

From Maltsev conditions to a duality theorem

These slides: <http://www.math.u-szeged.hu/~czedli/>
Source *paper*: <https://arxiv.org/abs/2406.15989>

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Gábor Czédli (University of Szeged)

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Online talk at the 16th International Summer School-Conference
“Problems Allied to Model Theory and Universal Algebra”

—
Novosibirsk, July 8, 2024

July 6, 2024

Notations. R is a ring with unit. $R\text{-Mod} = \{\text{all } R\text{-modules}\}$. An R -module is always a unital ($\forall x, 1x = x$) left module over R .
 $\text{Sub}(M)$: the submodule lattice of $M \in R\text{-Mod}$ ($\wedge = \cap, \vee = +$).
 The dual of a lattice identity: $\vee \leftrightarrow \wedge$ (i.e, turn \vee, \wedge upside down).

George Hutchinson's
 self-duality theorem



Hutchinson & Czédli, Algebra Universalis, 1978, but this is **ONLY his theorem**

For any lattice identity λ and for any ring R with 1, λ holds in $\text{Sub}(M)$ for all $M \in R\text{-Mod}$ if and only if so does the dual of λ .

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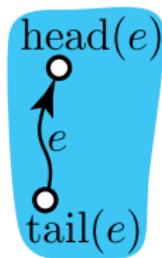
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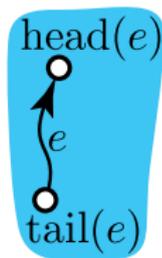
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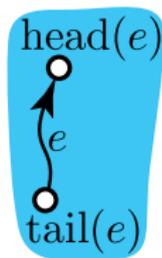
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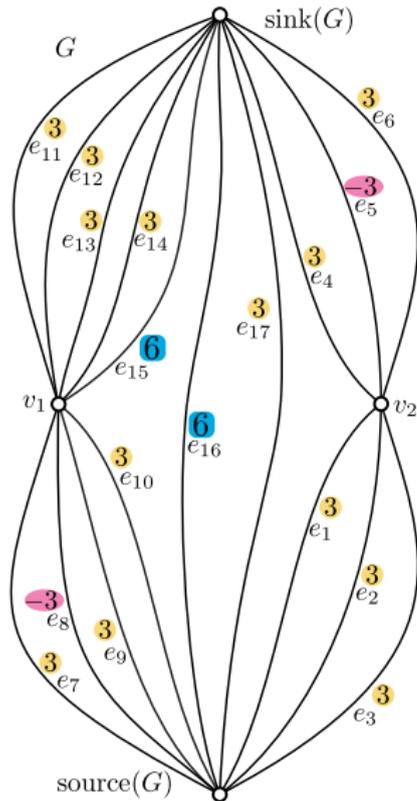


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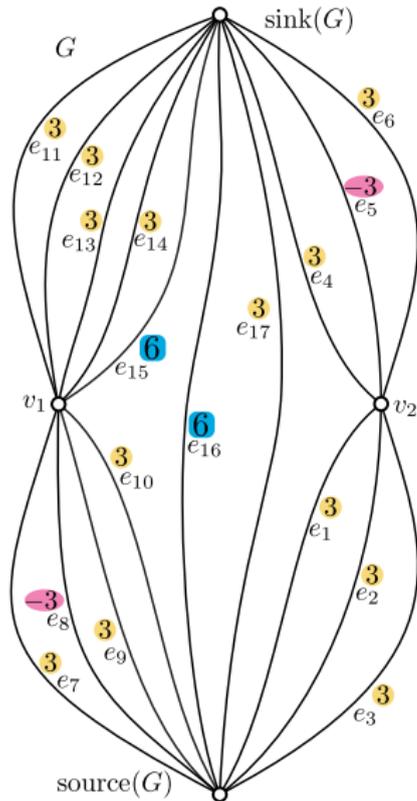
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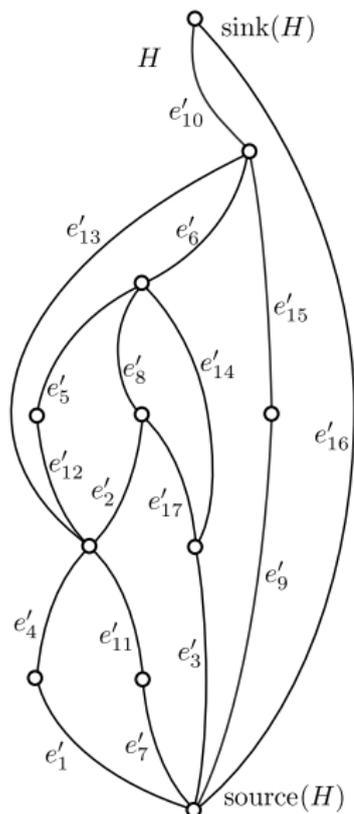
Such a graph G is a finite oriented **plane graph** (edges=non-crossing arcs in the plane) that has exactly one source (=vertex with no incoming edge into it), exactly one sink (a vertex with no outgoing edge from it), at least 2 vertices, and each edge of G has an **all-or-nothing-flow** capacity (those in the color-filled ovals). In our **hypothetical model**, the first graph, G , is a flow network; its edges can be:





