Quasiplanar poset diagrams

Gábor Czédli (SSAOS 51, Trojanovice, Sept. 1–7, 2013)

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Based on: Czédli: Quasiplanar diagrams and slim semimodular lattices, arXiv:1212.6904v1 (31 Dec 2012), 19 pages

All lattices, posets, and diagrams are assumed to be **finite**!

Lemma (Kelly and Rival, 1975)

*Let D be finite, planar lattice **diagram**, and let $x, y \in L$ be incomparable elements.*

(P) *If x is on the left of some maximal chain (of D) through y, then x is on the left of **every** maximal chain through y.*

Definition

Let D be a diagram of a **bounded** poset. If (P) holds for all $x \parallel y \in D$, then D is called **quasiplanar**.

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- In a quasiplanar diagram, $\boxed{x \lambda y}$ iff $x \parallel y$ and x is on the left of all maximal chains through y. (Relation „**left**".)
- Two quasiplanar diagrams are **similar** if there is an $\{\leq, \lambda\}$-isomorphism between them. Similar diagrams are treated as **equal**. Like D_4 and D_5 below:

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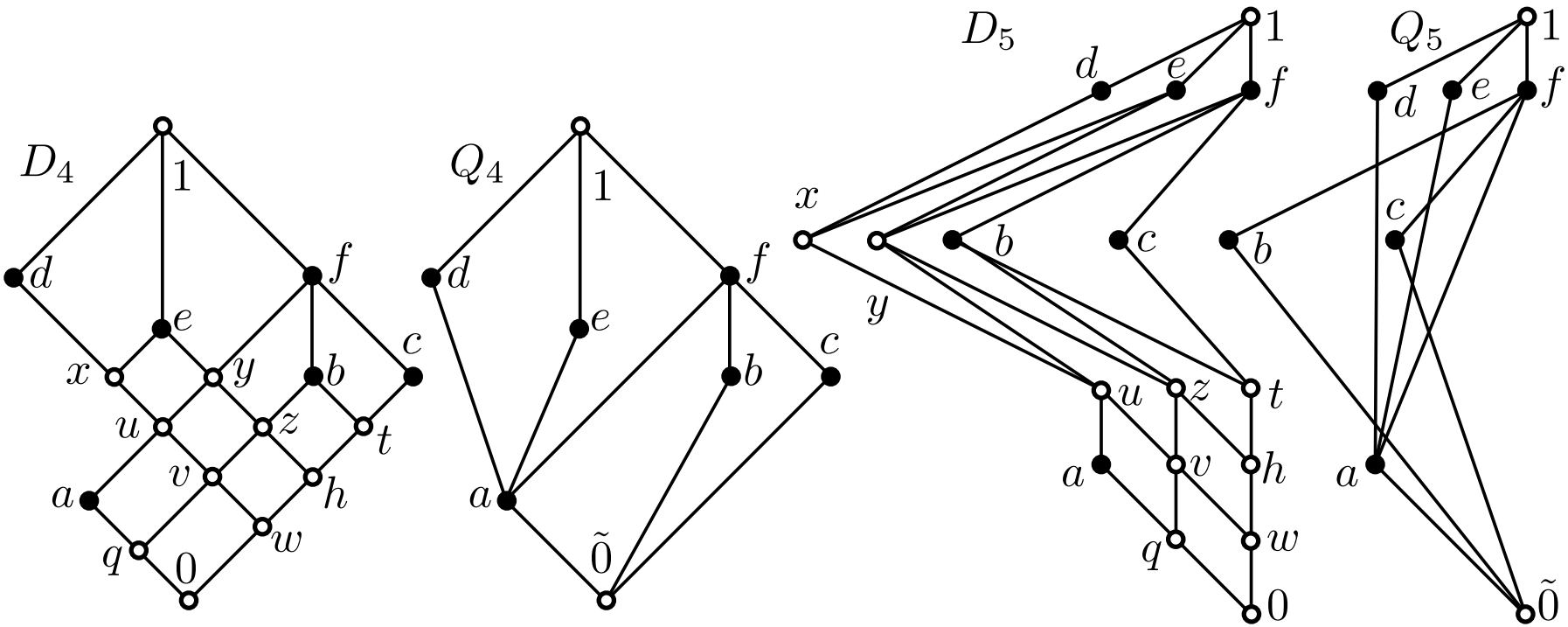
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1
d
f
e
c
x
y
b
u
z
t
v
a
h
q
0
w
Q_4
1
d
f
e
c
b
a
$\tilde{0}$
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Remarks

- „Quasiplanar" implies „bounded" (by definition).
- Planar bounded diagrams are quasiplanar (and they are lattice diagrams by Kelly and Rival 1975).
- A finite poset has a quasiplanar diagram iff it is of order dimension at most 2.
- Q_3 is quasiplanar but not planar:

