

# The asymptotic number of planar, slim, semimodular lattice diagrams

**Gábor Czédli** (AAA90, Novi Sad, June 5–7, 2015)  
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June 4, 2015

**All lattices and diagrams are assumed to be finite!**

Slim:  $J(L) = \text{Chain}_1 \cup \text{Chain}_2$ ;  $\Rightarrow$  planar. (Grätzer–Knapp 1997).

Semimodular:  $a \prec b \implies a \vee c \preceq b \vee c$ .

Natural example: if  $\vec{S}: \{1\} = S_0 \subset S_1 \subset \dots \subset S_n = G$  and  $\vec{T}$  are composition series of  $G$ , then  $(\{S_i \cap T_j\}; \supseteq)$ .

Application ( $\Rightarrow$  motivation): uniqueness part to the classical Jordan–Hölder theorem:

Theorem (uniqueness for the classical Jordan–Hölder theorem)

$\exists \pi \in S_n$  such that the quotient  $S_i/S_{i-1}$  is “down-and-up” perspective to  $T_{\pi(i)}/T_{\pi(i)-1}$ , for  $\forall i$ . (Grätzer and Nation, AU 2011). Furthermore, this  $\pi$  is unique. ( $\sim$  and Schmidt, AU 2011).

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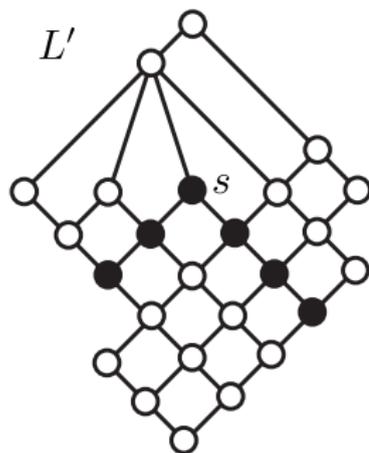
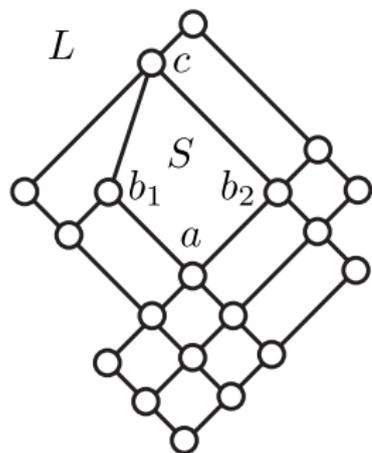
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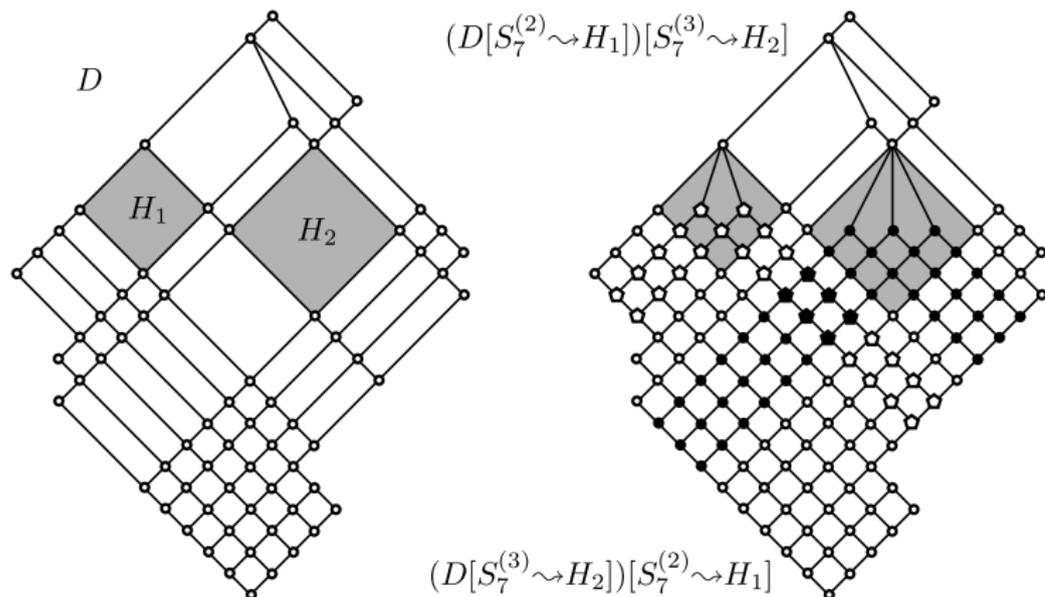
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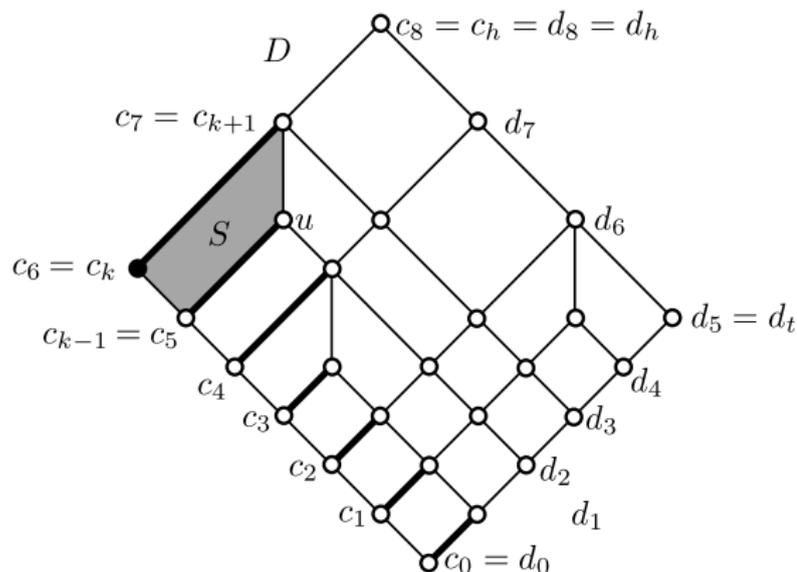




