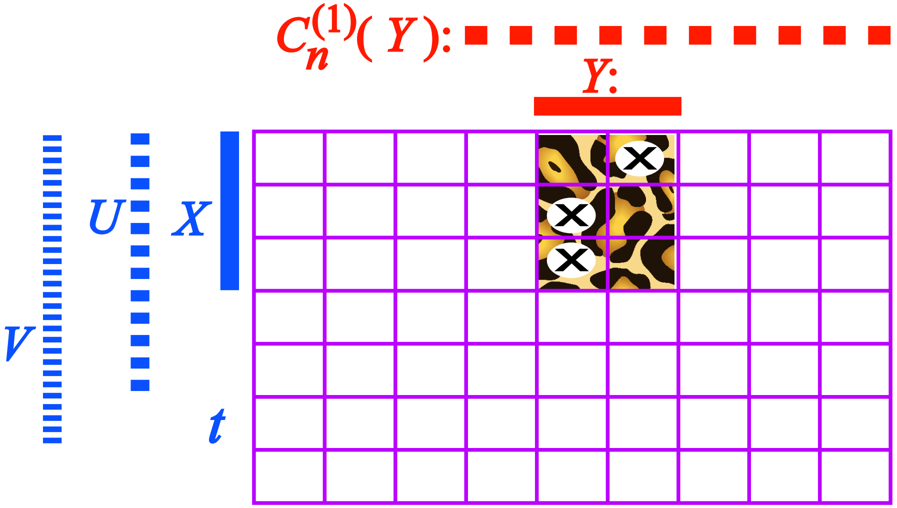
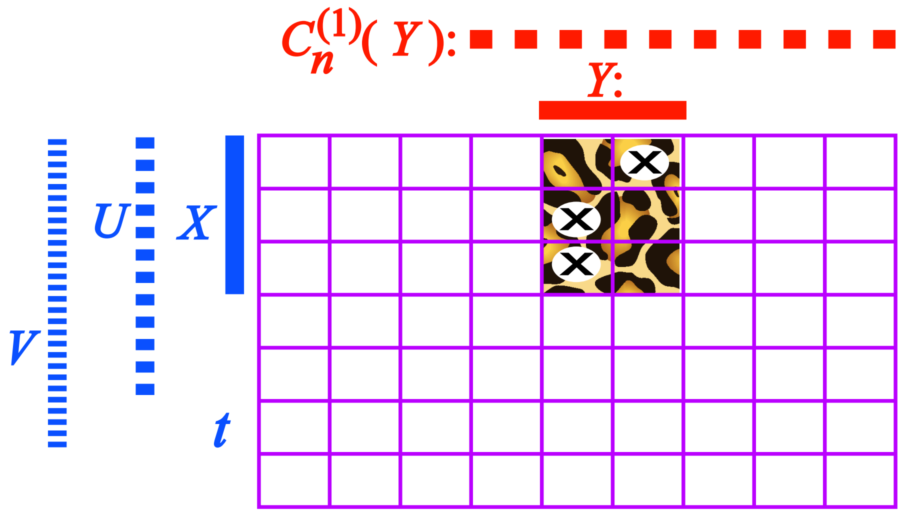
We want to show that they are **idempotent**. Picorially (for $i = 0$):



Here U is the $(n+1)$-th closure of X, and V is the $(n+1)$-th closure of U. We already know: $U \subseteq V$. **Aim:** $U = V$?



