Counting congruences: the largest and beyond for lattices, and the maximum for congruence-distributive algebras

These slides: http://www.math.u-szeged.hu/~czedli/
Audio pronunciation of my name (clickable link)

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Online talk at the 17th International Summer School-Conference
"Problems Allied to Model Theory and Universal Algebra"

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Novosibirsk, June 19, 2025

Let $L = (L; \vee, \wedge)$ be a lattice. We always assume that L is ***finite***. The lattice of all congruences of L is denoted by **Con(L)** (the lattice order is $\subseteq$). It is the **congruence lattice** of L. The *number of congruences* of L is denoted by **|Con(L)|**.

Similarly, the **sublattice lattice** of L is denoted by **Sub(L)**. It consists the $\{\vee, \wedge\}$-closed subsets of L, including $\emptyset$. When speaking of the *number of sublattices*, then we count the $\emptyset$, too.

A relation $\mu \subseteq L \times L$ is a *quasiorder* on L if it is reflexive, transitive, and compatible with $\vee$ and $\wedge$. Let **Quo(L)** stand for the **quasiorder lattice** of L. So **|Quo(L)|** is the *number of quasiorders* of L.

Mostly (according to the title of this talk), we will focus on $|\mathrm{Con}(L)|$. Tangentially, we will mention some results on $|\mathrm{Sub}(L)|$, as its study ran parallel, and on $|\mathrm{Quo}(L)|$.

Funayama and Nakayama (1942): for each lattice L, $\mathrm{Con}(L)$ is a distributive lattice. So, the statement below applies to lattices, too.

Observation (Czédli, Acta Univ. Matthiae Belii—Ser. Math., 2018, https://actamath.savbb.sk/pdf/aumb2602.pdf)

Let A be an n-element congruence-distributive finite algebra. (I.e., $\mathrm{Con}(A)$ is assumed to be distributive.) Then

$$|\mathrm{Con}(A)| \le 2^{n-1}.$$

Furthermore, if "$\le$" above is "$=$", then $\mathrm{Con}(A)$ is Boolean.

Proof.

For a chain $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ in a lattice, k is called the *length* of the chain. Assume that the chain above denotes one of the longest chains in $\mathrm{Con}(A)$, where $|A| = n$. The congruences α_i are also equivalences, so they can be described by partitions π_i. Since $\alpha_i \leq \alpha_{i+1}$, each block of π_{i+1} is the union of some blocks (=classes) of π_i. This fact and $\alpha_i \neq \alpha_{i+1}$ imply that π_{i+1} has fewer blocks than π_i. As π_0 has at most $n = |A|$ blocks and π_k at least 1 block, we obtain $k \leq n - 1$.
We have seen that every chain in the distributive lattice $\mathrm{Con}(A)$ is of length at most $n - 1$. Thus, it suffices to show that whenever the longest chain in a finite distributive lattice D is of length k, then $|D| \leq 2^k$. (to be cont'd on the next page) □

For a finite lattice L, $b \in L$ is a *join-irreducible element* if there is exactly one edge going upward from b in the Hasse diagram of L. Let **Ji(L)** $:= \{$**j**oin-**i**rreducible elements of $L\}$; it is a poset. For a poset P and $B \subseteq P$, we say that B is an *order-ideal* of P if $(\forall u \in B)(\forall v \in P)\,(v \le u \Rightarrow v \in B)$. Let **Idl($P$)** stand for the set of order-**id**ea**l**s of P; with respect to $\subseteq$, $\mathrm{Idl}(P)$ is a distributive lattice.

Theorem (The classical structure theorem of finite distributive lattices; see, e.g., G. Grätzer: General Lattice Theory, 1978)

For every distributive lattice D, $D \cong \mathrm{Idl}(\mathrm{Ji}(D))$

Lemma (See, e.g., G. Grätzer: General Lattice Theory, 1978)

The maximum length of a chain in a finite distributive lattice D is $|\mathrm{Ji}(D)|$.

Proof of the Observation (cont'd).