$\sim$  and Schmidt, Order 2012



Multifork extensions at distributive ideals ( $\rightsquigarrow$ , AU 2014)



This concept is due to Grätzer and Knapp, 2009.

Enumerating slim sm lattices of a given parameter (length, size)  $n$   
 $\approx$  how many ways can the intersections  $S_i \cap T_j$  be related, if  
 $\{1\} = S_0 \subset S_1 \subset \dots \subset S_n = G$  and  $\{1\} = T_0 \subset \dots \subset T_n = G$  are  
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 first composition series,  $\vec{S}$ , is distinguished from the second one.

### Definition

We enumerate planar diagrams up to *similarity*. I.e., the "from left to right" ordering of the upper and lower covers of every element is relevant but Euclidian distances and slopes are not. E.g.,  $N_5$  has exactly 2 planar diagrams while  $M_3$  has only 1.

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- M. Erné, J. Heitzig, J. Reinhold: On the number of distributive lattices, Electron. J. Combin. 9/1 (2002), Research Paper 24.
- J. Heitzig, J. Reinhold, Counting finite lattices, Algebra Universalis 48 (2002) 43–53 (of size  $\leq 18$ ).
- M.M. Pawar, B.N. Waphare, Enumeration of nonisomorphic lattices with equal number of elements and edges, Indian J. Math. 45 (2003) 315–323

# More recent enumerations (slim semimodular lattices)

## 10'/10

- $\sim$ , Ozsvárt, Udvari: How many ways can two composition series intersect, *Discr. Math.* (2012); recursive formula, lattices computed up to  $\ell = 100$ , asymptotically  $\sim \ell!/2$ .
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The number of planar **d** diagrams of slim semimodular lattices of size  $n$  is denoted by  $N_d(n)$ .