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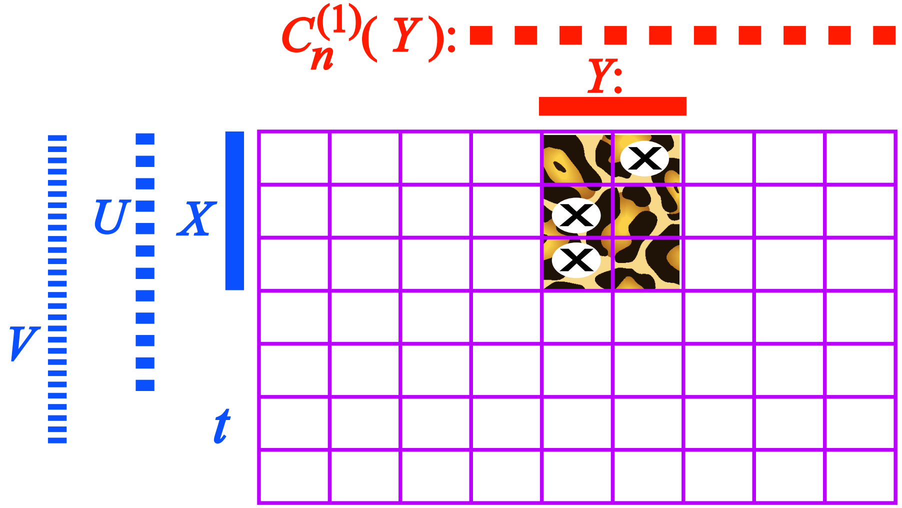
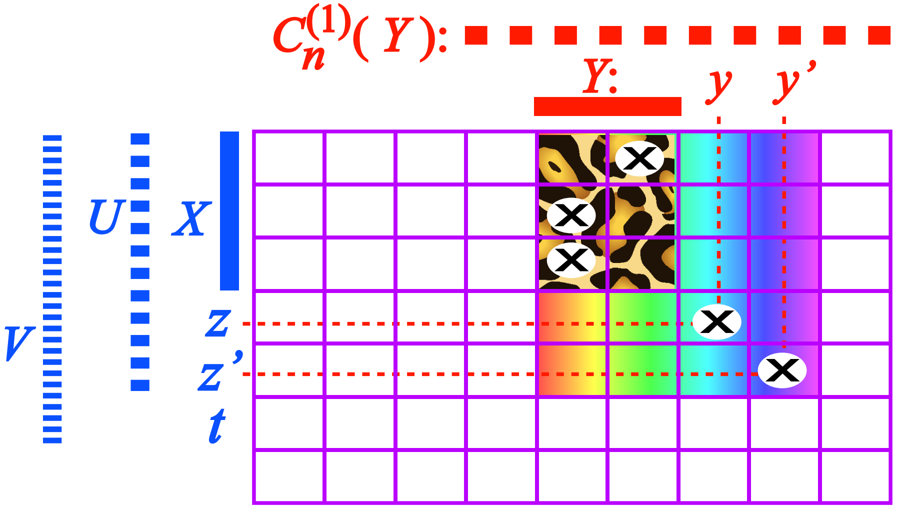
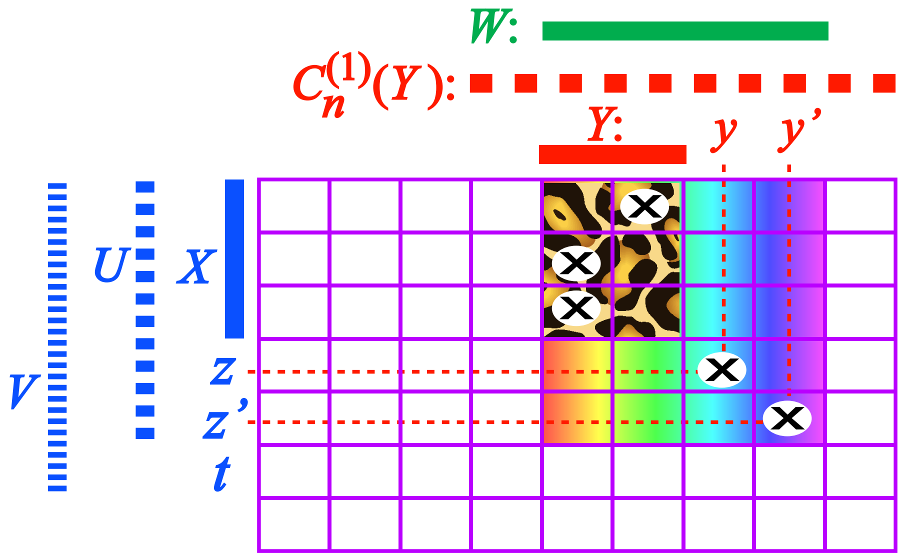


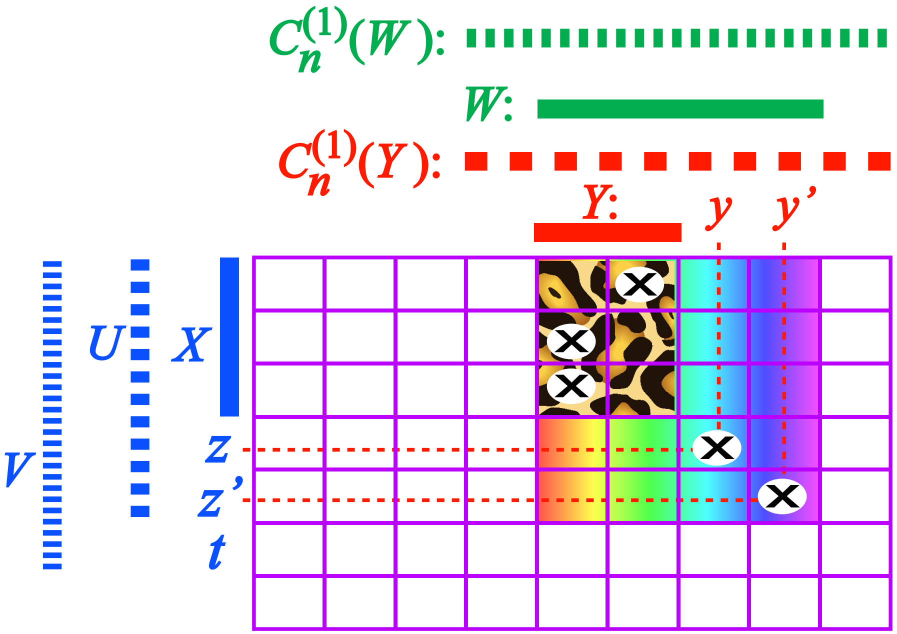
Consider an **arbitrary** ρ-image Y of X.



