From Maltsev conditions to a duality theorem

These slides: http://www.math.u-szeged.hu/~czedli/
Source ***paper***: https://arxiv.org/abs/2406.15989

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"Problems Allied to Model Theory and Universal Algebra"

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Notations. R is a ring with unit. R-**Mod** $= \{$all R-modules$\}$. An R-module is always a *unital* ($\forall x,\ 1x = x$) left module over R.
Sub(M): the *submodule lattice* of $M \in R$-**Mod** ($\wedge = \cap$, $\vee = +$).
The *dual* of a lattice identity: $\vee \leftrightarrow \wedge$ (i.e, turn $\vee, \wedge$ upside down).

George Hutchinson's **self-duality theorem**

, Hutchinson & Czédli, Algebra Universalis, 1978, but this is **ONLY his theorem**

For any lattice identity λ and for any ring R with 1*, λ holds in* Sub(M) *for all $M \in R$-**Mod** if and only if so does the dual of λ.*

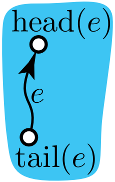
The lion's share of the proof belongs to the theory of *abelian categories* (in another paper by Hutchinson). *Maltsev conditions* are also needed in it.

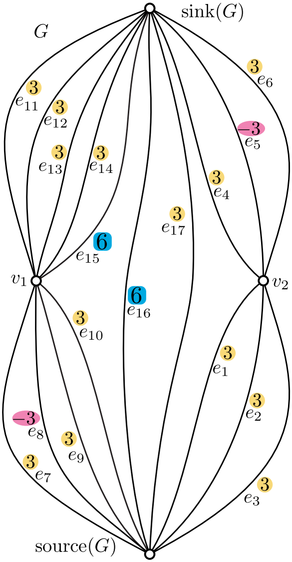


Named after: Anatoly Ivanovich *Maltsev* (Mal'cev), Novosibirsk, 1960–1967.

The goal of this talk is to give a *simpler proof*. A byproduct of the new proof could also be interesting. First, we deal with this byproduct.

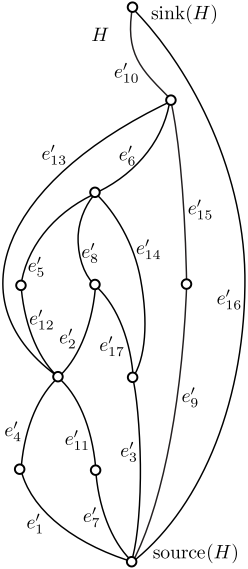
Our graphs will be *oriented graphs*; every edge will be directed *upward*; the arrowheads will be omitted.





Such a graph G is a finite oriented **plane graph** (edges=non-crossing arcs in the plane) that has exactly one source (=vertex with no incoming edge into it), exactly one sink (a vertex with no outgoing edge from it), at least 2 vertices, and each edge of G has an **all-or-nothing-flow** capacity (those in the color-filled ovals). In our hypothetical model, the **first graph**, G, is a **flow network**; its edges can be:



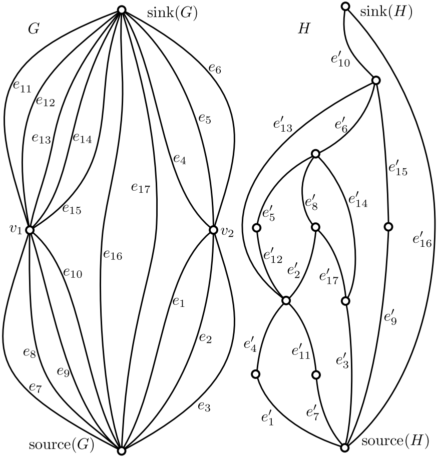



The second graph, H, is a ***control graph***; its edges are **switches** 

to activate the corresponding edges of G.