Theorem ( $\sim$ , arxiv:1206.3679, 16 June 2012)

*There exists a constant  $C$  such that  $0 < C < 1$  and  $N_d(n)$  is asymptotically  $C \cdot 2^n$ , that is,  $\lim_{n \rightarrow \infty} (N_d(n)/2^n) = C$ .*

We have also proved:  $0.42 \cdot 10^{-57} \leq C \leq 0.073$  .

We conjecture:  $0.023 \leq C \leq 0.073$  .

Lattices rather than diagrams of a given size  $n$ ? The earlier recursion is only effective up to  $n \approx 50$ . We do not even know if  $\lim_{n \rightarrow \infty} (N_l(n)/N_l(n-1))$  (conjecture: yes), and what is it.

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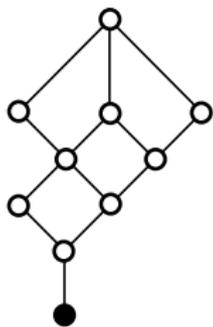
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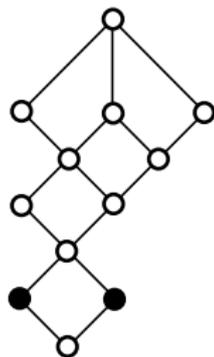
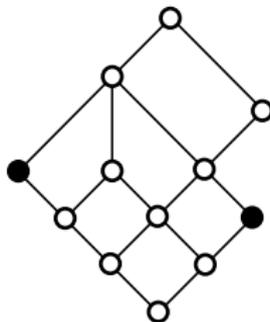


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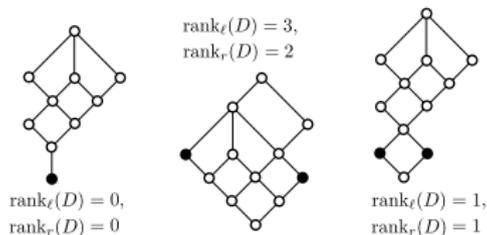
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We introduce the left and right rank of a slim sm. diagram. (Slim semimodular) diagrams of size  $n$  belong to three categories.

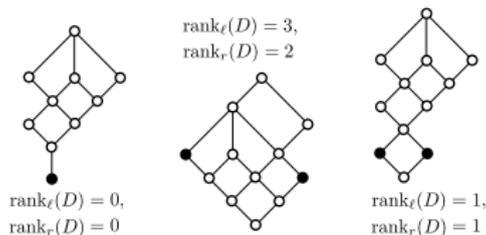


(a) left rank = right rank = 0: there are  $N_d(n - 1)$  of them.

(b) left rank = right rank = 1: there are  $N_d(n - 3)$  of them.

(c) both ranks are  $> 0$ : Removing the left corner,  $N_d(n - 1)$  diagrams remain ??? No, not every diagram of size  $n - 1$  is obtained in this way. Nevertheless, (a)+(b) and (a)+(c) yield

$$N_d(n - 1) + N_d(n - 3) \leq N_d(n) \leq 2 \cdot N_d(n - 1).$$

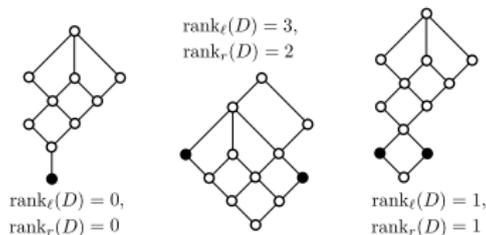


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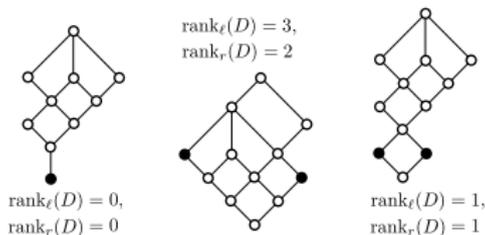


