

Slim semimodular lattices.

II. A description by patchwork systems *

Gábor Czédli and E. Tamás Schmidt

Podlesí, Svatka, Czech Republic, September 3–9, 2011
(SSAOS 2011, dedicated to Ivan Chajda's birthday)

Happy birthday, Ivan!

5th September 2011

*<http://www.math.u-szeged.hu/~czedli/>
<http://www.math.bme.hu/~schmidt/>

Why study planar semimodular lat's? Czedli-Schmidt, Sept.5, 2011 2'/18'

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(c) CzG+Ozsvárt+Udvari, using CzG+E.T. Schmidt (pure lattice theory), proved that the members of two composition series of length n can intersect approximately in $n!/2$ different ways.

What is a patch(work)?

Czédli-Schmidt, Sept.5, 2011

4'/16'

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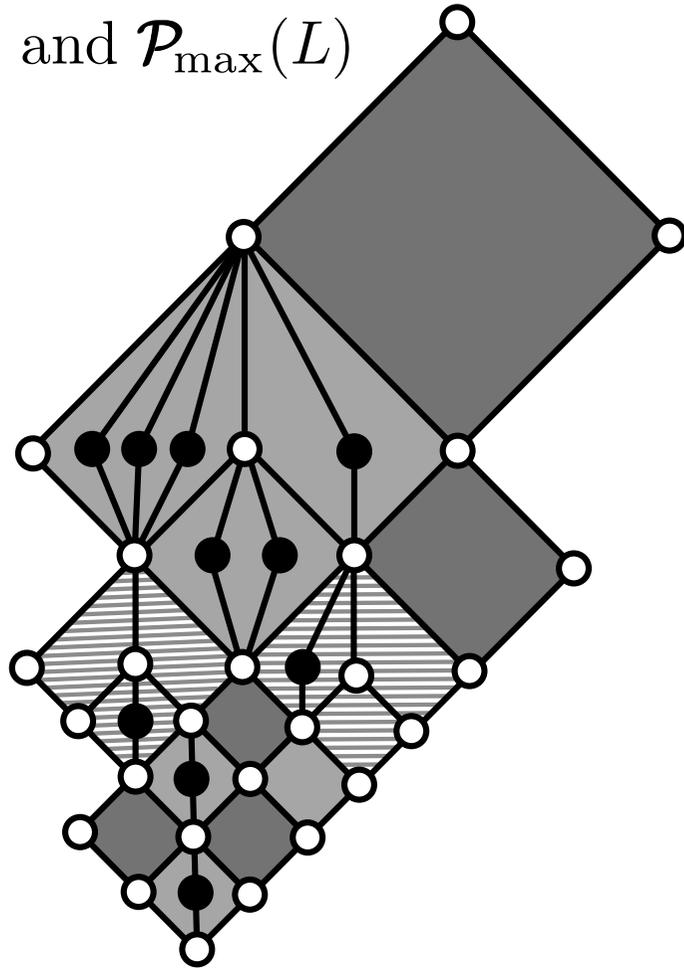
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Remark: for planar **modular lattices**, our main result coincides with Ch. Herrmann's S -glued sum construction (1973). However, opposed to Herrmann (and probably Joanna and [Katarzyna](#)* Grygiel's forthcoming talk), while Herrmann's construction gives rise to a lattice tolerance ρ and a quotient lattice L/ρ (see CzG 1982), called a skeleton, our construction in the semimodular case does not give rise to a similar tolerance.

*[The blue colored](#) corrections were added during or after the talk.

This is a patchwork (system)

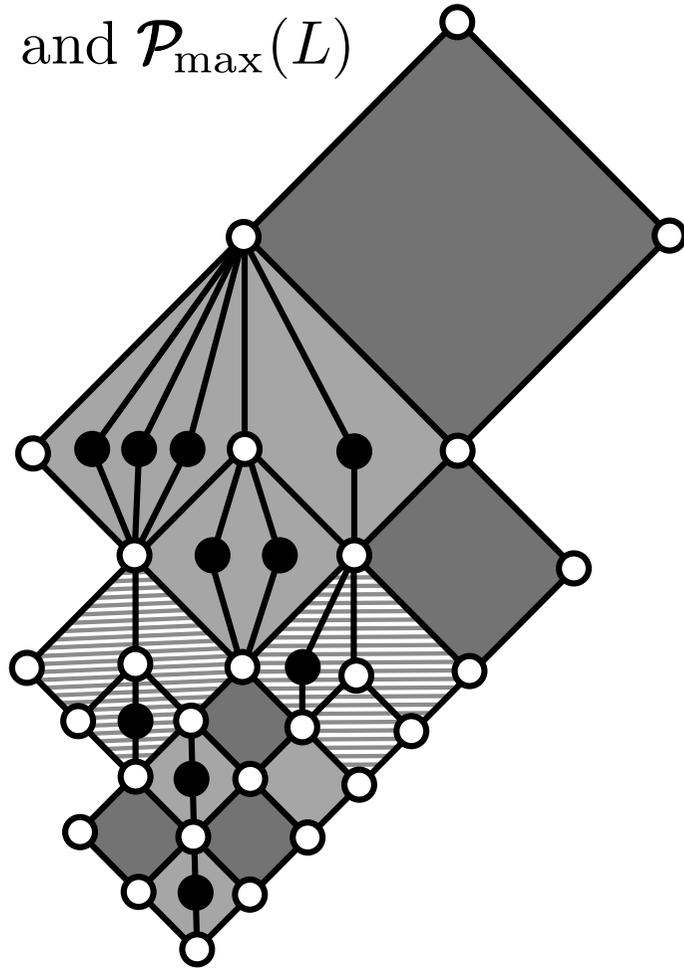
L and $\mathcal{P}_{\max}(L)$



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Another example: remove the black-filled elements.