Let $D := \mathrm{Con}(A)$. We have seen that the length of every chain in D is at most $n-1$. By the lemma, $|\mathrm{Ji}(D)| \leq n-1$. So $|\mathrm{Idl}(\mathrm{Ji}(D))| \leq 2^{n-1}$. By the structure theorem, $\mathrm{Con}(A) = D$ is isomorphic to $\mathrm{Idl}(\mathrm{Ji}(D))$, whereby $\mathrm{Con}(A)$ has at most 2^{n-1} elements. Straightforward but would need more time: if "$\leq$" is "$=$", then $\mathrm{Con}(A)$ is Boolean. □

Corollary (Ralph Freese, Proc. Am. Math. Soc. 125 (1997) 3457–3463)

For every $n \in \mathbb{N}^+$, the largest possible number of congruences of an n-element lattice is 2^{n-1}.

A preparation for a new proof: For $a, b \in L$, denote by **con(*a*, *b*)** the **C**ongruence **G**enerated by (a, b), i.e., smallest $\alpha \in \mathrm{Con}(L)$ such that $(a, b) \in \alpha$.

Let **$\mathrm{Inc}_1(n)$** denote the **L**argest possible **N**umber of **C**ongruences of an n-element lattice. (We will explain "1" in the subscript later.)

Lemma (Grätzer)

For a finite lattice L, $\mathrm{Ji}(L) = \{\mathrm{con}(a, b) : a \prec b \text{ in } L\}$.

Warning: $(a, b) \neq (a', b')$ with $\mathrm{con}(a, b) = \mathrm{con}(a', b')$ is possible.

Proof of the Corollary.

Let L be the n-element chain. It has $n-1$ edges. Let $a \prec b$ one of these edges. Using that L is a chain, it is not hard to see that (a, b) is the ONLY irreflexive pair (in particular, the ONLY edge) collapsed by $\mathrm{con}(a, b)$. Hence, the warning above does not apply, and $\mathrm{Ji}(L)$ is $(n-1)$-element and it is an *antichain*. Thus $\mathrm{Idl}(\mathrm{Ji}(L)) = \{\text{all subsets of } \mathrm{Ji}(L)\}$, so $\mathrm{Con}(L) \cong \mathrm{Idl}(\mathrm{Ji}(L))$ consists of exactly 2^{n-1} elements. Hence, $\mathrm{Inc}_1(n) \geq 2^{n-1}$. Conversely, the distributivity of $\mathrm{Con}(L)$ and the Observation proved earlier imply $\mathrm{Inc}_1(n) \leq 2^{n-1}$. □

The term ***k*th largest member** of a finite set is self-explanatory. For example, the 3rd largest member of $\{7, 1, 5, 4\}$ is 4, while the 1-st largest member is 7. That is, "1-st largest" means "the largest" = "the maximum".

Definition

$\mathbf{SCL}(n) = \{|\mathrm{Con}(L)| : L \text{ is an } n\text{-element lattice}\}$ will stand for the set of **S**izes of the **C**ongruence **L**attices of n-element lattices.
$\mathbf{Inc}_k(n)$ denotes the ***k***th **L**argest **N**umber of **C**ongruences of n-element lattices; it is the kth largest member of $\mathrm{SCL}(n)$.

Note that $\mathrm{SCL}(n)$ is finite, and $\mathrm{Inc}_k(n)$ exists $\iff k \leq |\mathrm{SCL}(n)|$.

Definition

SSL(n) $= \{|\mathrm{Sub}(L)| : L$ is an n-element lattice$\}$ will stand for the set of **S**izes of the **S**ublattice **L**attices of n-element lattices. (We treat $\emptyset$ as a sublattice!)
Ins$_k$(n) denotes the kth **L**argest **N**umber of **S**ublattices of n-element lattices; it is the kth largest member of SSL(n).

Definition

SQL(*n*) $= \{|\mathrm{Quo}(L)| : L$ is an n-element lattice$\}$ stands for the set of **S**izes of the **Q**uasiorder **L**attices of n-element lattices. Its kth largest member is denoted by $\mathbf{Inq}_k(n)$, which comes from "***k***th **L**argest **N**umber of **Q**uasiorders of n-element lattices".