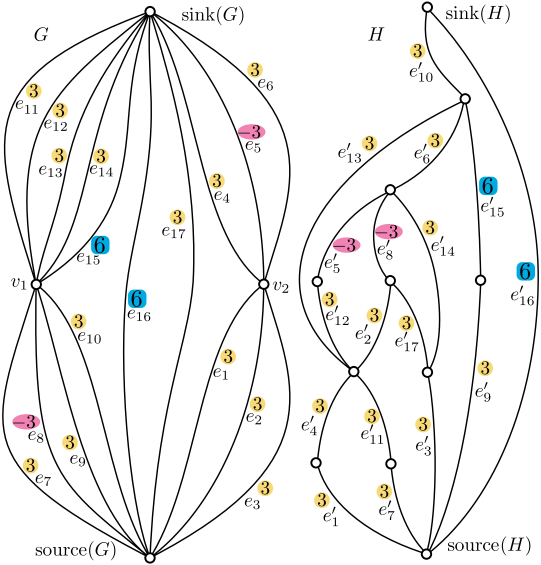
If $c_i \in \mathbb{A}$ is the (full-or-nothing-flow) capacity of $e_i \in E(G)$, then the **effect** of $e'_i \in E(H)$ is $f\colon V(G) \to \mathbb{A}$, $\mathrm{tail}(e_i) \mapsto -c_i$, $\mathrm{head}(e_i) \mapsto c_i$, else $\mapsto 0$. (I.e., e_i transports c_i.) Here $\mathbb{A}$ is an Abelian group; e.g., it can be the additive group $\mathbb{Z}$ of integers.

The **effect** of a set X of edges of H is the sum of the effects of its members.



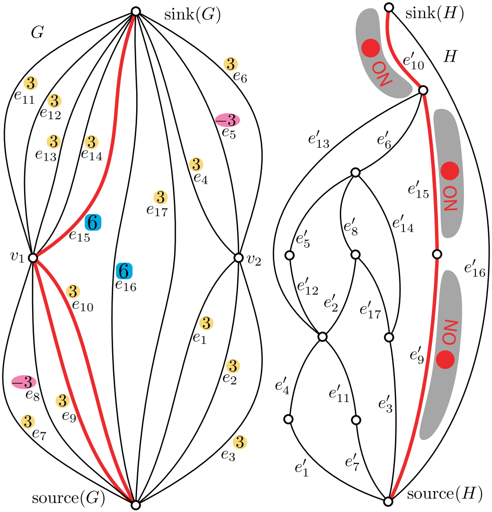
The **primal problem** P: G and H are as before, $\leftrightarrow$ is a bijection between their edge sets $E(G)$ and $E(H)$, $\mathbb{A}$ is an Abelian group, and $b \in \mathbb{A}$.

A **solution** of P is a vector $\vec{c} = (c_1, \ldots, c_n) \in \mathbb{A}^n$ of capacities such that the effect of every maximal path of H is source$(G) \mapsto -b$, sink$(G) \mapsto b$, other vertex of $G \mapsto 0$. (The maximal path of H in question is called a **navigation path for** G; it **transports** b from source(G) to sink(G).)



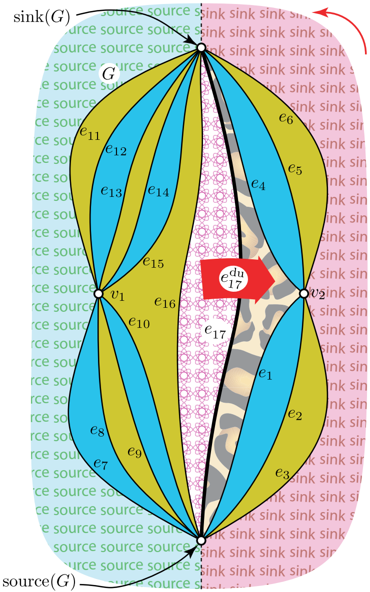
For $\mathbb{A} = \mathbb{Z}$ and $b = 6$, the numbers in the colored ovals form a solution of the primal problem.

I.e., the effect of each maximal directed path of H (=navigation path for G) is that we transport b (units of something) from source(G) to sink(G) so that the "contents" of other vertices do not change eventually.



We check only the "red" navigation path (e'_9, e'_{15}, e'_{10}).