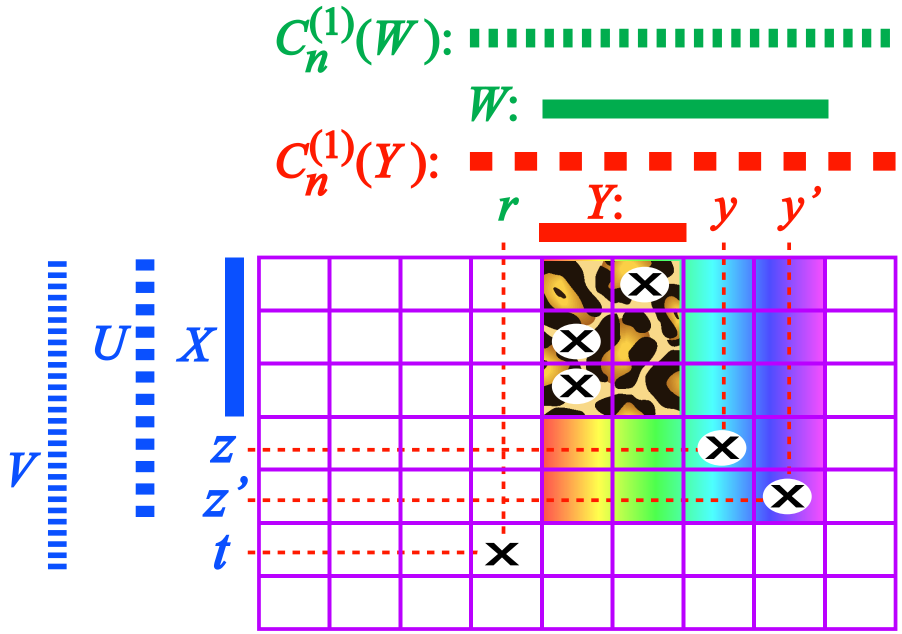
By the definition of U, for each $z \in U \setminus X$ there exists a column y in the n-th closure of Y such that the (z, y) entry is a cross. The same for (z', y').



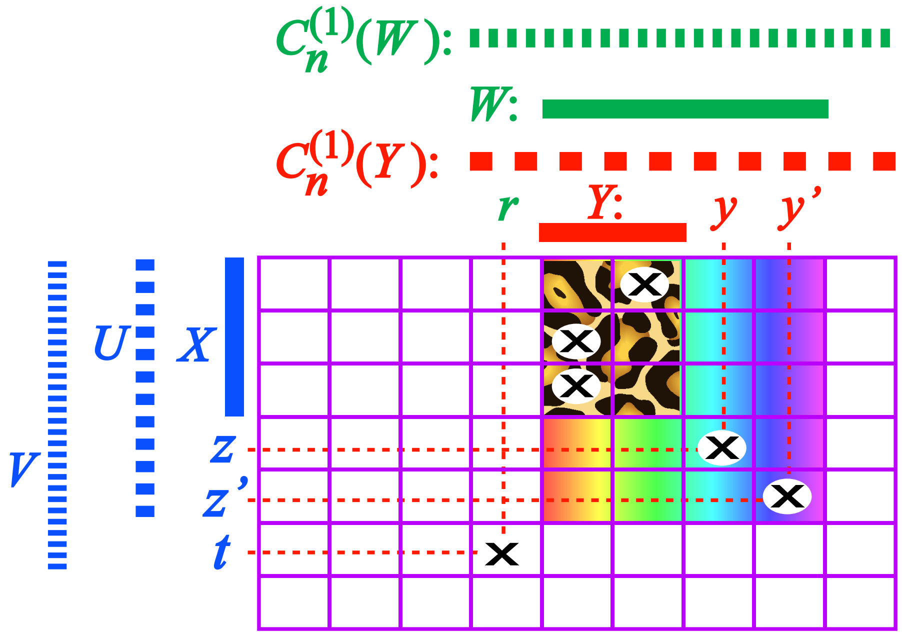
Add these columns $y, y', \ldots$ to Y. **This way we obtain** $W \subseteq \mathcal{C}_n{}^{(1)}(Y)$. By the induction hypothesis, the n-th closure of W is again $\mathcal{C}_n{}^{(1)}(Y)$ **(red dotted)**.



Along the thin red dotted lines (i.e., sending z to y, z' to y', ...) we extend the original surjection $X \to Y$ to a $U \to W$ surjection. Hence W is a ρ-image of U.



By the def. of V, there is a cross in row t whose column r belongs to $\mathcal{C}_n{}^{(1)}(W)$ **green dotted** $= \mathcal{C}_n{}^{(1)}(Y)$ **redd dotted**.



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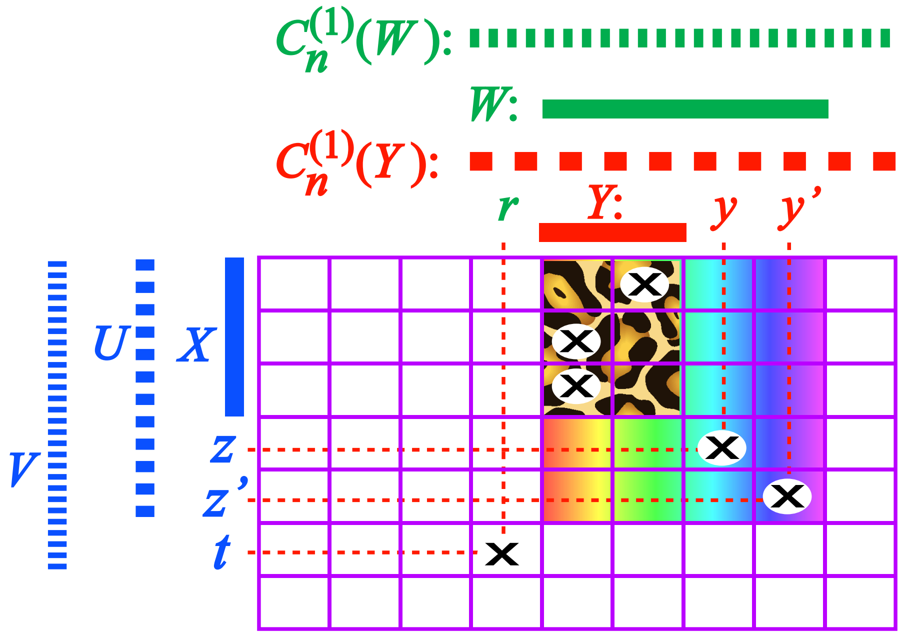