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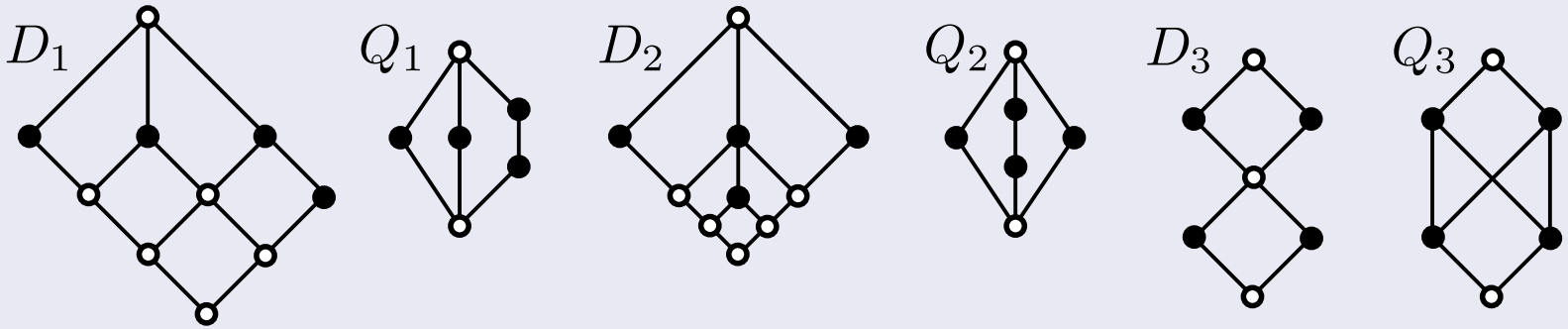

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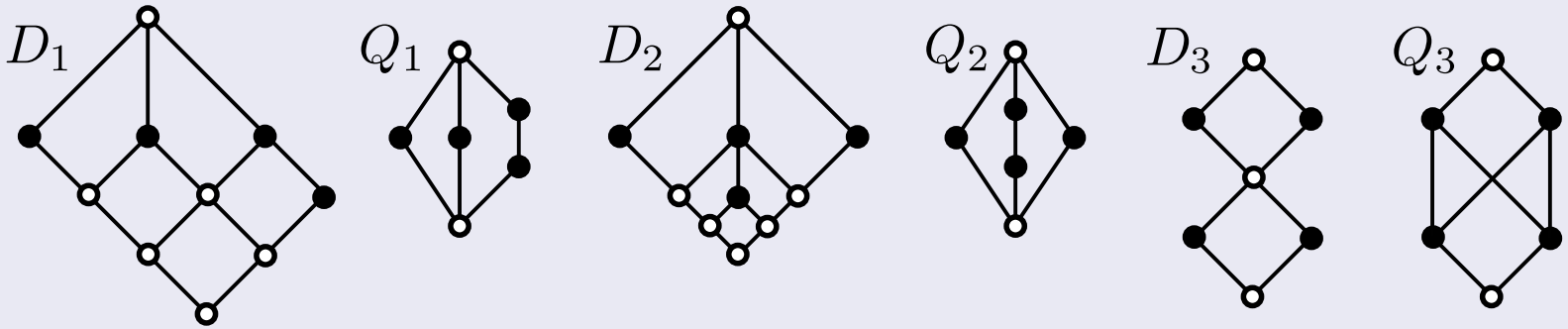

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- A lattice diagram is **semimodular** if $x \prec y \Rightarrow x \vee z \preceq y \vee z$.

Since 2007, ≈ 25 papers on planar and, in particular, slim semimodular lattices by Dékány, **Grätzer**, Klus, **Knapp**, Nation, Nguyen, Ozsvárt, **Schmidt**, Szakács, Udvari, Wares, and **Czédli**. Notably, the Czédli-Schmidt AU-2011 paper adds a uniqueness part to the classical Jordan-Hölder theorem.

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New description for slim semimodular lattices (by the poset $\mathrm{Mi}\, L$)

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Let D be a slim semimodular planar digram. ($\Rightarrow$ lattice diagram)

- $Q := \{1, \tilde{0}\} \cup \operatorname{Mi} D$, where $1 \in D$, $\tilde{0} \notin D$;
- for $x, y \in Q$, let $\boxed{x \leq y}$ in Q iff $x \leq y$ in D or $x = \tilde{0}$;
- for any two incomparable $x, y \in Q$, we let $\boxed{x \lambda y}$ in Q iff $x \lambda y$ in D.

If $\langle Q; \leq, \lambda \rangle$ above exists, then it is clearly unique up to similarity; it is denoted by $\boxed{\alpha(D)}$, called the **poset diagram** associated with D.