The action of putting them back is called **adding eyes**. (Adding one eye means that a covering square is changed to an M_3 .)

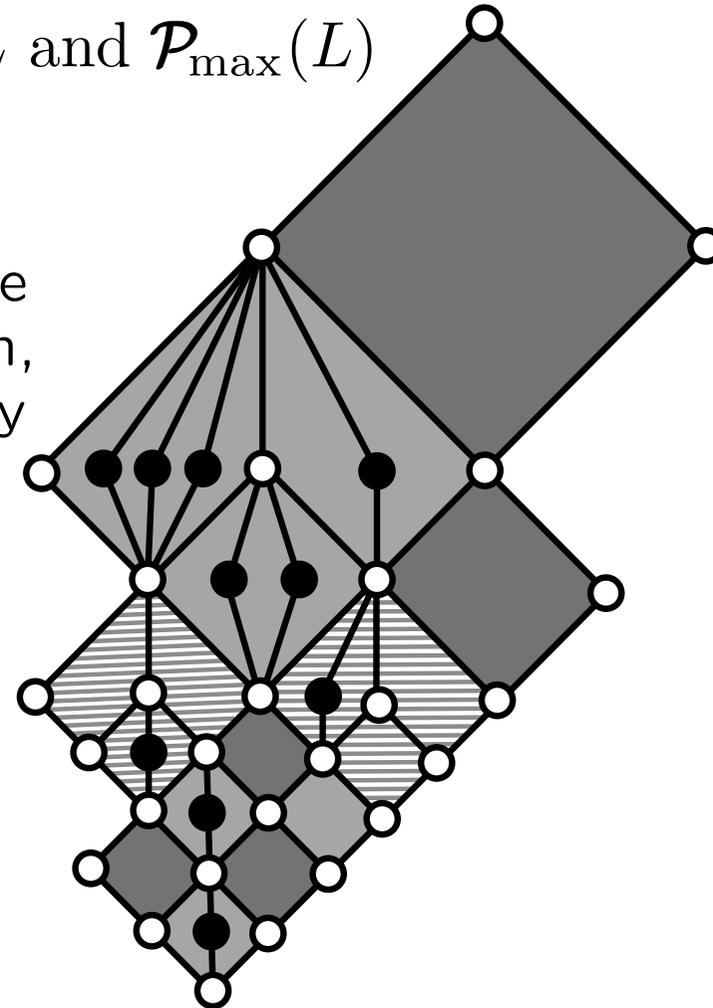
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L is *glued sum indecomposable* iff no $x \in L \setminus \{0, 1\}$ is comparable with all elements of L .

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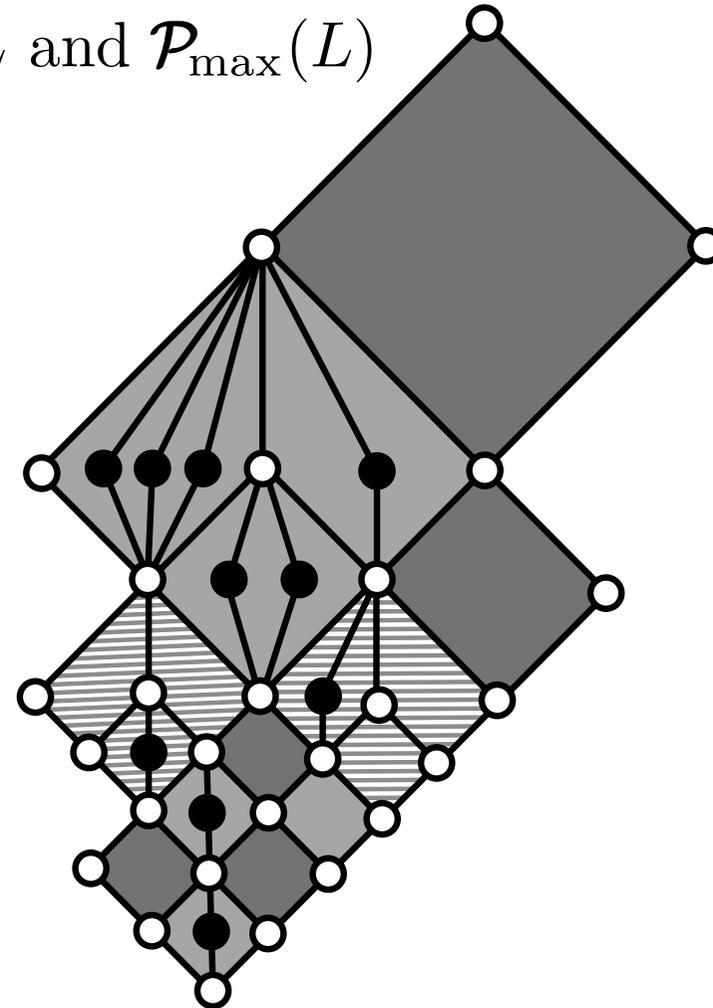
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Northern boundary