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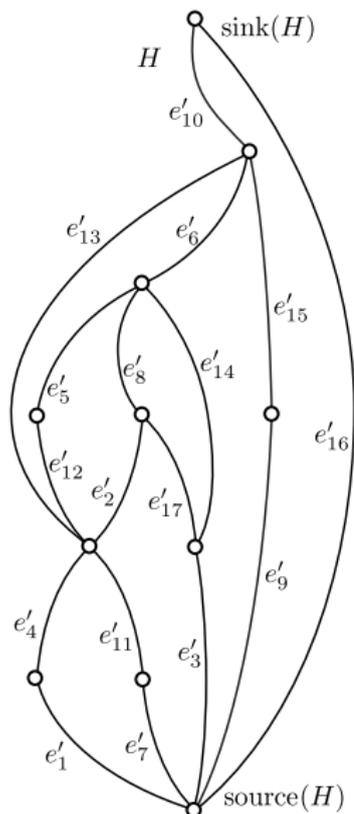


The second graph, H , is a **control graph**; its edges are **switches**

ON OFF to activate the corresponding edges of G .

If $c_i \in \mathbb{A}$ is the (full-or-nothing-flow) capacity of $e_i \in E(G)$, then the effect of $e'_i \in E(H)$ is $f: V(G) \rightarrow \mathbb{A}$, $\text{tail}(e_i) \mapsto -c_i$, $\text{head}(e_i) \mapsto c_i$, else $\mapsto 0$. (I.e., e_i transports c_i .) Here \mathbb{A} is an Abelian group; e.g., it can be the additive group \mathbb{Z} of integers.

The effect of a set X of edges of H is the sum of the effects of its members.



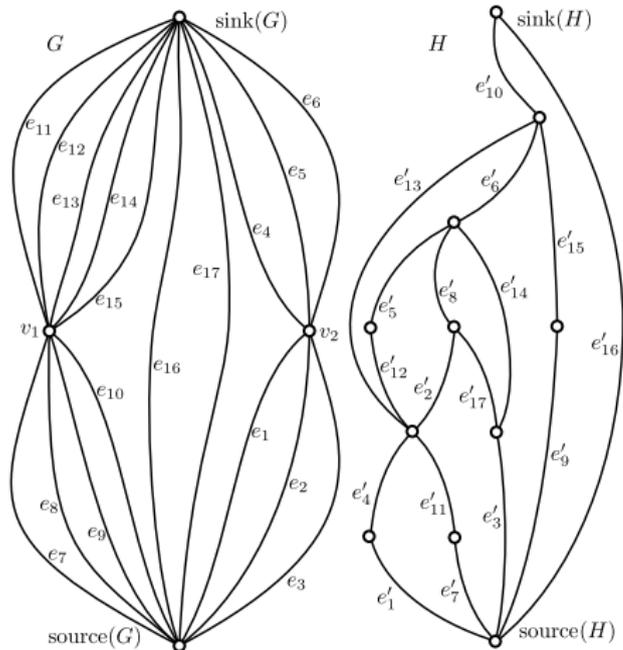
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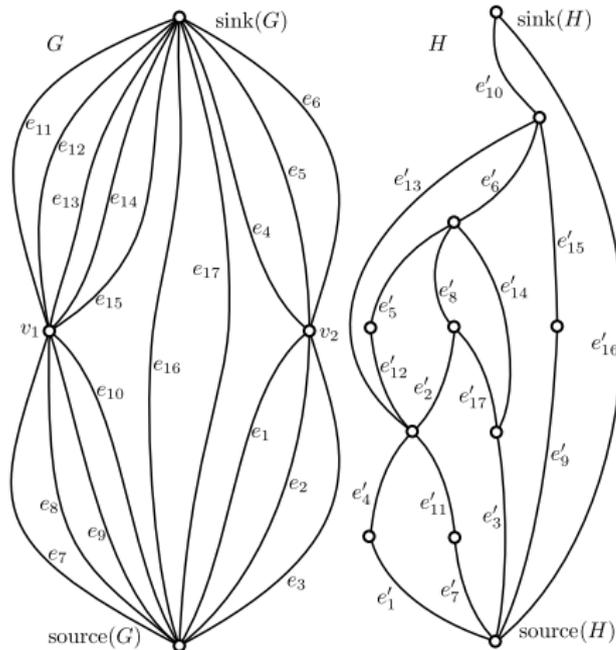
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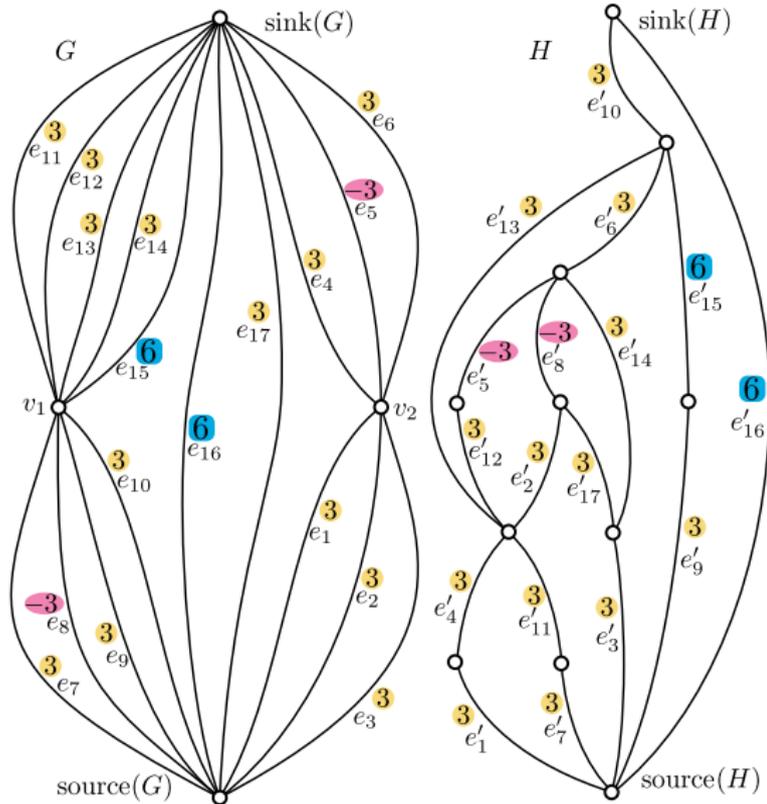
The **primal problem P** : G and H are as before, \leftrightarrow is a bijection between their edge sets $E(G)$ and $E(H)$, \mathbb{A} is an Abelian group, and $b \in \mathbb{A}$.