Initial idea (how to prove the existence of $\alpha(D)$?)

"Keep the meet-irreducibles, delete the rest, keep the ordering and the position in the plane!" ???

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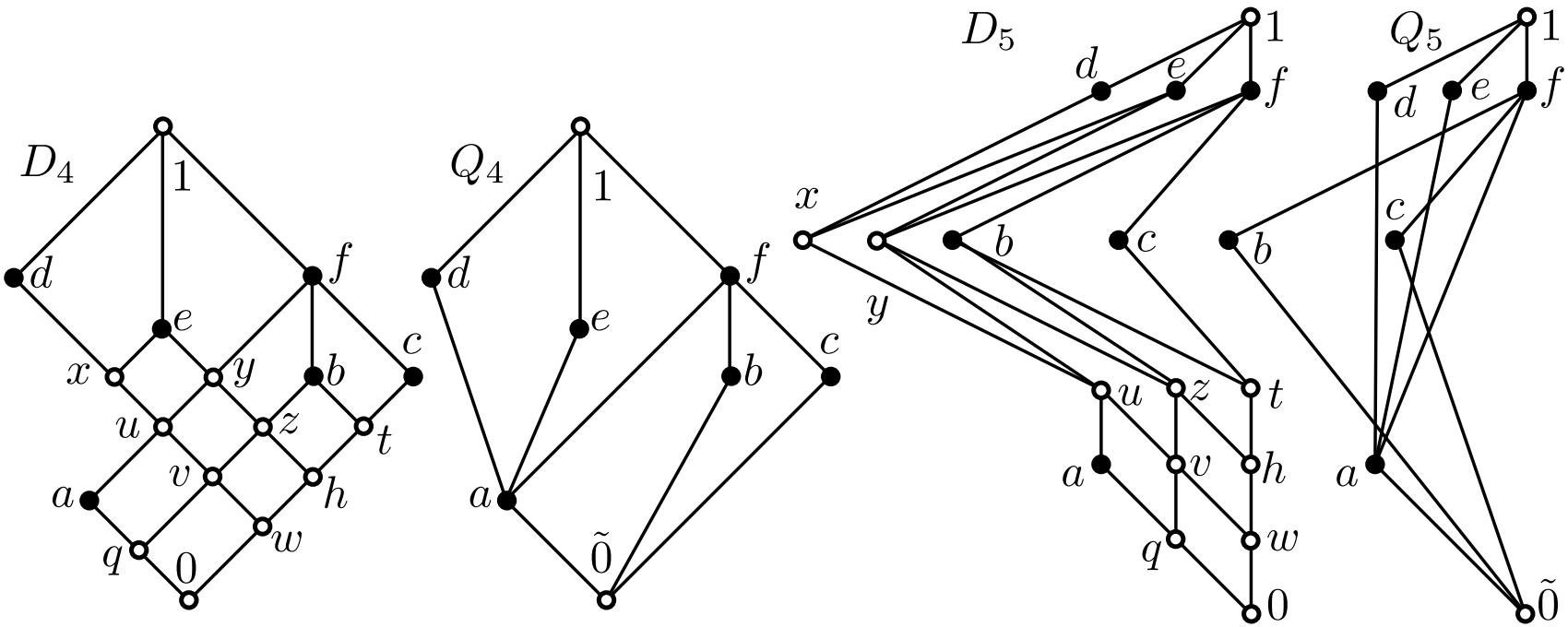
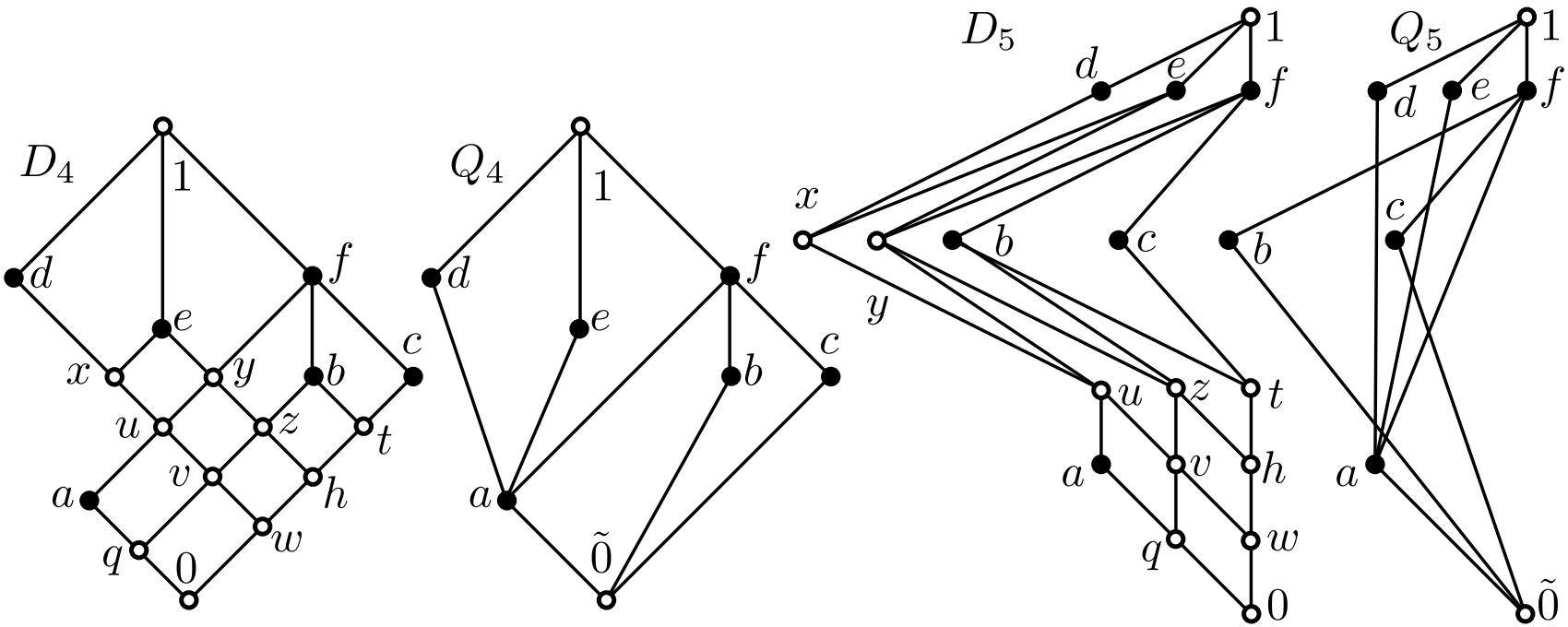