We **counted** $|\mathrm{Con}(L)|$ and $|\mathrm{Quo}(L)|$ for some small lattices L; this is how we could guess and prove the following theorem 3 decades ago:

Theorem (Czédli and Szabó, Acta Sci. Math. 60 (1995), 207–211.)

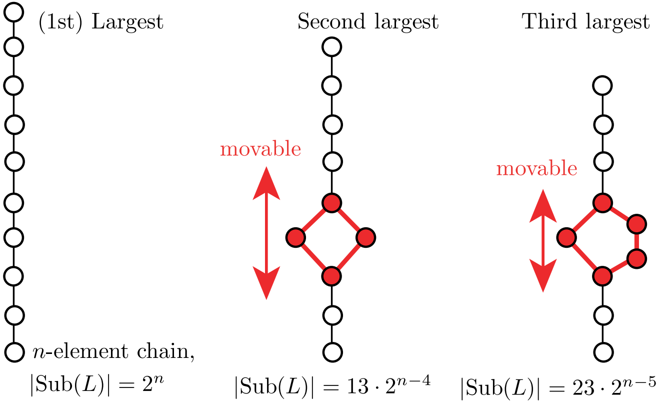
For every lattice L, $\mathrm{Quo}(L) \cong \mathrm{Con}(L) \times \mathrm{Con}(L)$ *(direct product).*

So $\mathrm{Inq}_k(n) = \mathrm{Inc}_k(n)^2$. This equality reduces the "number-of-quasiorders problems" to the corresponding "number-of-congruences problems". Thus, quasiorders will not occur in the rest of the talk.

Clearly, $\mathrm{Ins}_1(n) = 2^n$. Indeed, an n-element L cannot have more sublattices than subsets, and all subsets are sublattices when L is a chain.

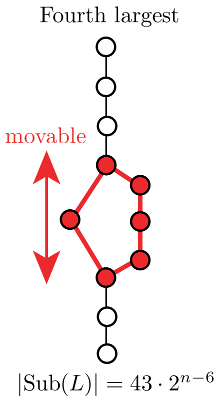
Theorem (Czédli and Eszter K. Horváth, Miskolc Mathematical Notes 20 (2019), No. 2, pp. 839–848)

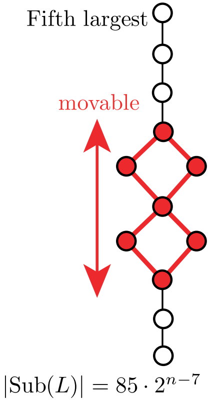
For $n \geq 5$, $\mathrm{Ins}_1(n) = 2^n$ (we mentioned this above),
$\mathrm{Ins}_2(n) = 13 \cdot 2^{n-4}$, *and*
$\mathrm{Ins}_3(n) = 23 \cdot 2^{n-5}$.
Furthermore, we described the n-element lattices witnessing these numbers; see the diagrams:


(1st) Largest
Second largest
Third largest
movable
movable
n-element chain,
$|\mathrm{Sub}(L)| = 2^n$
$|\mathrm{Sub}(L)| = 13 \cdot 2^{n-4}$
$|\mathrm{Sub}(L)| = 23 \cdot 2^{n-5}$

Theorem (Delbrin Ahmed and Eszter K. Horváth, Discus. Math. General Algebra and Appl. 39 (2019) 251–261)

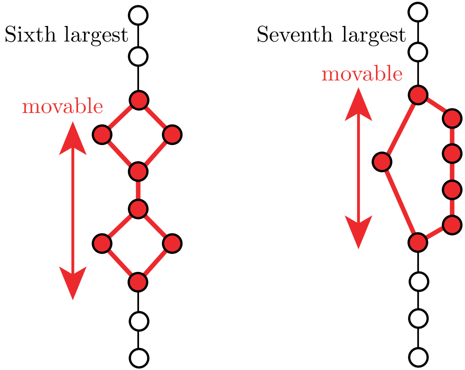
For $n \geq 6$, $\text{Ins}_4(n) = 43 \cdot 2^{n-6}$.
For $n \geq 7$, $\text{Ins}_5(n) = 85 \cdot 2^{n-7}$.
The witnesses are drawn on the next page.