A **solution of P** is a vector $\vec{c} = (c_1, \dots, c_n) \in \mathbb{A}^n$ of capacities such that the effect of every maximal path of H is $source(G) \mapsto -b$, $sink(G) \mapsto b$, other vertex of $G \mapsto 0$. (The maximal path of H in question is called a **navigation path for G** ; it transports b from $source(G)$ to $sink(G)$.)



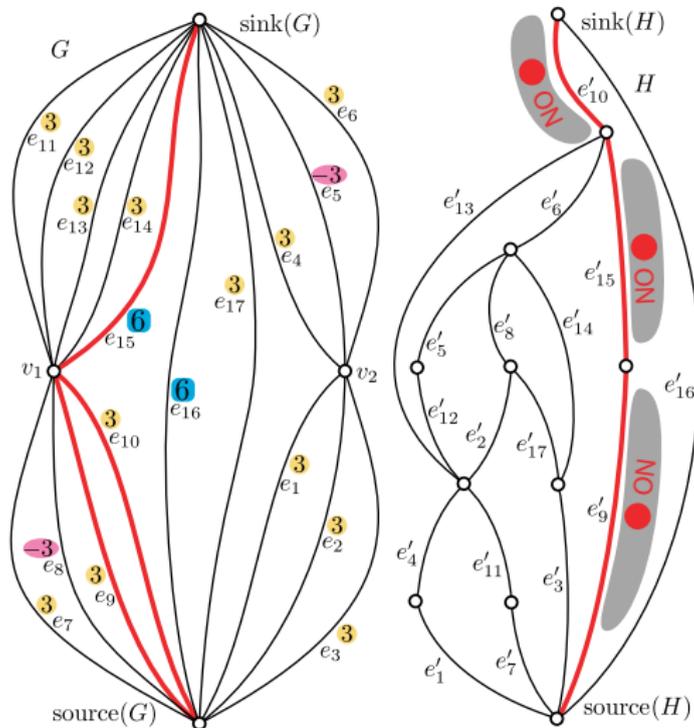
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For $\mathbb{A} = \mathbb{Z}$ and $b = 6$, the numbers in the colored ovals form a solution of the primal problem.

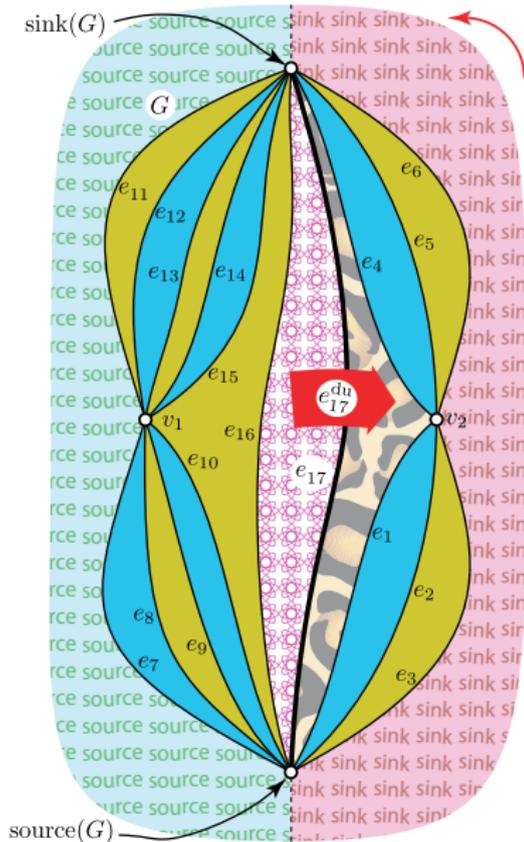
i.e., the effect of each maximal directed path of H (=navigation path for G) is that we transport b (units of something) from $\text{source}(G)$ to $\text{sink}(G)$ so that the “contents” of other vertices do not change eventually.