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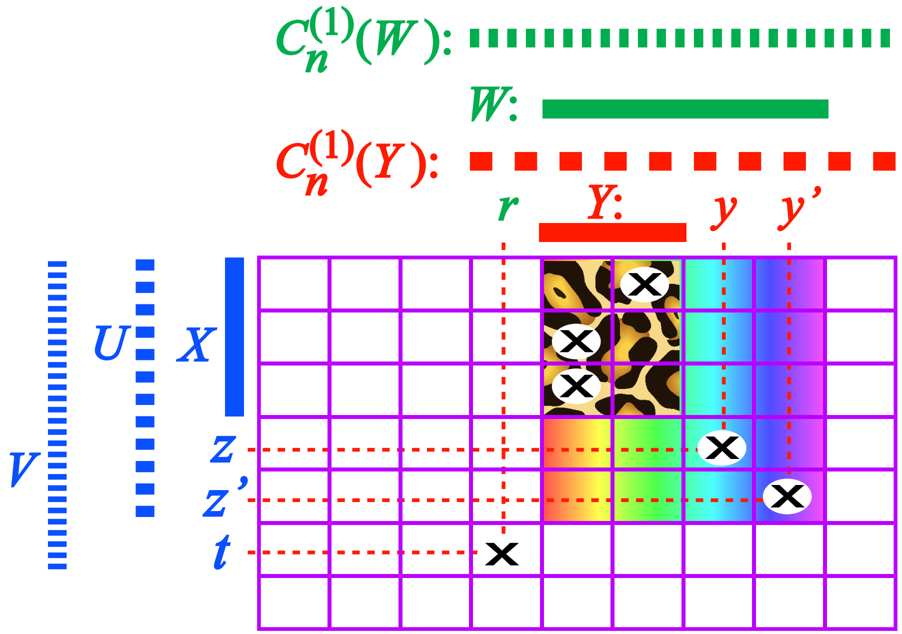

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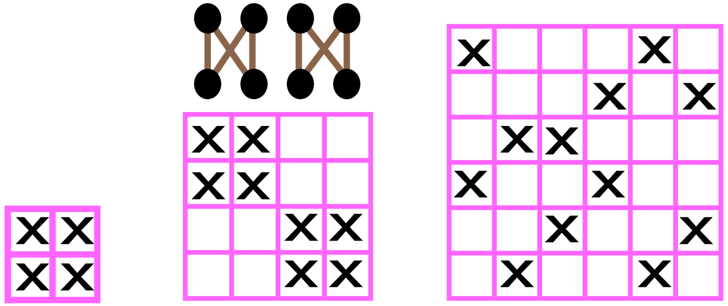
$$\mathbf{1} \notin \mathbf{typ}\, V \quad \Longleftrightarrow \quad \exists \textit{ idempotent Mal'cev condition}$$

then any two 2-uniform congruences of A permute.

Outline of the proof (how on earth does $\mathcal{C}_{new}$ come into the picture?)

$\bigcap_{Y \in \psi(X)}$ $\bigcup_{y \in \mathcal{C}_n{}^{(1-i)}(Y)}$ $\{y\}\rho_{1-i}$ 35' (C) Czédli 10'



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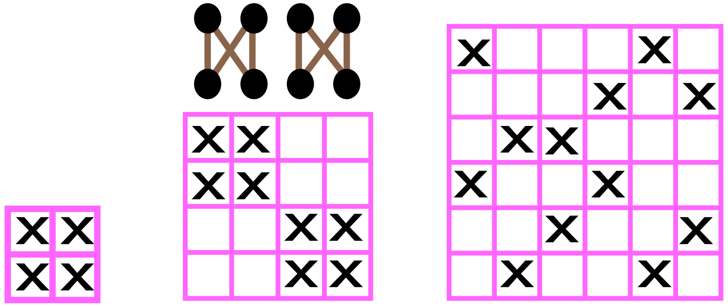
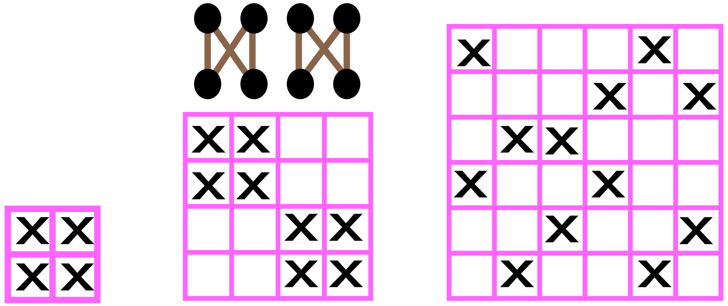
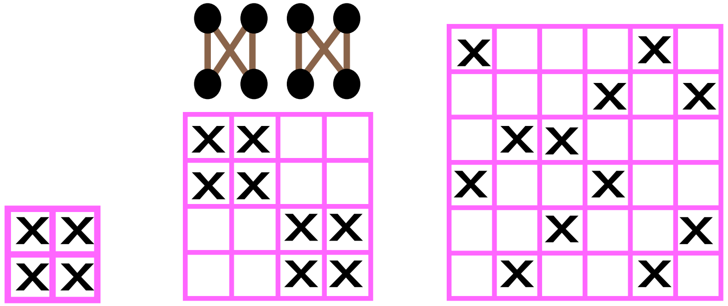