Fourth largest
movable
$|\mathrm{Sub}(L)| = 43 \cdot 2^{n-6}$


Fifth largest
movable
$|\mathrm{Sub}(L)| = 85 \cdot 2^{n-7}$

Theorem (Neven E. Zaya, Dilbak Haje, Delbrin Ahmed, Academic J. Nawroz Univ. (AJNU) 12/1, 143–148; 2023)

For $n \geq 8$, $\mathrm{lns}_6(n) = 169 \cdot 2^{n-8}$ *and* $\mathrm{lns}_7(n) = 83 \cdot 2^{n-7}$. *The witnesses are drawn on the next page.*



$|\mathrm{Sub}(L)| = 169 \cdot 2^{n-8}$ $|\mathrm{Sub}(L)| = 83 \cdot 2^{n-7}$

8th, 9th, . . . : unknown.

Theorem (Czédli: Eighty-three sublattices and planarity, Algebra Universalis, (2019) 80:45, 19 pages, arXiv version: 100 pages)

For a finite n-element lattice L, if $|\mathrm{Sub}(L)| > 83 \cdot 2^{n-8}$, *then L is planar.* *Sharpness: If* $n \geq 9$, *then there exists of an n-element lattice L such that* $|\mathrm{Sub}(L)| = 83 \cdot 2^{n-8}$ *and L is non-planar.*

Note that the $100 - 19 = 81$ pages are also needed, but they are computer-related. The paper is a combination of human considerations and the brute force of a computer.

83 plays an essential role in the theorem, which was published in the year when George Grätzer was 83. Thus, even though 83 is not divisible by 10, the paper was dedicated to him on his 83rd birthday.

Theorem (Czédli, Acta Sci. Math. (Szeged), 86 (2020), 117–165; 49 pages, arXiv 71 pages, built on the previous paper)

For a finite n-element (upper) nearlattice L, if $|\mathrm{Sub}(L)| > 83 \cdot 2^{n-8}$, *then L is planar. The statement is sharp: If* $n \geq 9$, *then there exists an n-element lattice L such that* $|\mathrm{Sub}(L)| = 83 \cdot 2^{n-8}$ *and L is non-planar.*

Definition

A finite structure $(L; \vee, \wedge)$ with a total operation $\vee$ and a partial operation $\wedge$ is a **finite nearlattice** if $(L; \vee)$ is a finite (join)-semilattice (so it determines an order $\leq$), and for each $b \in L$, $(\{x \in L : b \leq x\}; \vee, \wedge)$ is a lattice. (So, in a finite nearlattice, if $\{x, y\}$ has a lower bound, then $x \wedge y = \inf\{x, y\}$; $x \wedge y$ is undefined otherwise.) Sub(L) above is the collection of sub-nearlattices (including $\emptyset$).

Theorem (Czédli, One hundred twenty-seven subsemilattices and planarity, Order (2020) 37:559-569)

Let L be a finite n-element semilattice. If $|\mathrm{Sub}(L)| > 127 \cdot 2^{n-8}$, then L is planar. For $n > 8$, this result is sharp, since there is a non-planar semilattice with $|\mathrm{Sub}(L)| = 127 \cdot 2^{n-8}$.

Here, $\emptyset$ is in $\mathrm{Sub}(L)$, the lattice of subsemilattices.
Returning to lattices, the story of the number $|\mathrm{Con}(L)|$ ran parallel to that of $|\mathrm{Sub}(L)|$.

Theorem (Czédli, Algebra Universalis, 2019. No. 80:16, 11 pages; https://doi.org/10.1007/s00012-019-0589-1)

Let L be an n-element finite lattice. If $|\mathrm{Con}(L)| > 2^{n-5}$, then L is planar. This result is sharp, since for each natural number $n \geq 8$, there exists a non-planar lattice with exactly 2^{n-5} congruences.