We check only the "red" navigation path (e'_9, e'_{15}, e'_{10}).

Indeed, e_9 and e_{10} transport $3 + 3 = 6$ from $source(G)$ to v_1 , and e_{15} transports 6 further, into $sink(G)$.

Defining the dual of an upward bipolar plane graph G 10'/15

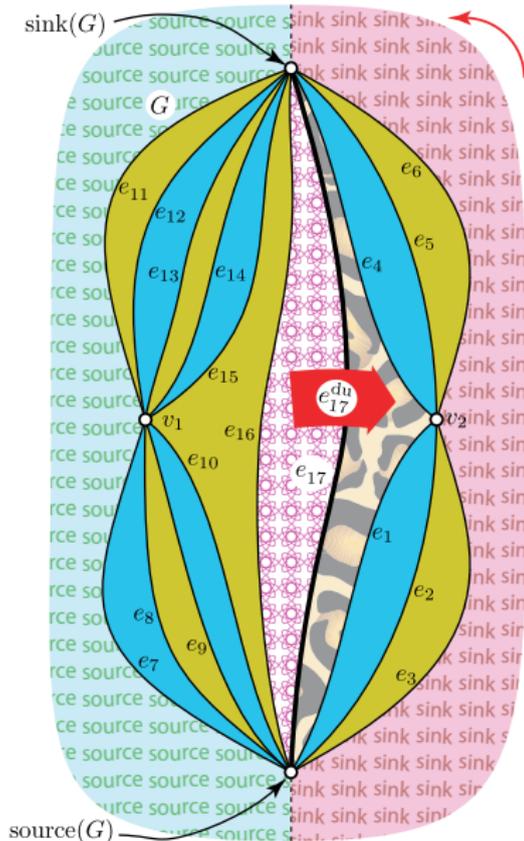


The vertices of G^{du} are the facets (\approx countries) of G , including the two outer facets as shown in the figure. The dual e^{du} of an edge e goes from the facet on the left of e to the facet on the right of e .

E.g., the **very thick red edge** $e^{\text{du}}_{17} \in E(G^{\text{du}})$ is the dual of the bold edge $e_{17} \in E(G)$; its tail and head are the **flower-filled facet** and the “leopard-filled” facet of G , respectively.

(To change “rightward” to upward, we could turn ∇ by 90° counterclockwise.)

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We have seen: The primal problem is $P = (G, H, \leftrightarrow, \mathbb{A}, b)$.
Define the dual problem as $P^{\text{du}} = (H^{\text{du}}, G^{\text{du}}, \leftrightarrow^{\text{du}}, \mathbb{A}, b)$, where $\leftrightarrow^{\text{du}}$ is given naturally by the indices.

Main Theorem (2024)

If (G, H) is a pair of upward bipolar plane graphs and \leftrightarrow is a bijection between their edge sets, then for any element b of any Abelian group \mathbb{A} , the primal problem P and the dual problem P^{du} have exactly the same solutions.

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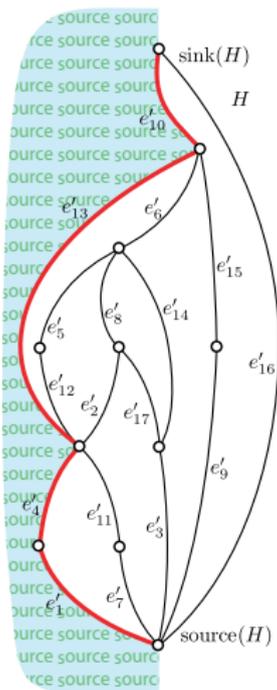
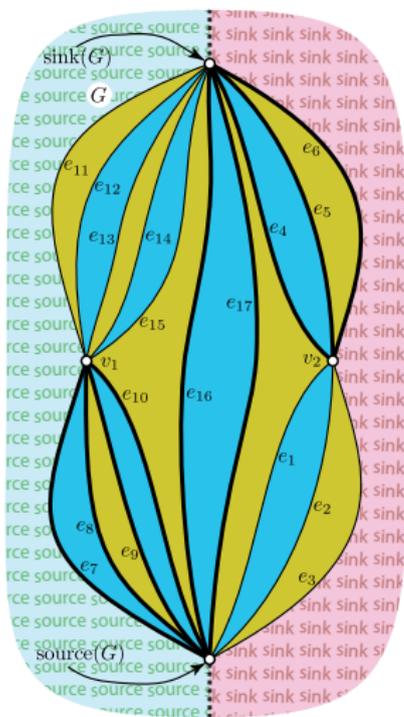
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Our example will indicate generality.