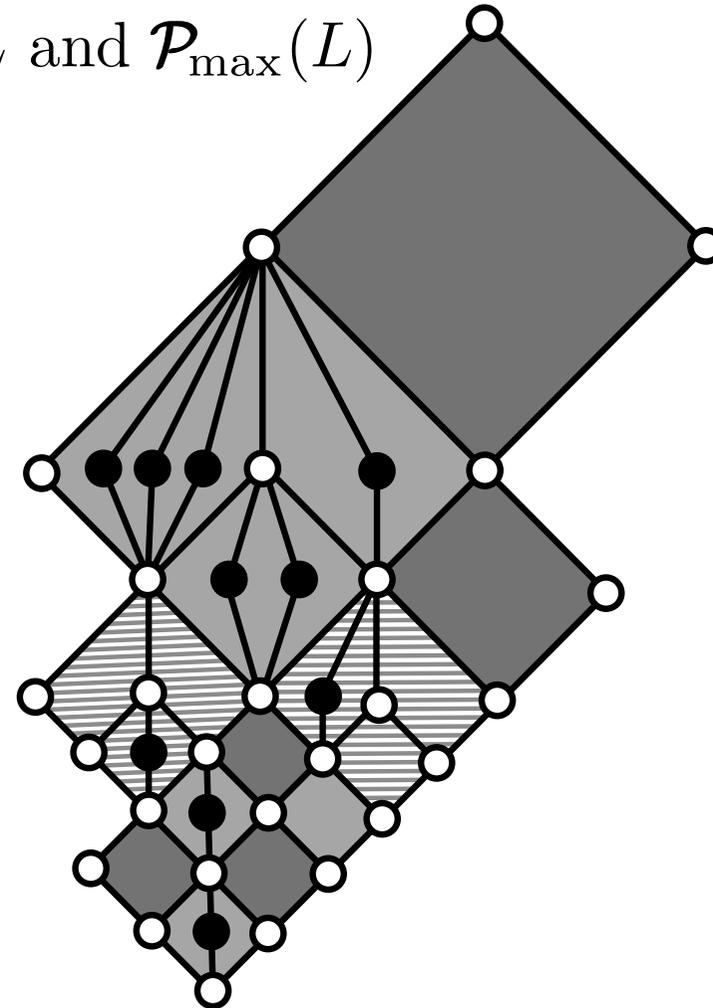
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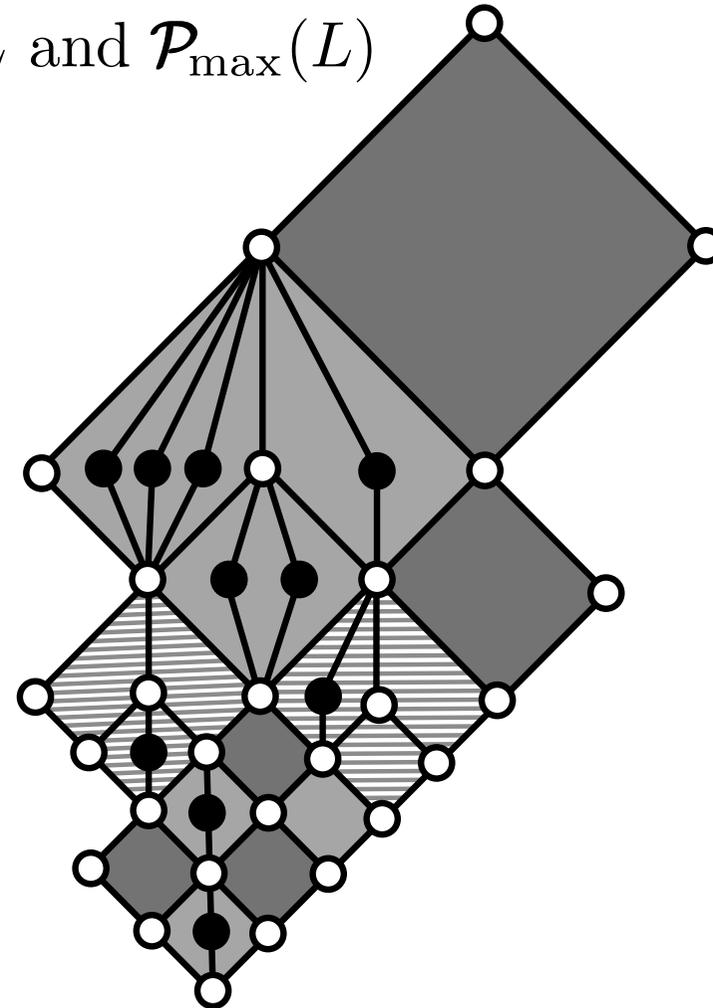
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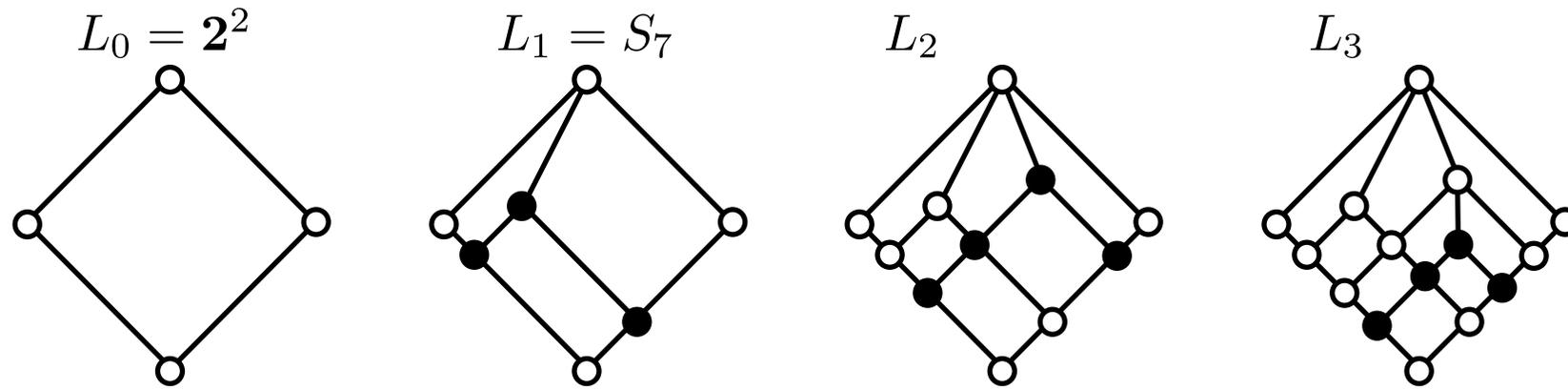
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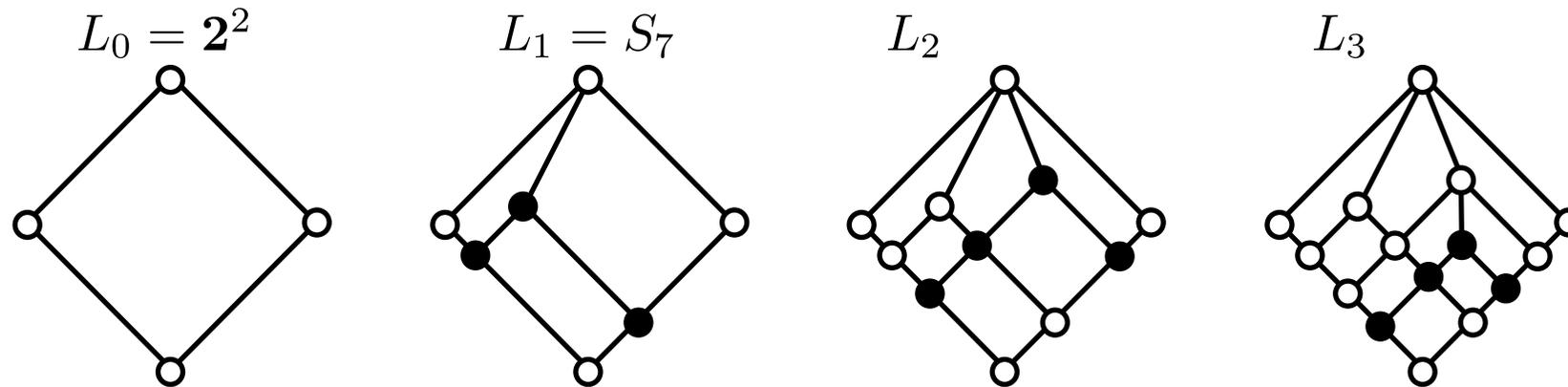
Remark (not at all trivial): \exists a diagram iff \forall planar diagrams.