Remark

The proofs of all the theorems about planarity that we have mentioned thus far heavily use David Kelly and Ivan Rival's deep and long paper: Planar lattices. Can. J. Math. 27, 636–665 (1975); this paper characterizes the planarity of a finite lattice L by stipulating that no lattice from an infinite list that the paper concretely describes is a subposet of L.

Definition

The **C**ongruence **D**ensity of a finite lattice L is defined to be **$\mathrm{cd}(L)$** $:= |\mathrm{Con}(L)|/\mathrm{Inc}_1(|L|) = |\mathrm{Con}(L)|/2^{|L|-1}$. The kth **L**argest **C**ongruence **D**ensity, denoted by **$\mathrm{lcd}_k(n)$**, is the kth largest member of $\{\mathrm{cd}(L) : |L| = n\}$.

Remark

Since $\mathrm{Inc}_k(n) = \mathrm{lcd}_k(n) \times 2^{n-1}$, *the study of* $\mathrm{Inc}_k(n)$ *and that of* $\mathrm{lcd}_k(n)$ *are equivalent. In what follows, we will work only with* $\mathrm{lcd}_k(n)$, *which is easier to work with.*

Definition

For $4 \le n \in \mathbb{N}^+$, **$\mathrm{Circle}(n)$** is the set of n-element non-chain lattices such that $L = C \cup C'$, C and C' are chains, and $C \cap C' = \{0_L, 1_L\}$. So the diagram of L as a graph is a circle, Figures will occur later.

Theorem (Freese, Czédli, and mainly Mureșan and Kulin)

(1) $\mathrm{lcd}_1(n) = \mathbf{64/64} = 1$; *R. Freese or see the Observation. Witness: the n-element chain.*

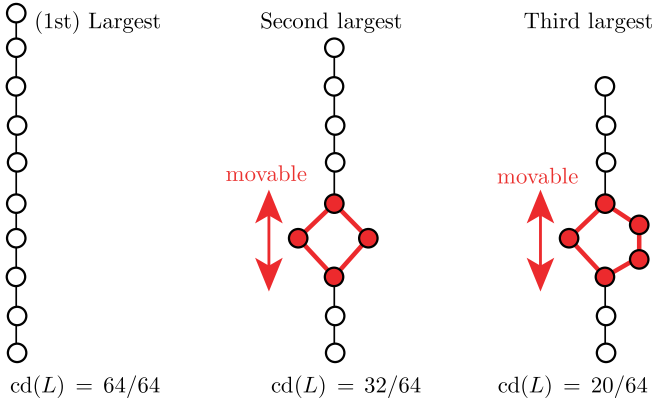
(2) For $4 \le n \in \mathbb{N}^+$, $\mathrm{lcd}_2(n) = \mathbf{32/64} = 1/2$ *(Czédli, Acta Univ. Matthiae Belii—Ser. Math., 2018, https://actamath.savbb.sk/pdf/aumb2602.pdf); witnesses: see the figure (later).*

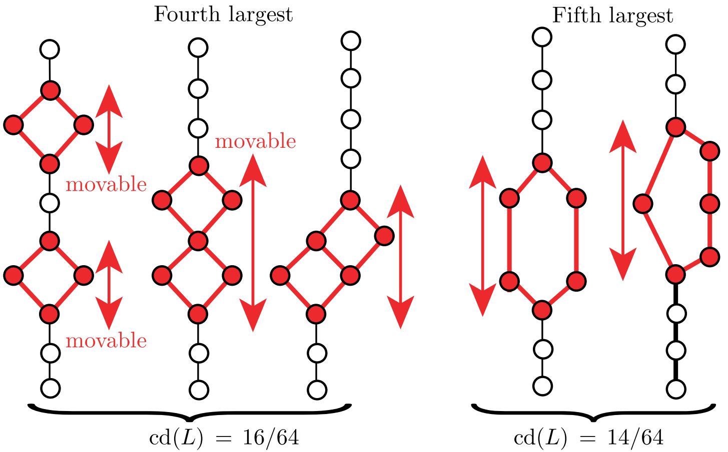
(3) For $5 \le n \in \mathbb{N}^+$, $\mathrm{lcd}_3(n) = \mathbf{20/64}$ *(Mureșan and Kulin, Order, 2020. Vol. 37. P. 445–460, https://doi.org/10.1007/s11083-019-09514-2); witnesses: see the figure.*