Assume that (c_1, \dots, c_n) is a solution of the primal problem P . To show that it is a solution of the dual problem P^{du} ,

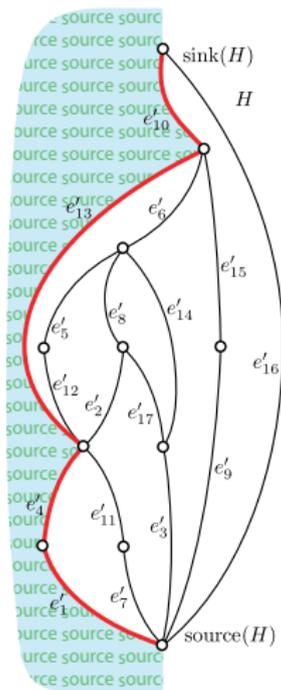
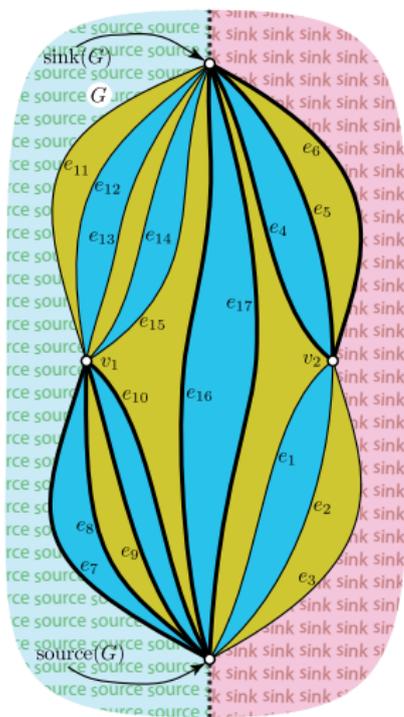
it is now G^{du} from which we take a navigation path, which we denote by α . So α is a navigation path for H^{du} .



Say, the navigation path α consists of the duals of the black bold edges.

When investigating the effect of α on each facet of H (i.e., vertex of H^{du}), there are three cases; we deal only with the left-most = source facet of H .

Let β consist of those edges of H that are on the right boundary of this facet; β consists of the red bold edges of H .

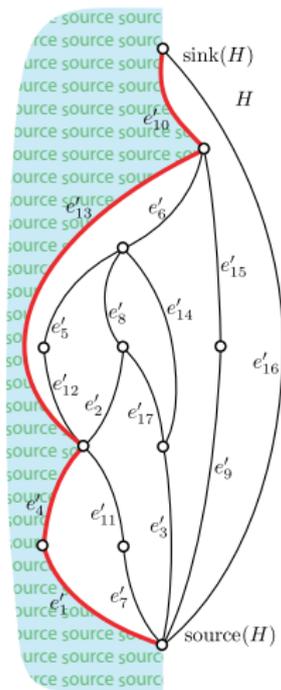
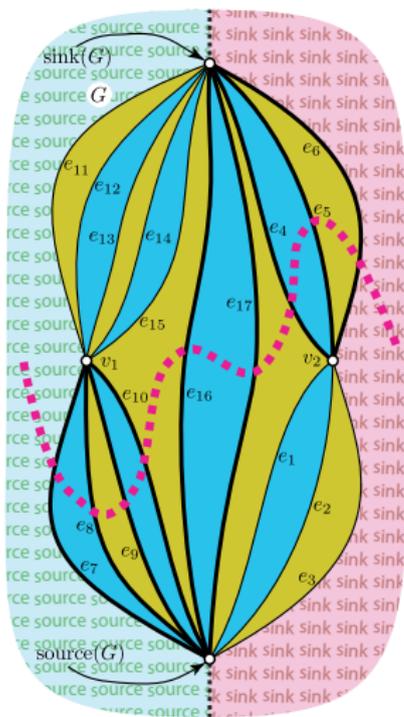


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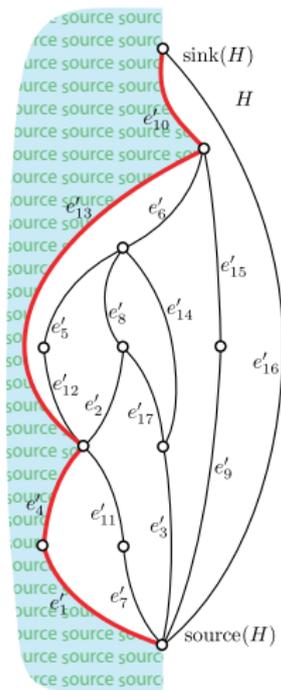
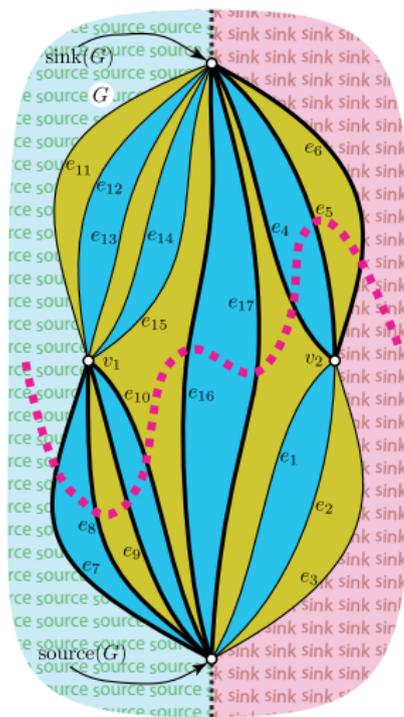
α, β : navigation paths for H^{du} and G , respectively 14'/11



So $\alpha = \text{black bold}^{\text{du}}$ is a navigation path for H^{du} , and β is a navigation path for G .