Adding a fork:



L_i

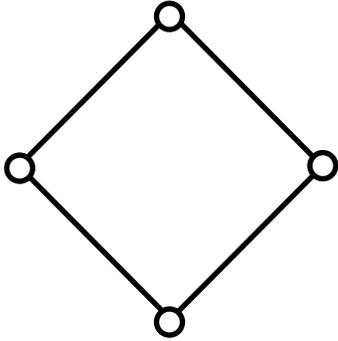
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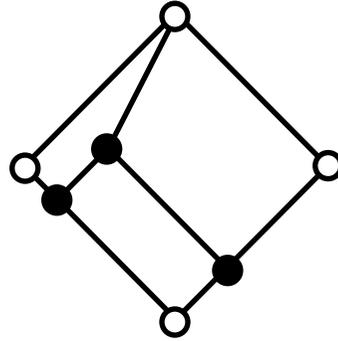
L is called *patchwork irreducible* if it has a patchwork system H , and all of its patchwork systems are singletons.

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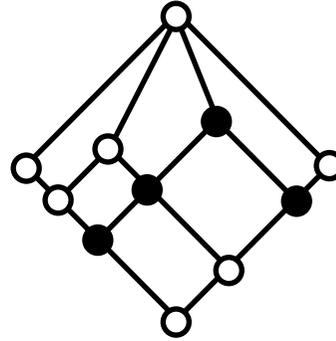
$$L_0 = \mathbf{2}^2$$



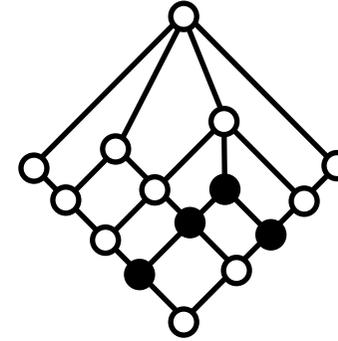
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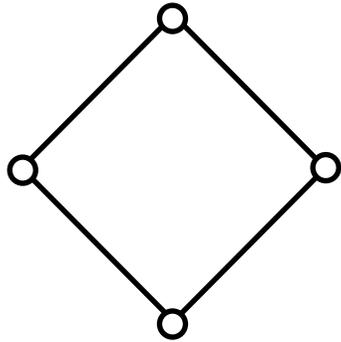
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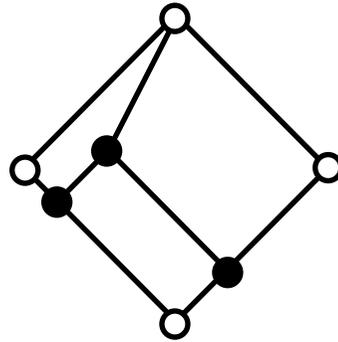
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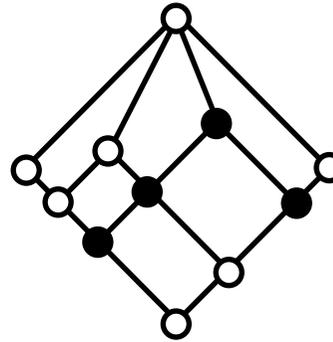
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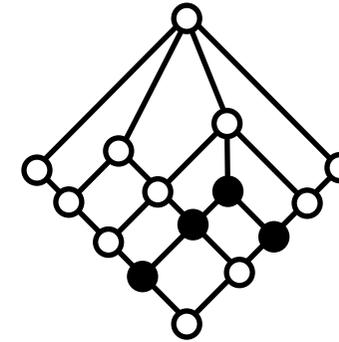
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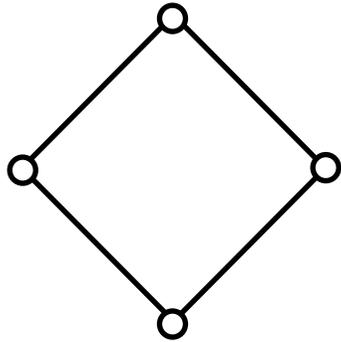
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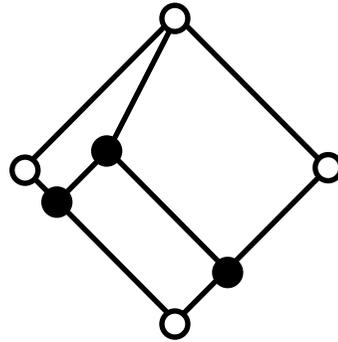
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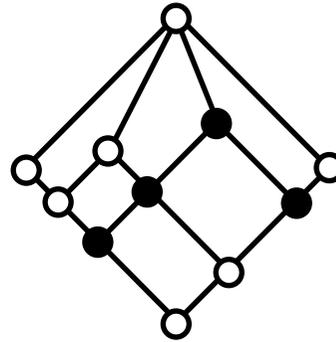
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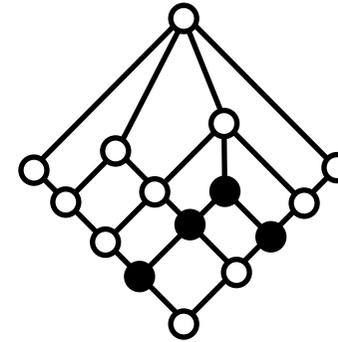
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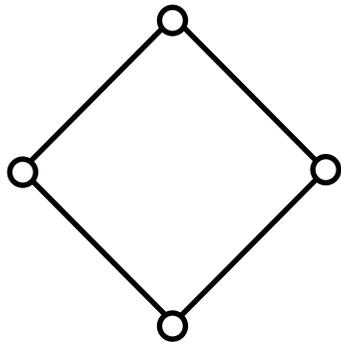
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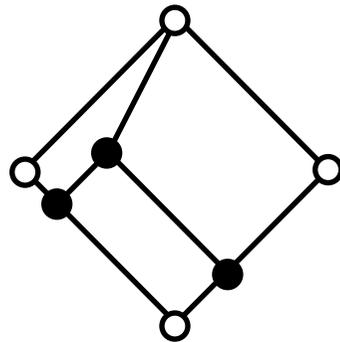