$D_4 = D_5$, $Q_4 = \alpha(D_4)$, but Q_5 is not even quasiplanar, since $a \parallel c$, but c is on the left of $\{\tilde{0}, a, f, 1\}$ but **not** on the left of $\{\tilde{0}, a, d, 1\}$.



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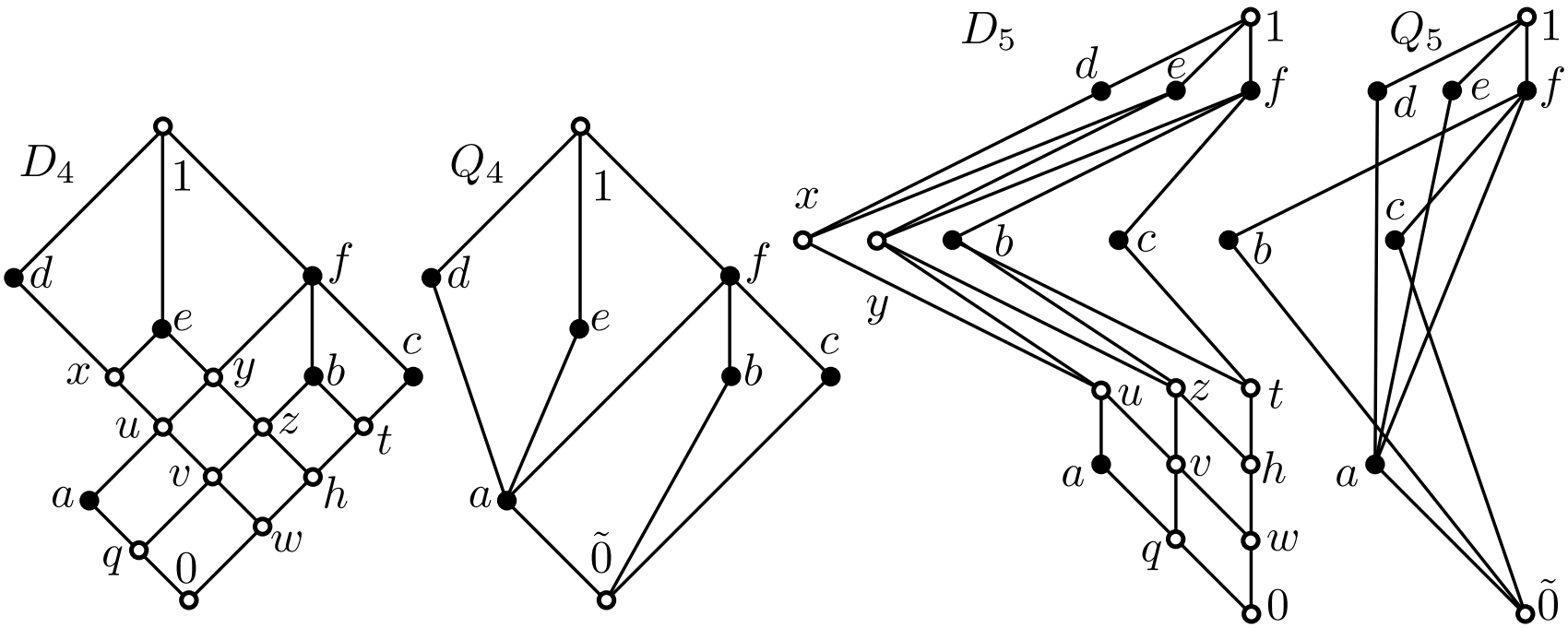