Let $X := \{i : e_i^{\text{du}} \in \alpha\}$ and $Y := \{i : e'_i \in \beta\}$.

Key step: Call the dotted thick magenta curve crossing the bold edges (and only those) on the left as the **Equator**.



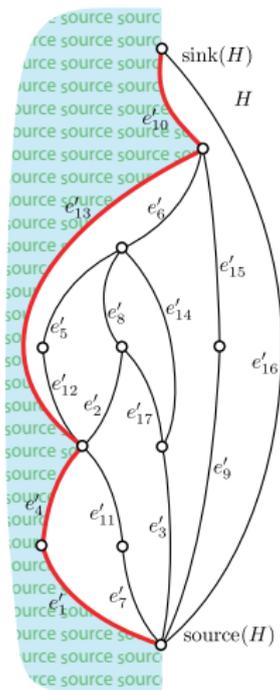
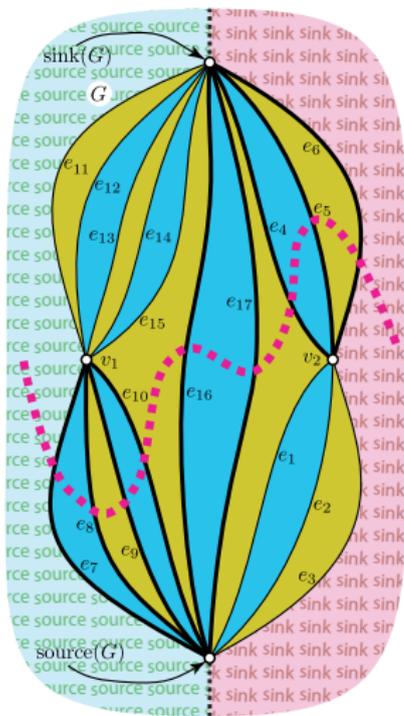
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15'/10



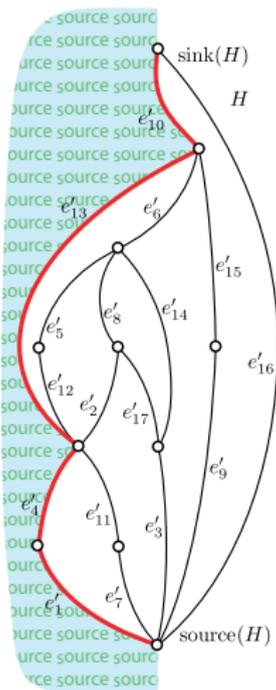
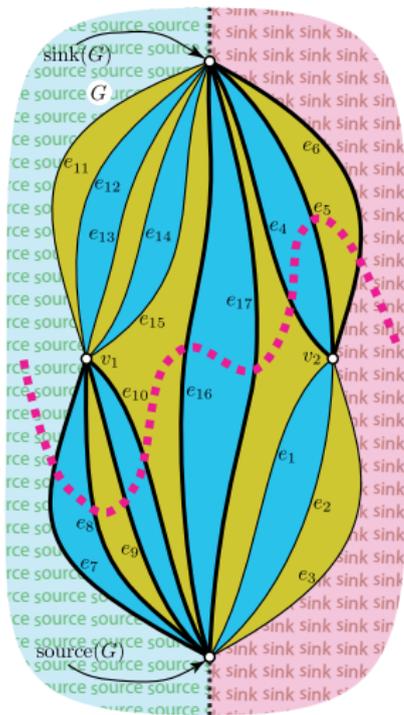
The navigation by β transports $b = 6$ from $\text{source}(G)$ (“South Pole”) to $\text{sink}(G)$ (“North Pole”). Hence exactly b units cross the **Equator**.

For which subscripts i do these crossings happen? Clearly, $i \in X$ (since crossing) and $i \in Y$ (since we navigate by β).

Thus, $\sum_{i \in X \cap Y} c_i = b$.