Indeed, e_9 and e_{10} transport $3 + 3 = 6$ from source(G) to v_1, and e_{15} transports 6 further, into sink(G).



The vertices of G^{du} are the **facets** (≈countries) of G, including the **two** outer facets as shown in the figure. The dual e^{du} of an edge e goes from the facet on the left of e to the facet on the right of e. E.g., the **very thick red edge** $e_{17}^{\mathrm{du}} \in E(G^{\mathrm{du}})$ is the dual of the bold edge $e_{17} \in E(G)$; its tail and head are the flower-filled facet and the "leopard-filled" facet of G, respectively.

(To change "rightward" to upward, we could turn $\forall$ by 90° counterclockwise.)

We have seen: The primal problem is $P = (G, H, \leftrightarrow, \mathbb{A}, b)$.
Define the *dual problem* as $P^{\mathrm{du}} = (H^{\mathrm{du}}, G^{\mathrm{du}}, \leftrightarrow^{\mathrm{du}}, \mathbb{A}, b)$, where $\leftrightarrow^{\mathrm{du}}$ is given naturally by the indices.

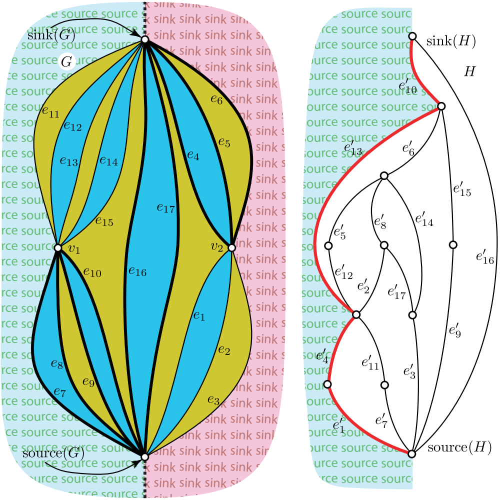
Main Theorem (2024)

If (G, H) is a pair of upward bipolar plane graphs and $\leftrightarrow$ is a bijection between their edge sets, then for any element b of any Abelian group $\mathbb{A}$, the primal problem P and the dual problem P^{du} have exactly the same solutions.

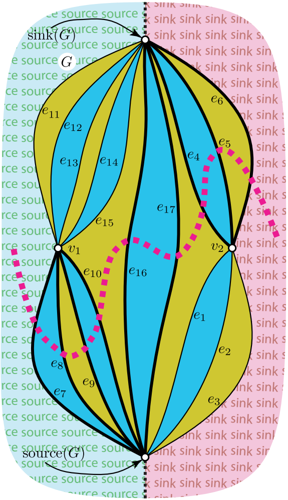
Our example will indicate generality.

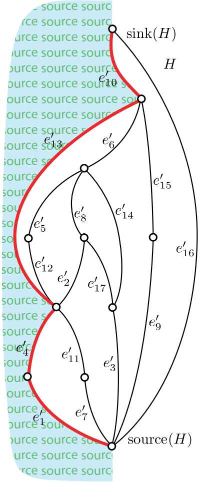
Assume that $(c_1, \ldots, c_n)$ is a solution of the primal problem P. To show that it is a solution of the dual problem P^{du},

it is now G^{du} from which we take a navigation path, which we denote by α. So α is a navigation path for H^{du}.



Say, the navigation path $\boldsymbol{\alpha}$ consists of the duals of the black bold edges. When investigating the effect of $\boldsymbol{\alpha}$ on each facet of H (i.e., vertex of H^{du}), there are three cases; we deal only with the leftmost = source facet of H. Let $\boldsymbol{\beta}$ consist of those edges of H that are on the right boundary of this facet; $\boldsymbol{\beta}$ consists of the red bold edges of H.