Straightforward: to re



Straightforward: to reduce to the case when A is idempotent, α and β are 2-uniform and $\alpha \vee \beta = 1$ and $\alpha \wedge \beta = 0$.



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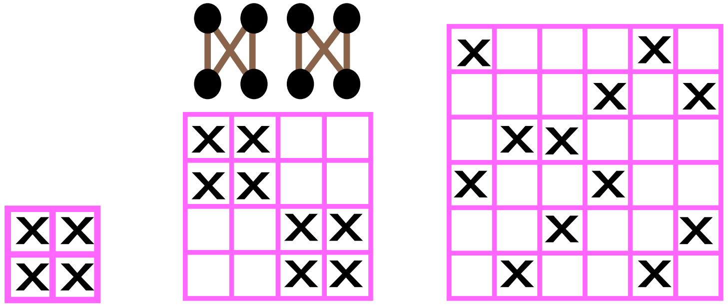

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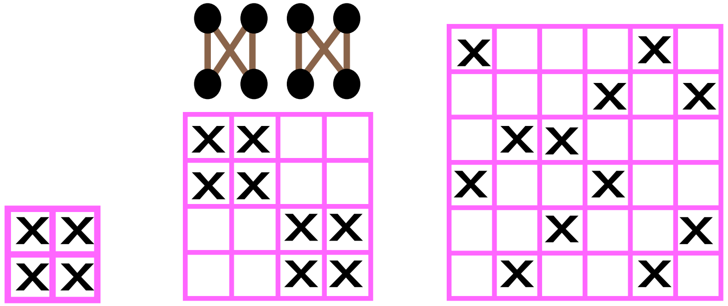
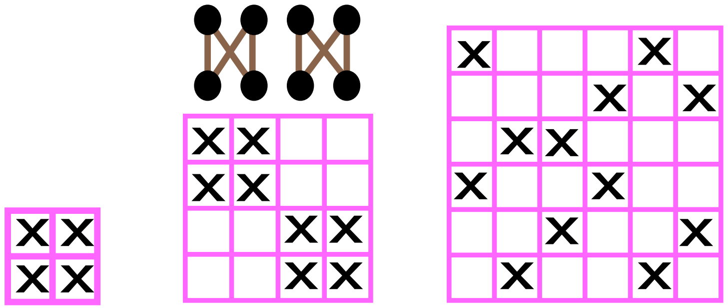


Straightforward: to reduce to the case when A is idempotent, α and β are 2-uniform and $\alpha \vee \beta = 1$ and $\alpha \wedge \beta = 0$. This leads to a so-called **2-uniform** context. Here $A^{(0)} = A/\alpha$ and $A^{(1)} = A/\beta$. Only the rightmost, the so-called **indecomposable** case is interesting (for otherwise α and β trivially permute).



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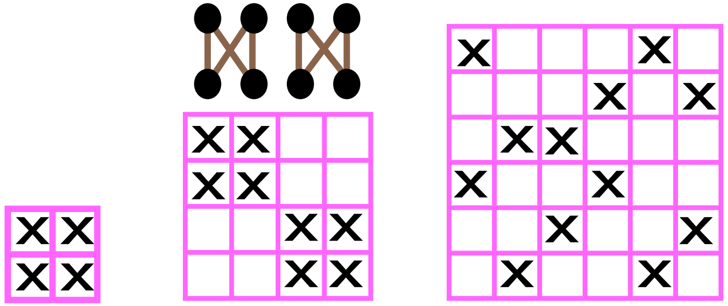

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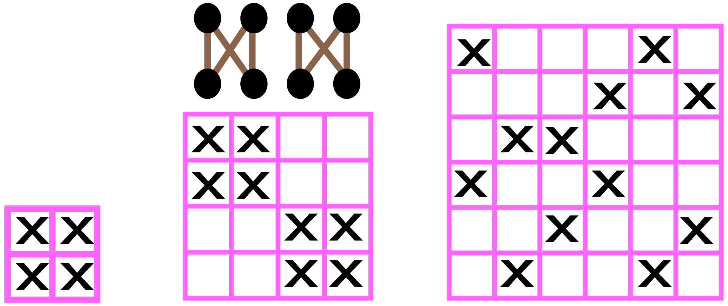