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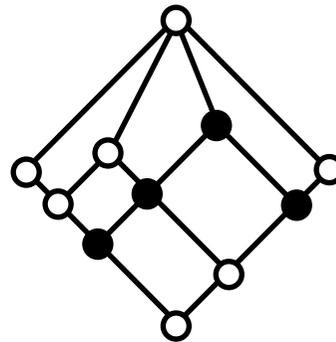
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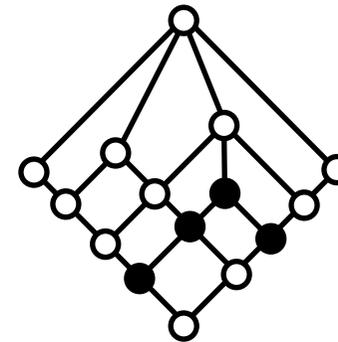
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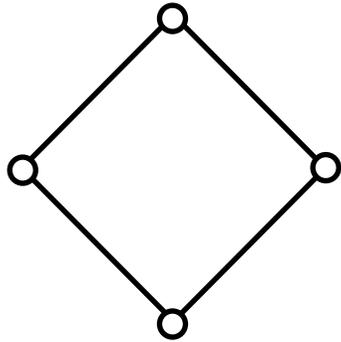
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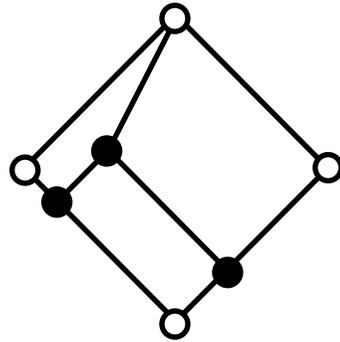
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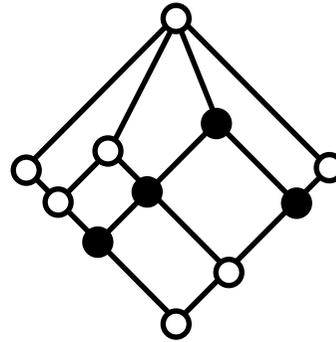
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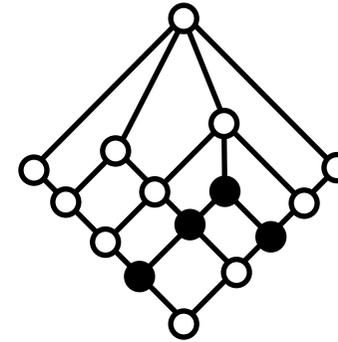
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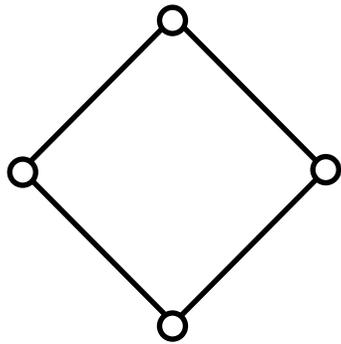
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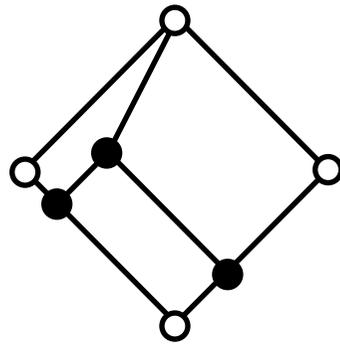
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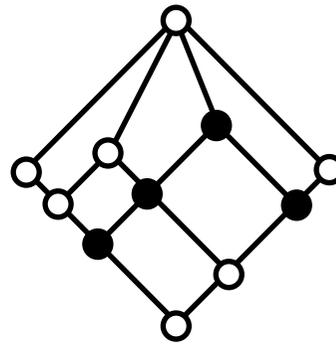
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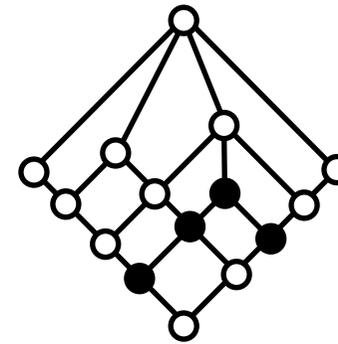
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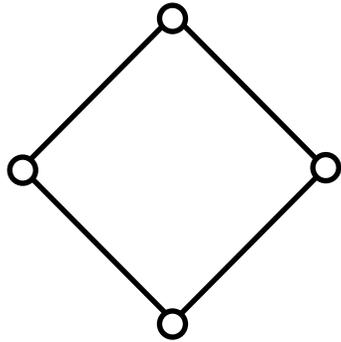
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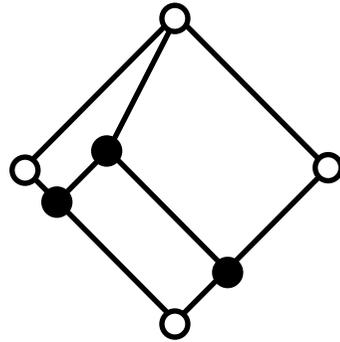
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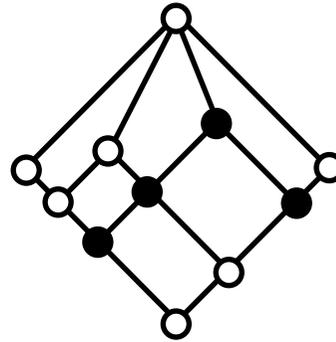
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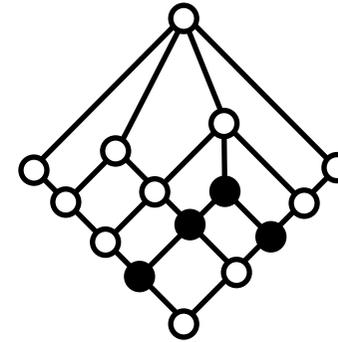
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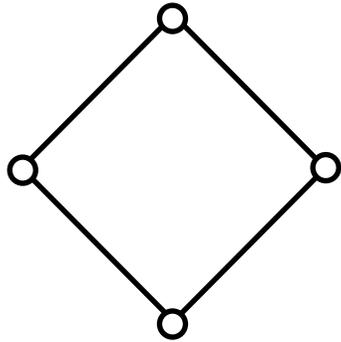