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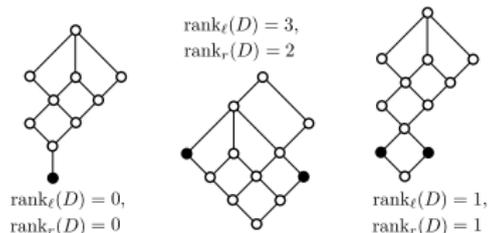


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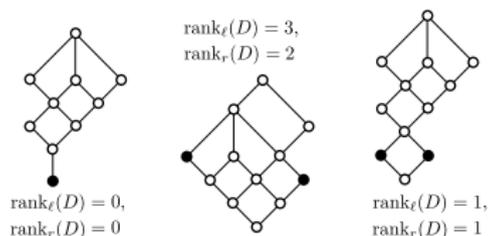
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### Lemma

*$D'$  with  $|D'| = n - 1$  is obtained from some  $D$  by omitting the left corner iff the left corner of  $D'$  is not a coatom.*

Proof of the easier direction: length(left corner) decreases by 1.  
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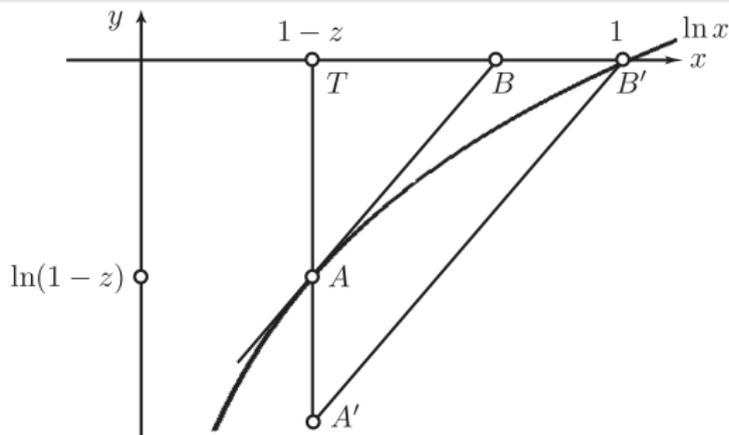
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## Lemma

If  $0 < z \leq z_0 < 1$ , then

$$-\ln(1-z) \leq z/(1-z) \leq z/(1-z_0).$$



Proof:

With  $\kappa_n := N_d(n)/N_d(n-1)$ ,  $\rho_n = \prod_{j=m+1}^n (\kappa_j/2)$ ,  
 $N_d(n)/2^n = \rho_n \cdot N_d(m)/2^m$ ,  $s_n = -\ln \rho_n$ , and  $\alpha := 4/5$ ,

$$\begin{aligned} 0 < s_n &= \sum_{j=m+1}^n (-\ln(\kappa_j/2)) \leq' \sum_{j=m+1}^n (1 - \kappa_j/2)/(1 - z_0) \\ &\leq^* \mu \cdot \sum_{j=m+1}^n \alpha^{\lceil \sqrt{j-1} \rceil - 2} \leq \mu \cdot \sum_{j=m+1}^n \alpha^{\sqrt{j-1} - 1} \leq \mu \cdot \sum_{k=m}^{n-1} \alpha^{\sqrt{k} - 1} \\ &= \nu \cdot \sum_{k=m}^{n-1} \alpha^{\sqrt{k}} \leq \nu \cdot \int_{x=m-1}^{n-1} \alpha^{\sqrt{x}} dx \leq \nu \cdot (F(\infty) - F(m-1)), \end{aligned}$$

where  $F(x) = -2 \cdot \delta^{-2} \cdot (1 + \delta\sqrt{x}) \cdot \alpha^{\sqrt{x}}$ .  $\exists F(\infty) \implies$  Q.e.d.

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