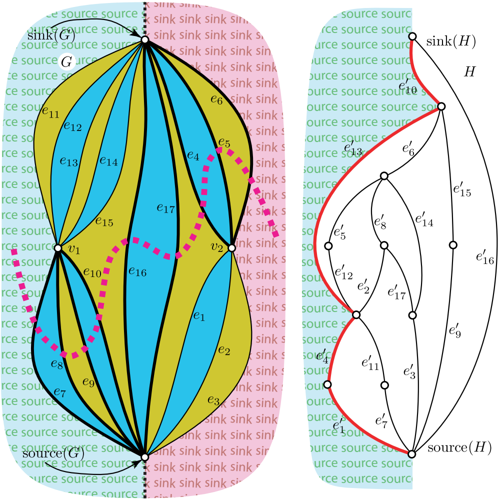




So $\boldsymbol{\alpha}$ = **black bold**$^{\mathrm{du}}$ is a navigation path for H^{du}, and $\boldsymbol{\beta}$ is a navigation path for G.

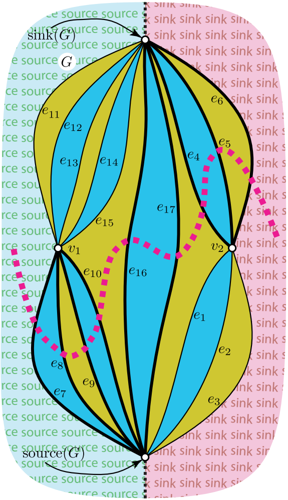
Let $X := \{i : e_i^{\mathrm{du}} \in \boldsymbol{\alpha}\}$ and $Y := \{i : e'_i \in \boldsymbol{\beta}\}$.

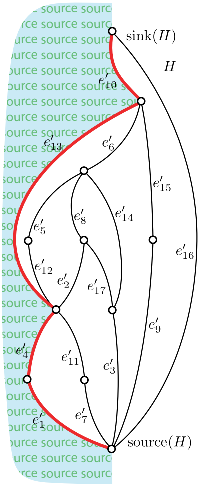
Key step: Call the dotted thick magenta curve crossing the bold edges (and only those) on the left as the **Equator**.



The navigation by $\color{red}{\boldsymbol{\beta}}$ transports $b = 6$ from source(G) ("South Pole") to sink(G) ("North Pole"). Hence exactly b units cross the **Equator**. For which subscripts i do these crossings happen? Clearly, $i \in X$ (since crossing) and $i \in \color{red}{Y}$ (since we navigate by $\color{red}{\boldsymbol{\beta}}$). Thus, $\sum_{i \in X \cap \color{red}{Y}} c_i = b$.

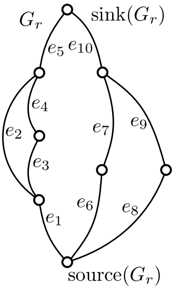
By the previous slide, $\sum_{i\in X\cap Y} c_i = b$





Next, the effect of $\boldsymbol{\alpha}$ on source(H^{du}) includes those subscripts i that belong to X, because of $\boldsymbol{\alpha}$, and belong also to Y (as otherwise neither the tail nor the head of $e_i'^{\,\mathrm{du}}$ is source(H^{du})). Thus, this effect is $-\sum_{i\in X\cap Y} c_i$. Hence, by the previous slide, the effect of $\boldsymbol{\alpha}$ on source(H^{du}) is $-b$, as required. ... Q.e.d.

Repetition-free lattice term: each of its variables occurs only once.



With each such lattice term r, we associate an upward bipolar plane graph G_r by induction. Namely: variable $\mapsto$ two-element-one-edge bipolar graph. $G_{r_1 \vee r_2}$ is the *series connection* of G_{r_1} and G_{r_2}. (I.e., we put G_{r_2} atop G_{r_1} and glue them.) $G_{r_1 \wedge r_2}$ is the *parallel connection* of G_{r_1} and G_{r_2}. The indexing of the edges := the indexing of the variables. Example: below and on the left.

$$r = \Big(x_1 \vee \big(x_2 \wedge (x_3 \vee x_4)\big) \vee x_5\Big) \wedge \Big(\big((x_6 \vee x_7) \wedge (x_8 \vee x_9)\big) \vee x_{10}\Big)$$

Easy Fact (An easy induction yields it.)

$G_{r^{\mathrm{du}}} \cong G_r^{\mathrm{du}}$.

Easy Lemma (Proof: induction, omitted.)