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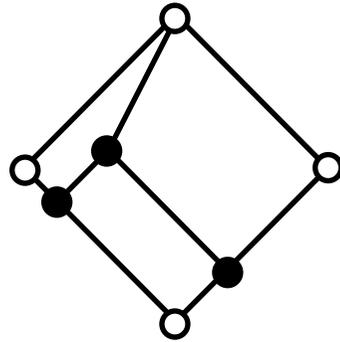
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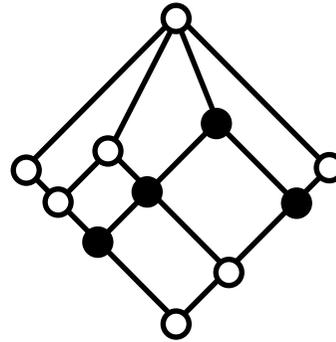
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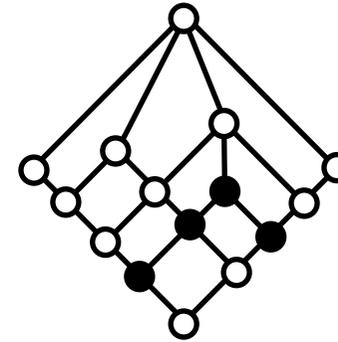
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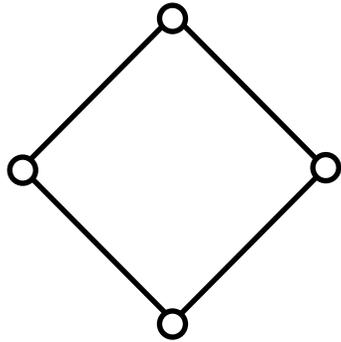