(4) For $6 \le n \in \mathbb{N}^+$, $\mathrm{lcd}_4(n) = \mathbf{16/64}$ *(Mureșan and Kulin. ibid.), witnesses: see next page.*

(5) For $6 \le n \in \mathbb{N}^+$, $\mathrm{lcd}_5(n) = \mathbf{14/64}$ *(Mureșan and Kulin. ibid.); witnesses: see the figure.*

Witnesses of $\mathrm{lcd}_k(n)$ for $k = 1, 2, 3$



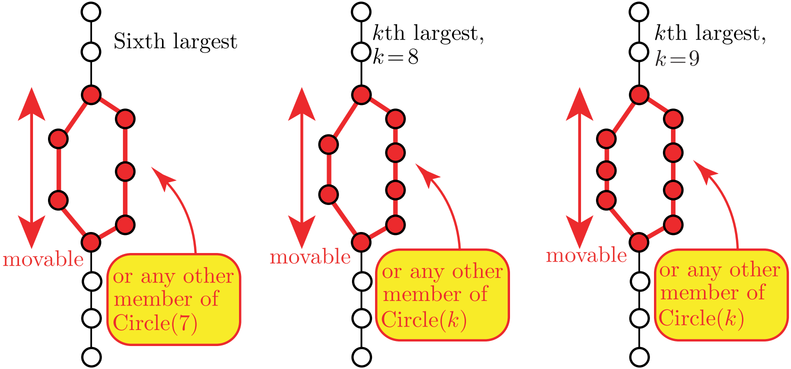

Fourth largest
Fifth largest
movable
movable
movable
cd(L) = 16/64
cd(L) = 14/64

Theorem (Czédli, https://tinyurl.com/czg-bsconl)

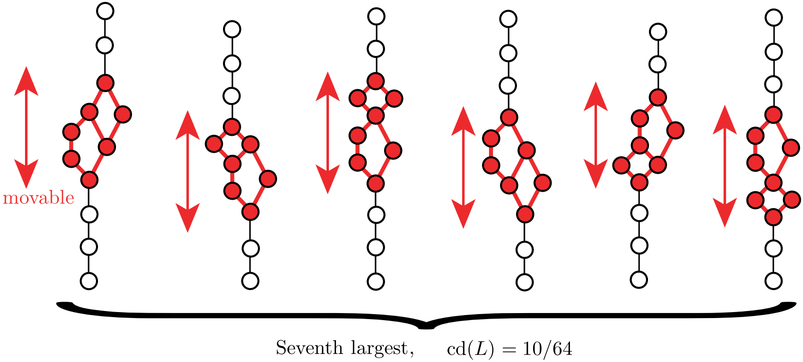
(6) *For* $7 \leq n \in \mathbb{N}^+$, $\mathrm{lcd}_6(n) = \mathbf{11/64}$; *witnesses: see the figure later.*

(7) *For* $7 \leq n \in \mathbb{N}^+$, $\mathrm{lcd}_7(n) = \mathbf{10/64}$; *witnesses: see the figure later.*

(k ≥ 8) *For* $8 \leq k \leq n \in \mathbb{N}^+$, $\mathrm{lcd}_k(n) = \mathbf{(8 + 3/2^{k-7})/64}$; *witnesses: see the figure later.*



$\mathrm{cd}(L) = 11/64$ $\mathrm{cd}(L) = (8 + 3/2^{k-7})/64$ $\mathrm{cd}(L) = (8 + 3/2^{k-7})/64$



Finally, these slides and many of the referenced papers are available at http://www.math.u-szeged.hu/~czedli/

Thank you for your attention!