FROM MALTSEV CONDITIONS TO A DUALITY THEOREM

GÁBOR CZÉDLI

(Szeged, Hungary)

The dual of a lattice identity is obtained by interchanging the operation symbols $\vee$ (join) and $\wedge$ (meet) in it. George Hutchinson, using Maltsev conditions (named after Анатолий Иванович Мальцев) and, mainly, his deep earlier results on abelian categories, proved the following theorem.

Hutchinson's Self-duality Theorem (G. Hutchinson, 1978[1]). *For any ring R with unit element and any lattice identity λ, λ holds in the submodule lattice of every unital left R-module if and only if so does the dual of λ.*

Our goal is to give a short and elementary proof of this theorem. This proof yields a **byproduct**, a duality result for graphs. Below, we present only some introductory ideas of the new graph theoretic duality result; for more details, one can look into the slides https://tinyurl.com/czg-ns2024 or into the source paper https://arxiv.org/abs/2406.15989 (these are clickable links).

Graphs are often considered flow networks; this is exemplified by, say, the classical max-flow min-cut theorem. However, our setting is different in several aspects.

Let G and H be acyclic, upward bipolarly oriented plane graphs with the same number n of edges. While G can symbolize a flow network, H has only a controlling role. Let $E(G) = \{e_1, \ldots, e_n\}$ and $E(H) = \{e'_1, \ldots, e'_n\}$ denote their edge sets; in particular, $': E(G) \to E(H)$ is a bijection. Let b be an element of an Abelian group $\mathbb{A}$. We say that an n-tuple $\vec{c} = (c_1, \ldots, c_n) \in \mathbb{A}^n$ is a *solution* of the *primal problem* $P := (G, H, ', \mathbb{A}, b)$ if whenever c_i is the "all-or-nothing-flow" capacity of the edge e_i for $i = 1, ..., n$ and $Y \subseteq E(H)$ is a maximal directed path in H, then by fully exploiting the capacities of the edges of $X := \{e_i : e'_i \in Y\}$ but neglecting the rest of the edges of G we have a flow process transporting b from the source (vertex) of G to the sink of G. The vertices of the *dual* G^{du} of G are the *facets* (in another terminology, the *countries*) of G, including *two outer facets* (rather than one). Each edge e_i of G determines a *dual edge* e_i^{du}, which goes from the facet on the left of e_i to the facet on the right of e_i. These dual edges form $E(G^{\mathrm{du}})$. The graph H^{du} is defined in the same way. The *dual problem* is $P^{\mathrm{du}} := (H^{\mathrm{du}}, G^{\mathrm{du}}, ', \mathbb{A}, b)$.

Main Theorem. *With the assumptions and notations above, the primal problem P an the dual problem P^{du} have exactly the same solutions.*

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[1]This theorem was published in a joint paper *George Hutchinson and Gábor Czédli: A test for identities satisfied in lattices of submodules, Algebra Universalis* **8** *(1978), 269–309*, but **this is only his theorem**.