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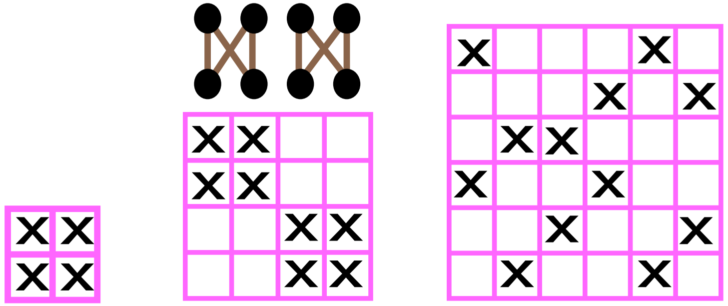

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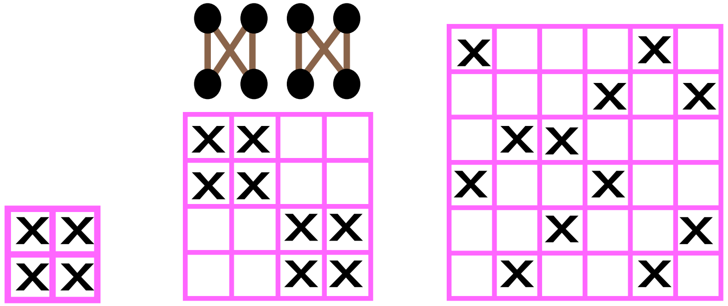

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2. Proposition. *Let* $A^{(0)}$ *and* $A^{(1)}$ *be algebras with a majority term* m*, and let* $\rho \leq_{sd} A^{(0)} \times A^{(1)}$ *be a subdirect product of them.*

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3. Proposition. *Each finite 2-uniform indecomposable context* $(A^{(0)}, A^{(1)}, \rho)$ *with* $|A^{(i)}| \geq 3$ *contains a nondegenerate triangle.*

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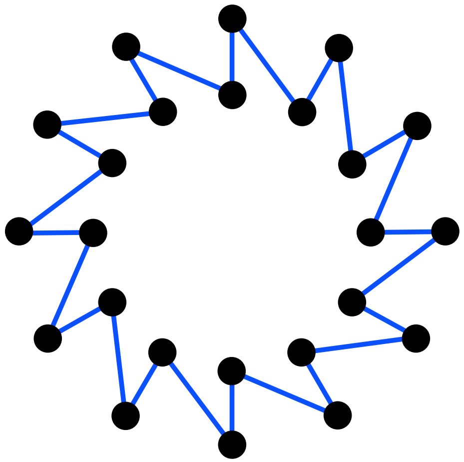
Proof We prove more, namely: $\mathcal{C}_{new}$ **is trivial** (the identity map)! Since the context (as a bipartite graph) is 2-regular and connected,it has a Hamiltonian circle. So it can be depicted this way:  The inner vertices of this „circular saw" constitute $A^{(0)}$ while the outer ones form $A^{(1)}$.

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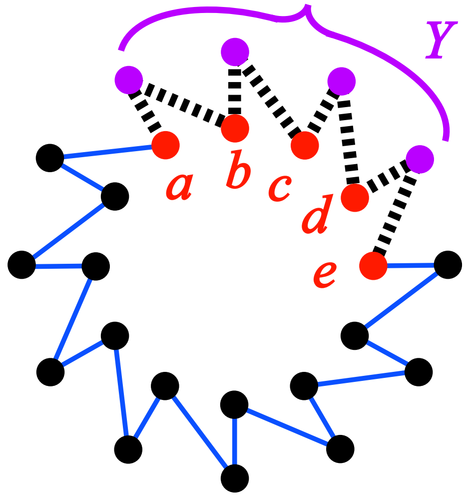

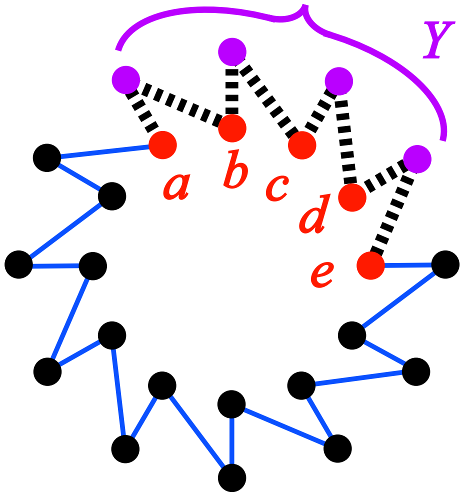

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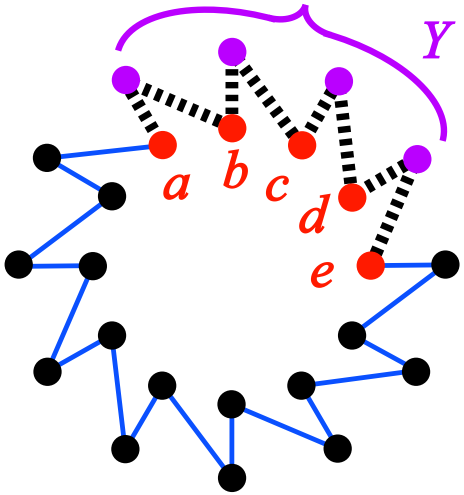