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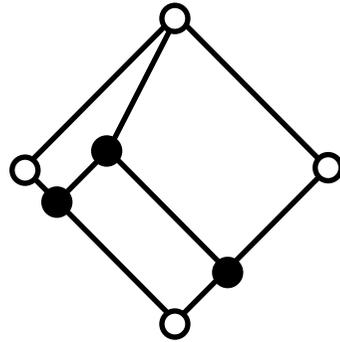
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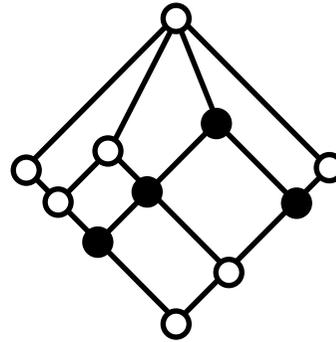
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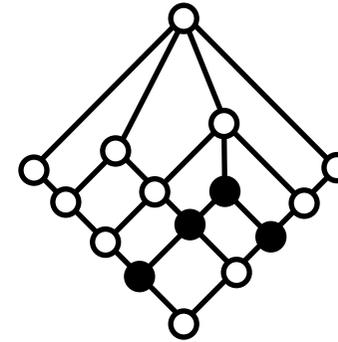
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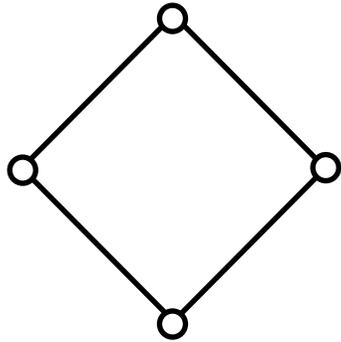