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15' / 10



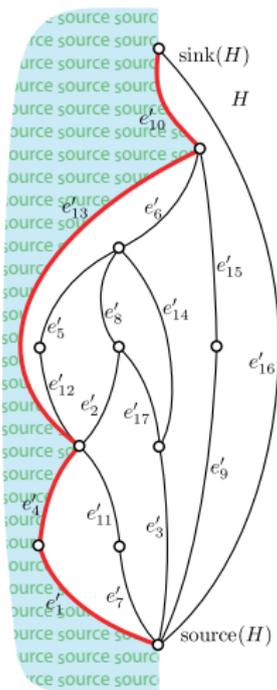
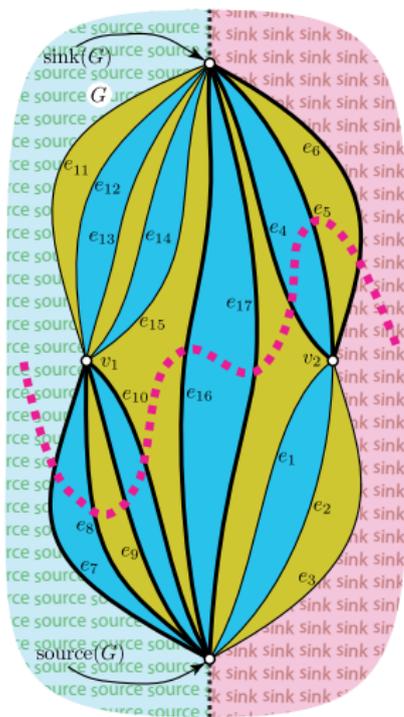
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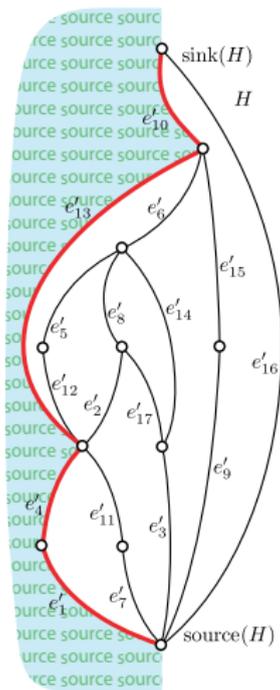
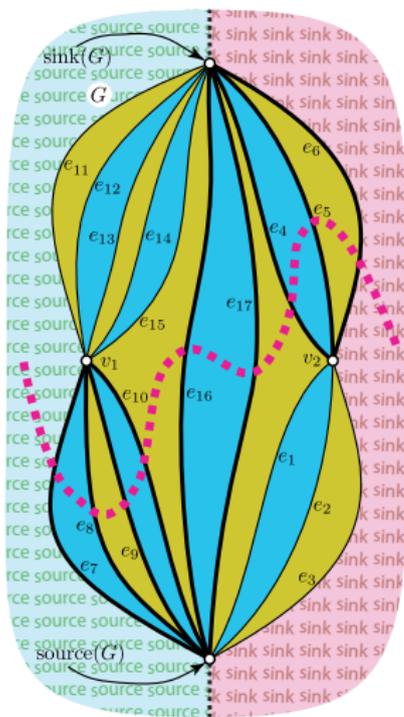
17'/8



Next, the effect of α on $\text{source}(H^{\text{du}})$ includes those subscripts i that belong to X , because of α , and belong also to Y (as otherwise neither the tail nor the head of $e'_i{}^{\text{du}}$ is $\text{source}(H^{\text{du}})$). Thus, this effect is $-\sum_{i \in X \cap Y} c_i$. Hence, by the previous slide, the effect of α on $\text{source}(H^{\text{du}})$ is $-b$, as required. ... Q.e.d.

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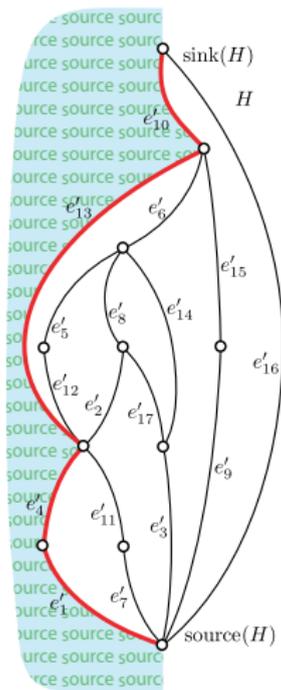
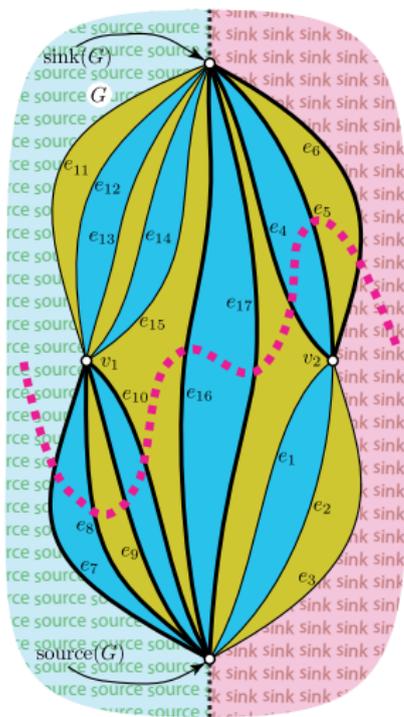


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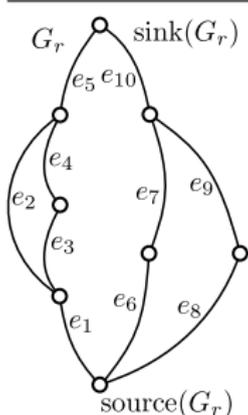
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Repetition-free lattice term: each of its variables occurs only once.



With each such lattice term r , we associate an upward bipolar plane graph G_r by induction. Namely: variable \mapsto two-element-one-edge bipolar graph.