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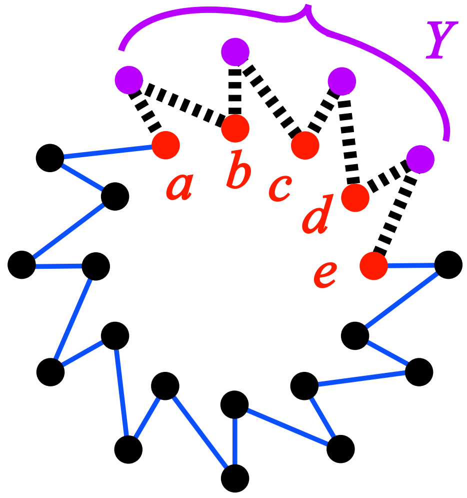

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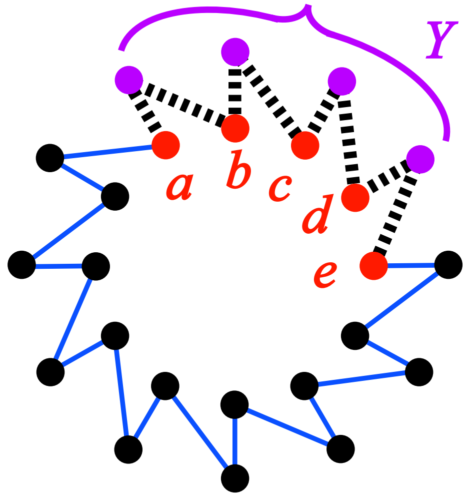

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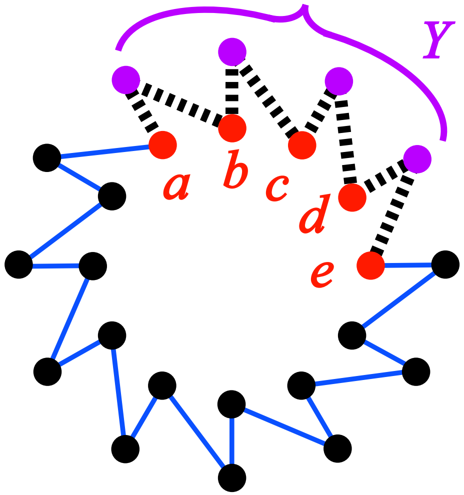

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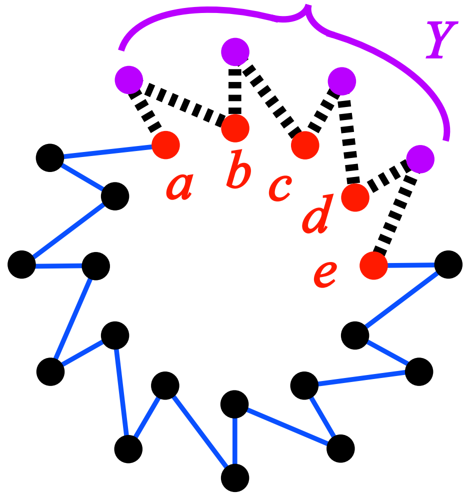

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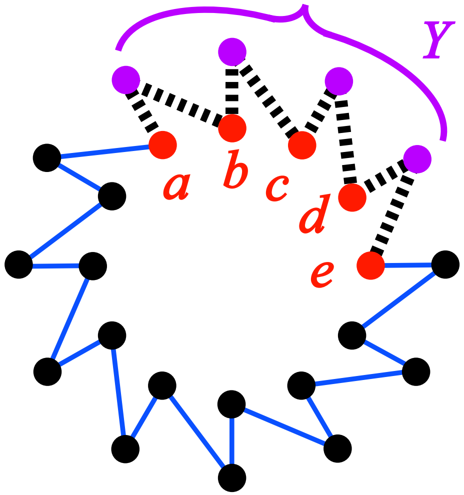

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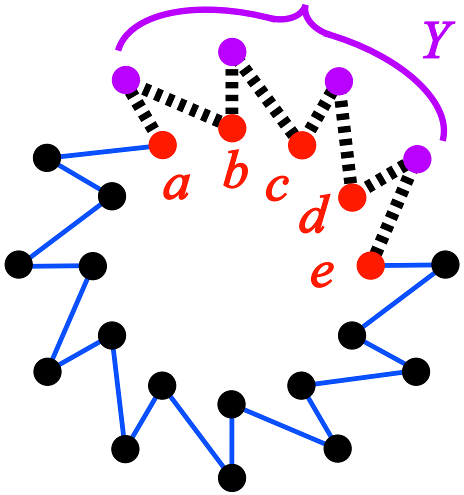

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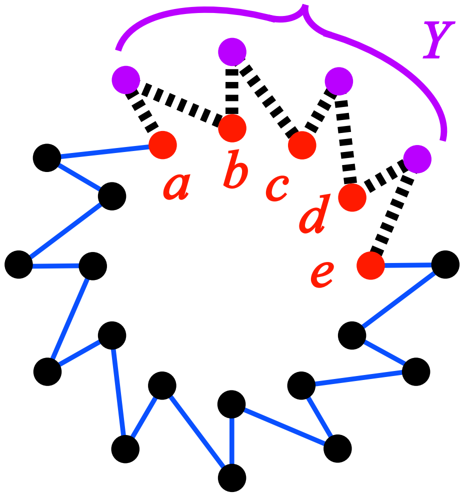