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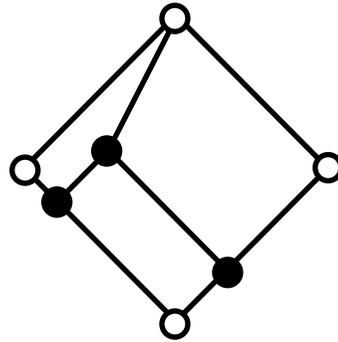
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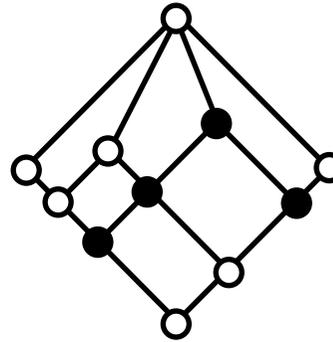
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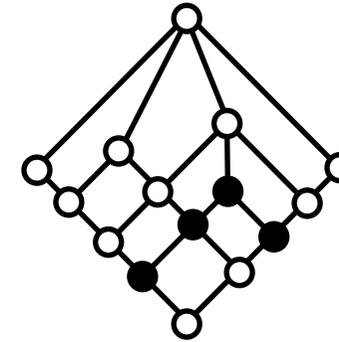
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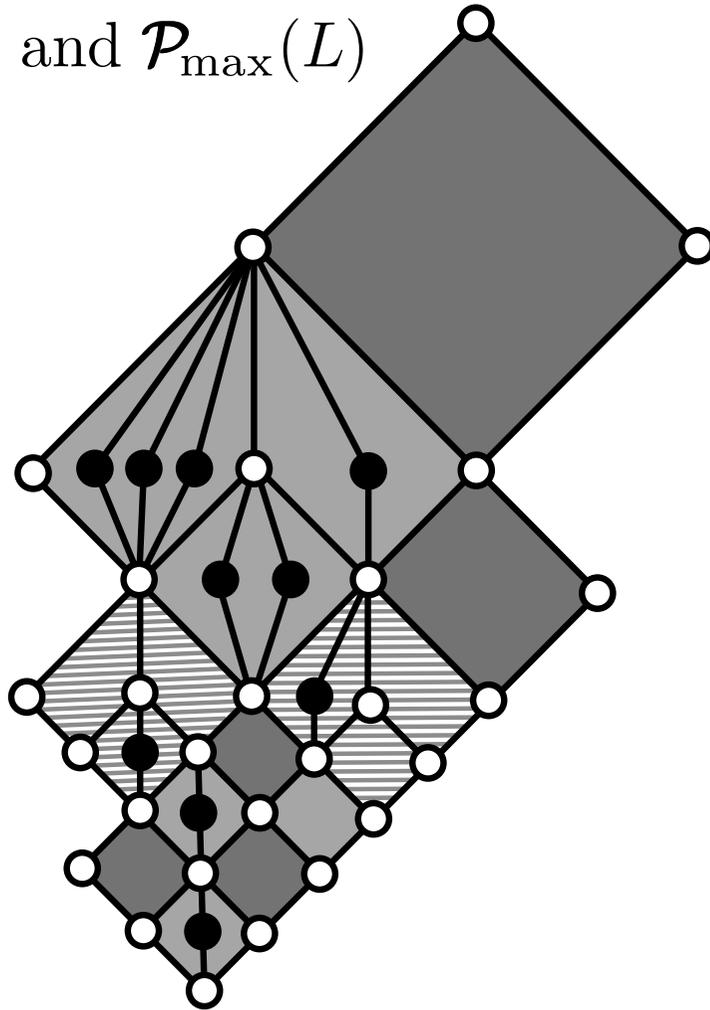


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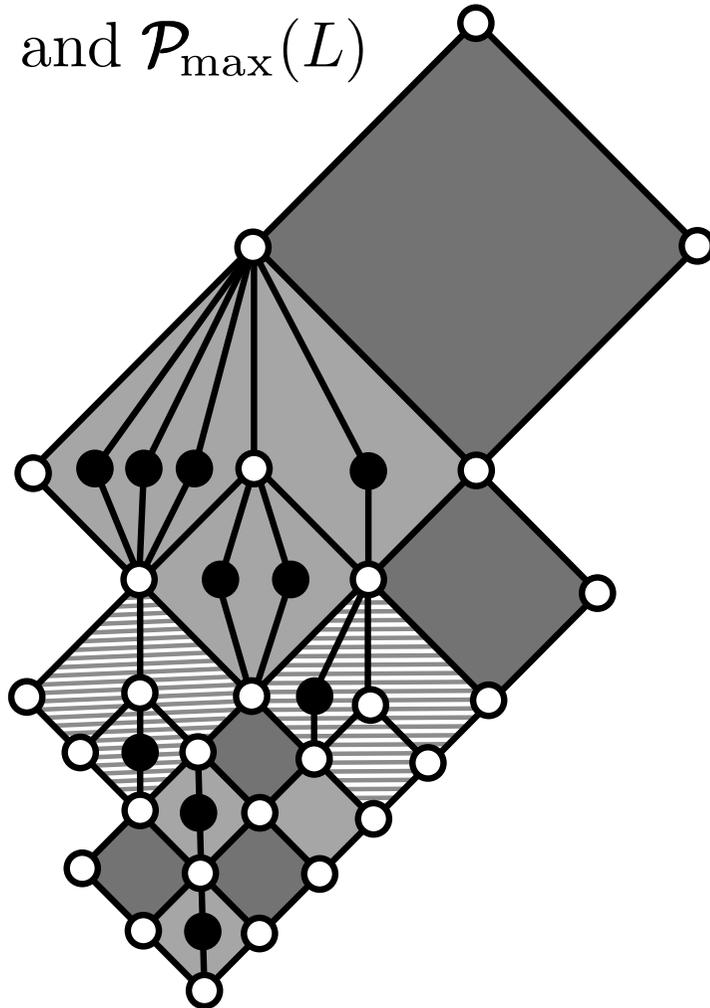
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- L is indecomposable with respect to the Hall-Dilworth gluing over chains.
- L is rectangular and its corners are coatoms. (\exists iff \forall .)
- L has two coatoms whose meet is 0.

We call these lattices *patch lattices*. They are our building stones since, using either the HD-gluing over chains, or the patchwork construction, we obtain all planar semimodular lattices from them. (This was two new structure theorems.)

L and $\mathcal{P}_{\max}(L)$ **Main Theorem**

Let L be a glued sum indecomposable planar semimodular lattice. Then the collection of the maximal patch lattice intervals forms a patchwork system for L (see the figure).

Bibli

L and $\mathcal{P}_{\max}(L)$ 

Main Theorem

Let L be a glued sum indecomposable planar semimodular lattice. Then the collection of the maximal patch lattice intervals forms a patchwork system for L (see the figure).

Bibliography

This file **will be** (next week), and the corresponding paper (31 pages, same title) **is** here:

<http://www.math.u-szeged.hu/~czedli/>

<http://www.math.bme.hu/~schmidt/>