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- Let $\boxed{\mathrm{E(Q)}} = \{\langle x, y\rangle \in Q^+ \times Q^+ : x \lambda^{=} y\}$. (Eligible pairs)
- For $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in E(Q)$, we define
$$\langle x_1, y_1\rangle \leq \langle x_2, y_2\rangle \overset{\text{def}}{\iff} x_1 \lambda^{\leq} x_2 \text{ and } y_2 \lambda^{\geq} y_1, \text{ and}$$
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- Let $\boxed{\beta(Q)}$ be the unique planar diagram (of) $\langle E(Q); \leq, \lambda\rangle$, where „$\leq$" and „$\lambda$" are defined above. ($\exists$?; uniqueness $\surd$)

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Theorem (Main Theorem)

Let D be a finite, slim, semimodular, planar lattice diagram, and let Q be a finite quasiplanar diagram. Then the following hold.

- *$\alpha(D)$ is a finite quasiplanar diagram.*
- *$\beta(Q)$ is a finite, slim, semimodular, planar lattice diagram.*
- *Up to similarity, $\beta(\alpha(D))$ equals D.*
- *Up to similarity, $\alpha(\beta(Q))$ equals Q.*

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Up to similarity, the number of n-element quasiplanar diagrams is $(n-2)!$.

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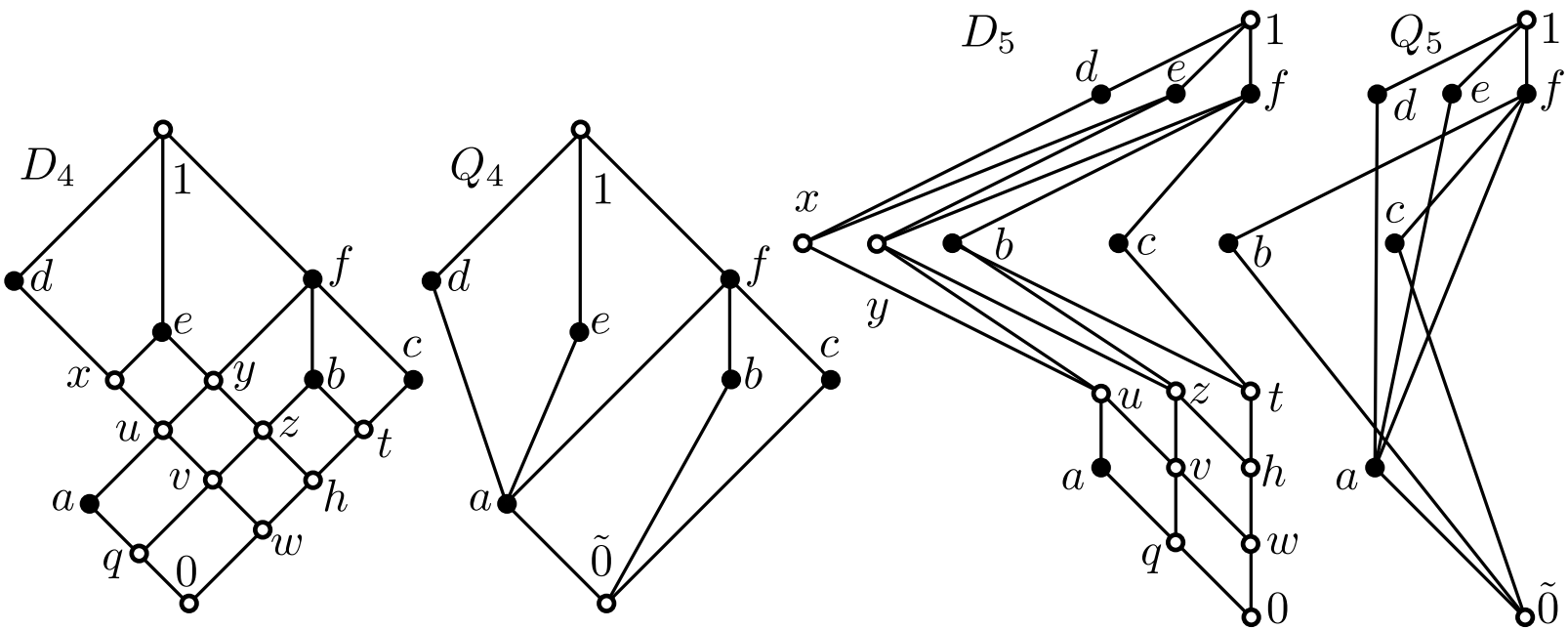
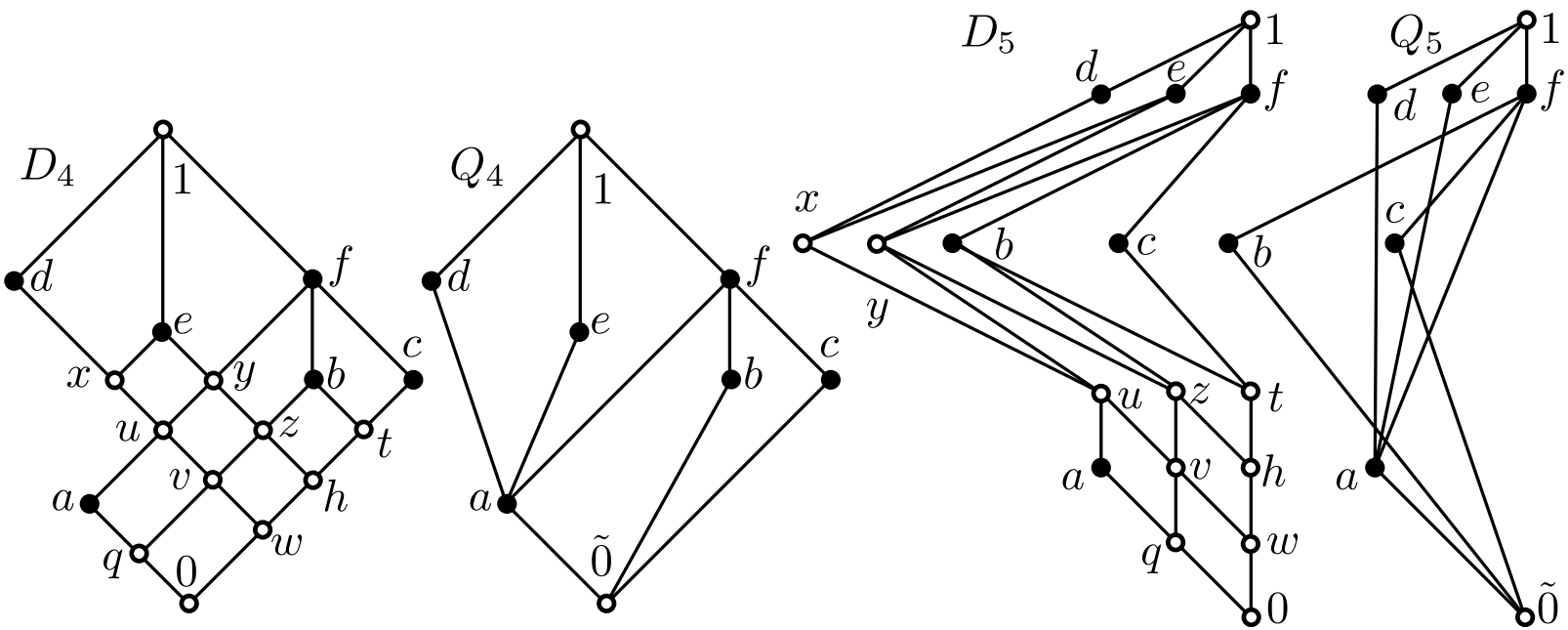
An illustration



Here $Q_4 = \alpha(D_4)$ and $D_4 = \beta(Q_4)$.



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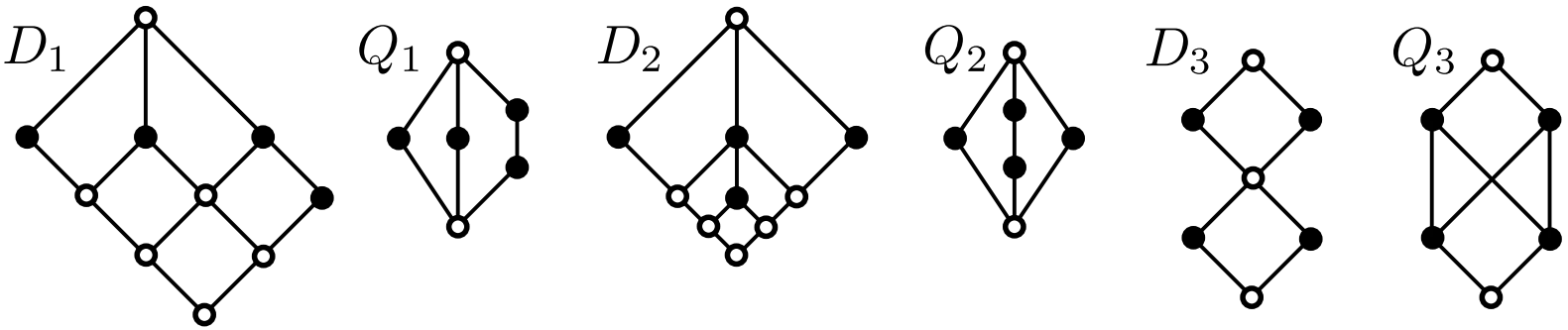