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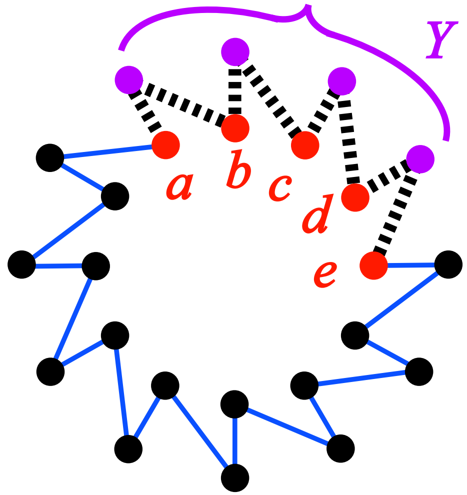

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Hence any triangle of three distinct elements is nondegenerate. This proves the proposition and the theorem.

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And another open problem: How far can Kaarli's result be generalized? Majority algebras? CD?

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(In spite of several asymptotic results on the length of the longest circle in the covering graph of the poset

$$(\{X : X \subseteq \{1, 2, \ldots, 2k+1\} \text{ and } |X| \in \{k, k+1\}\}; \subseteq),$$

it is not known if this graph has a Hamiltonian circle.)

Proof of Proposition 2: Let $a_0, b_0, c_0 \in A^{(0)}$, $u = m(a_0, b_0, c_0)$ and $X := \{a_0, b_0\}$. By symmetry, it suffices to show that

$$u \in \mathcal{C}_n{}^{(0)}(X), \qquad \text{for all } n.$$

$\bigcap_{Y \in \psi(X)} \quad \bigcup_{y \in \mathcal{C}_n{}^{(1-i)}(Y)} \{y\}\rho_{1-i}$ 45' (C) Czédli 0'

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Case $n = 0$: To show

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let $y \in X\rho_0$ be arbitrary. I.e., $(a_0, y), (b_0, y) \in \rho_0 = \rho$. Since ρ is a subdirect product, $\exists\, z$ with $(c_0, z) \in \rho$. Then

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shows $u \in \{y\}\rho_1$. This settles the case $n = 0$.

$$\mathcal{C}_{n+1}^{(0)}(X) := \mathcal{C}_n{}^{(0)}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n{}^{(1)}(Y)} \{y\}\rho_1$$

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$$\mathcal{C}_{n+1}^{(0)}(X) := \mathcal{C}_n{}^{(0)}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n{}^{(1)}(Y)} \{y\}\rho_1$$

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Step $n \to n+1$: We already know that $u \in \mathcal{C}_n{}^{(0)}(X)$. Let $Y = \{a_1, b_1\} \in X\psi$ via the surjection $\varphi : a_0 \mapsto a_1$, $b_0 \mapsto b_1$. Since $\varphi \subseteq \rho$, $(a_0, a_1), (b_0, b_1) \in \rho$. Since ρ is a subdirect product, $\exists\, c_1 \in A^{(1)}$ with $(c_0, c_1) \in \rho$. Let $y := m(a_1, b_1, c_1)$. By the ind. hyp.,

$$y := m(a_1, b_1, c_1) \in \mathcal{C}_n{}^{(1)}(Y)$$

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Step $n \to n+1$: We already know that $u \in \mathcal{C}_n{}^{(0)}(X)$. Let $Y = \{a_1, b_1\} \in X\psi$ via the surjection $\varphi : a_0 \mapsto a_1,\ b_0 \mapsto b_1$. Since $\varphi \subseteq \rho$, $(a_0, a_1), (b_0, b_1) \in \rho$. Since ρ is a subdirect product, $\exists\, c_1 \in A^{(1)}$ with $(c_0, c_1) \in \rho$. Let $y := m(a_1, b_1, c_1)$. By the ind. hyp.,

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$(m(a_0, b_0, c_0), m(a_1, b_1, c_1)) =$

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$$(m(a_0, b_0, c_0), m(a_1, b_1, c_1)) = m((a_0, a_1), (b_0, b_1), (c_0, c_1)) \in \rho,$$

i.e., $u \in \{y\}\rho_1$. Thus,

$$\mathcal{C}_{n+1}^{(0)}(X) := \mathcal{C}_n^{(0)}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n^{(1)}(Y)} \{y\}\rho_1$$

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Step $n \to n+1$: We already know that $u \in \mathcal{C}_n^{(0)}(X)$. Let $Y = \{a_1, b_1\} \in X\psi$ via the surjection $\varphi : a_0 \mapsto a_1$, $b_0 \mapsto b_1$. Since $\varphi \subseteq \rho$, $(a_0, a_1), (b_0, b_1) \in \rho$. Since ρ is a subdirect product, $\exists\, c_1 \in A^{(1)}$ with $(c_0, c_1) \in \rho$. Let $y := m(a_1, b_1, c_1)$. By the ind. hyp.,

$$y := m(a_1, b_1, c_1) \in \mathcal{C}_n^{(1)}(Y). \qquad \text{Further, } (u, y) =$$
$$(m(a_0, b_0, c_0), m(a_1, b_1, c_1)) = m((a_0, a_1), (b_0, b_1), (c_0, c_1)) \in \rho,$$

i.e., $u \in \{y\}\rho_1$. Thus, $u \in \mathcal{C}_{n+1}^{(0)}(X)$. Q.e.d.

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