Let $r = r(x_1, \ldots, x_n)$ be a repetition-free lattice term. For submodules $B_1, \ldots, B_n$ and elements u, v of an R-module M,

$$v - u \in r(B_1, \ldots, B_n)$$

if and only if there exists an $S\colon V(G_r) \to M$ such that $S(\mathrm{source}(G_r)) = u$, $S(\mathrm{sink}(G_r)) = v$, and for $i \in \{1, \ldots, n\}$ (that is, for every edge e_i of G_r), $S(\mathrm{head}(e_i)) - S(\mathrm{tail}(e_i)) \in B_i$.

Pure and easy lattice theoretic considerations (1.3 page long) show that it suffices to prove Hutchinson's self-duality theorem only for *1-balanced* identities of the form $p \leq q$. (I.e., each variable x_i occurs exactly once in $p = p(x_1, \ldots, x_n)$ and exactly once on q.)

Key Lemma (Its proof is about two pages)

Let R be a ring with $1 = 1_R$ and let $p \leq q$ be a 1-balanced lattice identity. Then the following two conditions are equivalent.

1. *For every (unital left) R-module M, $p \leq q$ holds in* $\mathrm{Sub}(M)$.
2. *The primal problem $P = (G_p, G_q, \leftrightarrow, (R, +), 1)$ is solvable.*

The idea of the proof, which resembles Maltsev's.

Assume (1). Let F be the free R-module generated by the vertices of G_p. For a variable x_i, let B_i be the submodule of F generated by the difference $\mathrm{head}(e_i) - \mathrm{tail}(e_i)$. Letting S be the identity map, the Easy Lemma gives that $\mathrm{sink}(G_p) - \mathrm{source}(G_p) \in p(B_1, \ldots, B_n)$. By $p \leq q$, $\mathrm{sink}(G_p) - \mathrm{source}(G_p) \in q(B_1, \ldots, B_n)$. Apply the Easy Lemma again and compute to obtain (2). Assume (2). Read the earlier argument backward to obtain that $\mathrm{Sub}(F) \models p \leq q$. Then we can tailor Maltsev's idea to the situation to get (1). $\square$

By the Key Lemma, the following two are equivalent

1. *For every (unital left) R-module M, $p \le q$ holds in* $\mathrm{Sub}(M)$.
2. *The primal problem* $P = (G_p, G_q, \leftrightarrow, (R, +), 1)$ *is solvable.*

In short, $p \le q \iff P$ is solvable. By the Easy Fact, $G_{q^{\mathrm{du}}} \cong G_q^{\mathrm{du}}$ and $G_{p^{\mathrm{du}}} \cong G_p^{\mathrm{du}}$. Thus, applying the Key Lemma to $q^{\mathrm{du}} \le p^{\mathrm{du}}$, $q^{\mathrm{du}} \le p^{\mathrm{du}} \iff P^{\mathrm{du}}$ is solvable. However, by the Main Theorem, P is solvable $\iff P^{\mathrm{du}}$ is solvable. The three red facts in this slide imply that $p \le q \iff q^{\mathrm{du}} \le p^{\mathrm{du}}$, that is, $p \le q$ in $\mathrm{Sub}(M)$ for all $M \in R$-**Mod** if and only if $q^{\mathrm{du}} \le p^{\mathrm{du}}$ in $\mathrm{Sub}(M)$ for all $M \in R$-**Mod**. **So, by avoiding category theory, we have completed the simple proof** of Hutchinson's self-duality theorem.

The use of category theory in the study of submodule lattices remains useful and probably inevitable in another joint paper: G. Czédli and G. Hutchinson, Submodule lattice quasivarieties and **exact embedding functors** for rings with prime power characteristic; Algebra Universalis, 35 (**1996**), 425–445.

Finally, let me repeat that these slides are available at

https://tinyurl.com/czg-ns2024 ; the hand-out version (without pauses) can be found at http://www.math.u-szeged.hu/~czedli/ .

The **source paper** is here: https://arxiv.org/abs/2406.15989

THANK YOU FOR YOUR ATTENTION!