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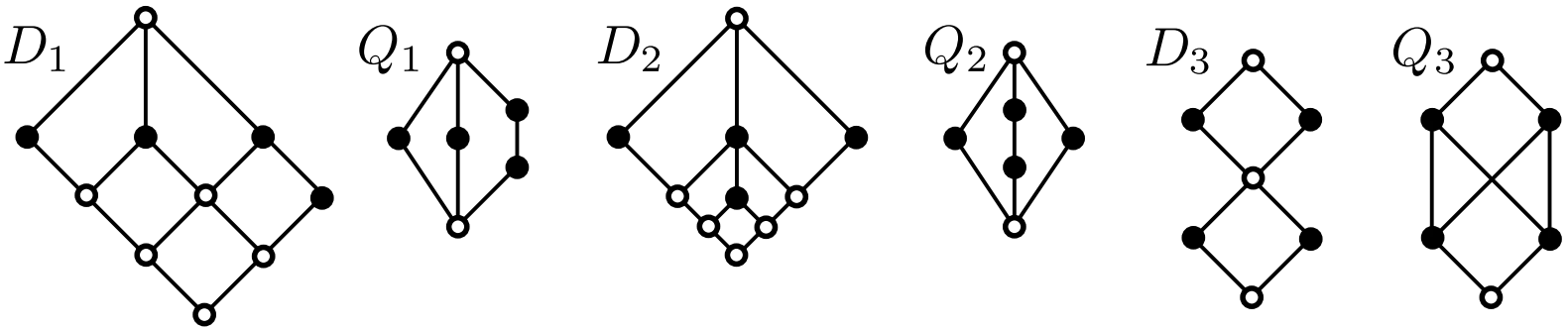

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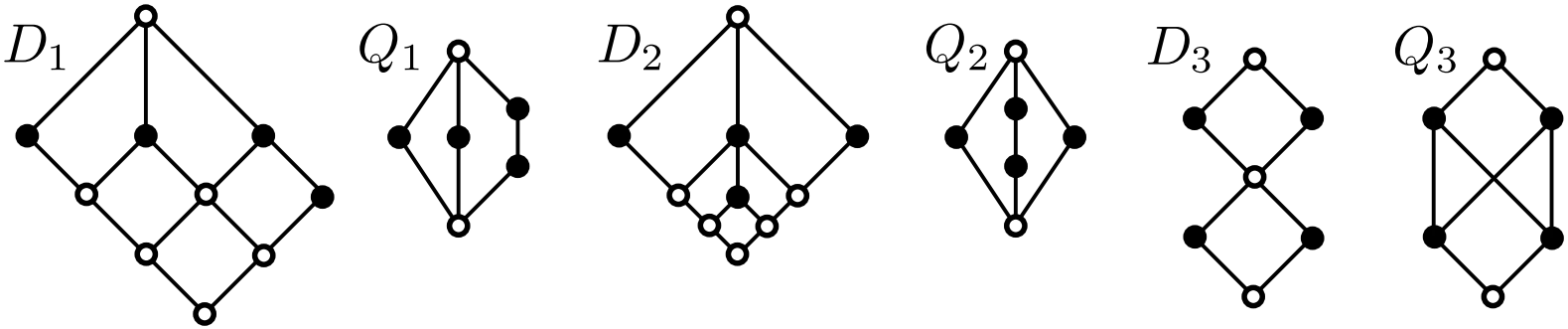
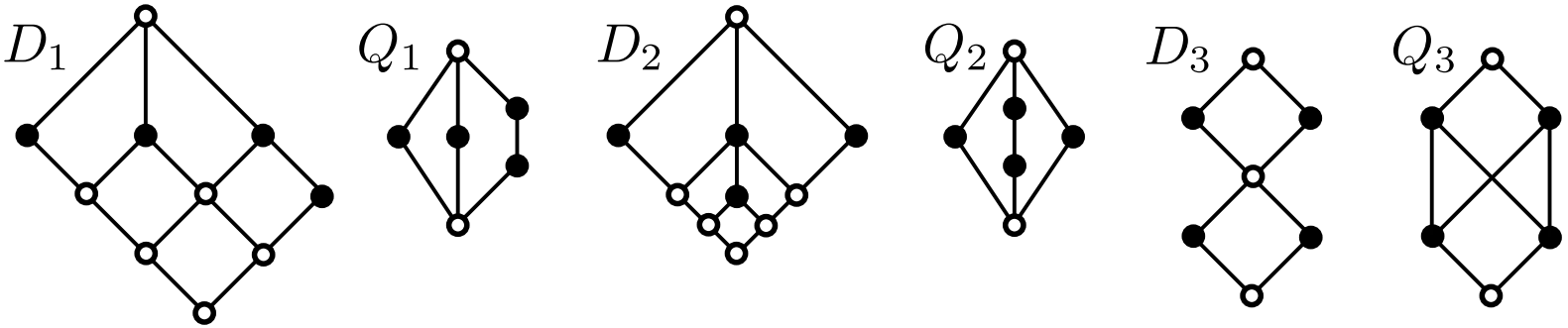