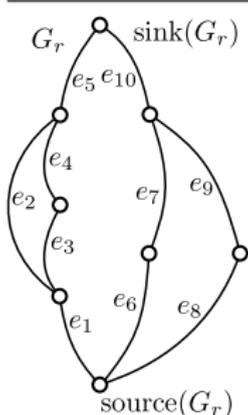
$G_{r_1 \vee r_2}$ is the series connection of G_{r_1} and G_{r_2} . (i.e., we put G_{r_2} atop G_{r_1} and glue them.) $G_{r_1 \wedge r_2}$ is the parallel connection of G_{r_1} and G_{r_2} . The indexing of the edges := the indexing of the variables. Example: below and on the left.

$$r = \left(x_1 \vee (x_2 \wedge (x_3 \vee x_4)) \vee x_5 \right) \wedge \left(((x_6 \vee x_7) \wedge (x_8 \vee x_9)) \vee x_{10} \right)$$

Easy Fact (An easy induction yields it.)

$$G_{r\text{-du}} \cong G_r^{\text{du}}.$$

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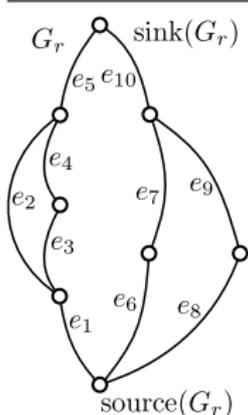
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Easy Lemma (Proof: induction, omitted.)

Let $r = r(x_1, \dots, x_n)$ be a repetition-free lattice term. For submodules B_1, \dots, B_n and elements u, v of an R -module M ,

$$v - u \in r(B_1, \dots, B_n)$$

if and only if there exists an $S: V(G_r) \rightarrow M$ such that $S(\text{source}(G_r)) = u$, $S(\text{sink}(G_r)) = v$, and for $i \in \{1, \dots, n\}$ (that is, for every edge e_i of G_r), $S(\text{head}(e_i)) - S(\text{tail}(e_i)) \in B_i$.

Pure and easy lattice theoretic considerations (1.3 page long) show that it suffices to prove Hutchinson's self-duality theorem only for 1-balanced identities of the form $p \leq q$. (I.e., each variable x_i occurs exactly once in $p = p(x_1, \dots, x_n)$ and exactly once on q .)

Easy Lemma (Proof: induction, omitted.)

Let $r = r(x_1, \dots, x_n)$ be a repetition-free lattice term. For submodules B_1, \dots, B_n and elements u, v of an R -module M ,

$$v - u \in r(B_1, \dots, B_n)$$

if and only if there exists an $S: V(G_r) \rightarrow M$ such that $S(\text{source}(G_r)) = u$, $S(\text{sink}(G_r)) = v$, and for $i \in \{1, \dots, n\}$ (that is, for every edge e_i of G_r), $S(\text{head}(e_i)) - S(\text{tail}(e_i)) \in B_i$.

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Key Lemma (Its proof is about two pages)

Let R be a ring with $1 = 1_R$ and let $p \leq q$ be a 1-balanced lattice identity. Then the following two conditions are equivalent.

- ① For every (unital left) R -module M , $p \leq q$ holds in $\text{Sub}(M)$.
- ② The primal problem $P = (G_p, G_q, \leftrightarrow, (R, +), 1)$ is solvable.

The idea of the proof, which resembles Maltsev's.

Assume (1). Let F be the free R -module generated by the vertices of G_p . For a variable x_i , let B_i be the submodule of F generated by the difference $\text{head}(e_i) - \text{tail}(e_i)$. Letting S be the identity map, the Easy Lemma gives that $\text{sink}(G_p) - \text{source}(G_p) \in p(B_1, \dots, B_n)$. By $p \leq q$, $\text{sink}(G_p) - \text{source}(G_p) \in q(B_1, \dots, B_n)$. Apply the Easy Lemma again and compute to obtain (2). **Assume (2)**. Read the earlier argument backward to obtain that $\text{Sub}(F) \models p \leq q$. Then we can tailor Maltsev's idea to the situation to get (1). \square

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In short, $p \leq q \iff P$ is solvable. By the Easy Fact, $G_{q^{\text{du}}} \cong G_q^{\text{du}}$ and $G_{p^{\text{du}}} \cong G_p^{\text{du}}$. Thus, applying the Key Lemma to $q^{\text{du}} \leq p^{\text{du}}$, $q^{\text{du}} \leq p^{\text{du}} \iff P^{\text{du}}$ is solvable. However, by the Main Theorem, P is solvable $\iff P^{\text{du}}$ is solvable. The three red facts in this slide imply that $p \leq q \iff q^{\text{du}} \leq p^{\text{du}}$, that is, $p \leq q$ in $\text{Sub}(M)$ for all $M \in R\text{-Mod}$ if and only if $q^{\text{du}} \leq p^{\text{du}}$ in $\text{Sub}(M)$ for all $M \in R\text{-Mod}$. So, by avoiding category theory, we have completed the simple proof of Hutchinson's self-duality theorem.

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The use of category theory in the study of submodule lattices remains useful and probably inevitable in another joint paper: G. Czédli and G. Hutchinson, Submodule lattice quasivarieties and **exact embedding functors** for rings with prime power characteristic; Algebra Universalis, 35 (1996), 425–445.

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