Here Q_1 and Q_2 determine isomorphic posets but non-isomorphic lattices. $\Rightarrow$ Goal 2??? : We cannot describe L by the poset $\langle \mathrm{Mi}\, L; \leq \rangle$ without λ, but we can do this with $\langle \mathrm{Mi}\, L; \leq, \lambda \rangle$.

THANK YOU FOR YOUR ATTENTION!



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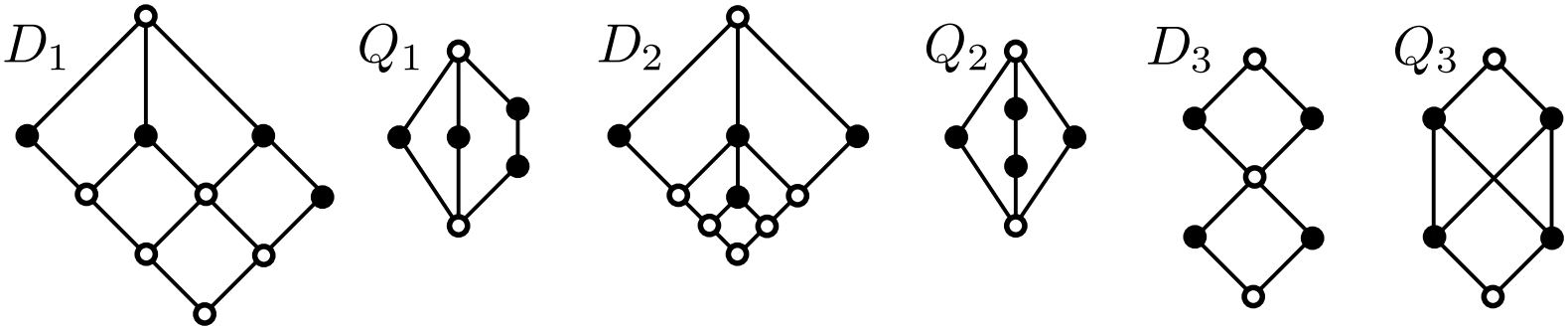

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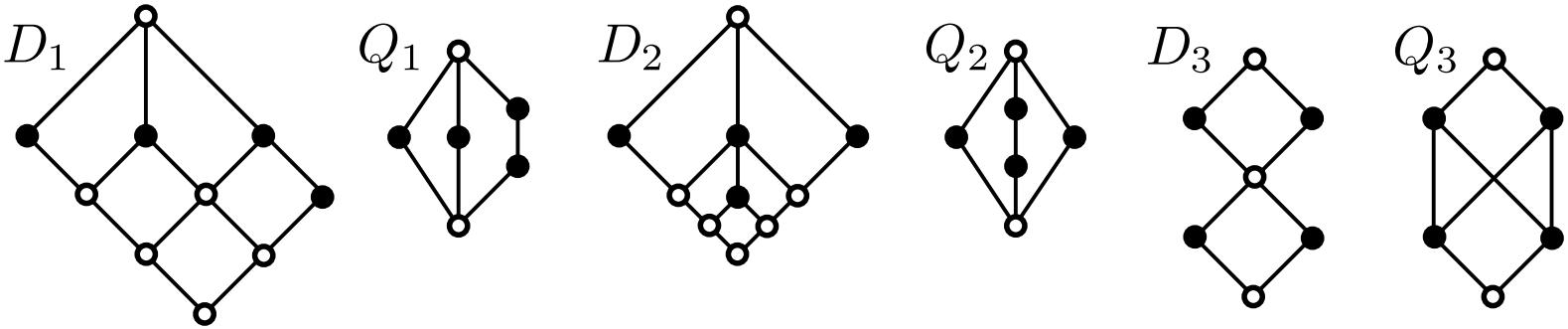

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