

From congruence lattices to circles  
and an online game  
(Dedicated to Professor Branimir Šešelja)

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These slides: <http://www.math.u-szeged.hu/~czedli/>

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How did I start to deal with  $\text{Con}(A)$  (for a single  $A$ )?  
or: What did I learn in Novi Sad ten years ago?

0' / 50

Novi Sad, 2007, invited by Branimir Šešelja and Andreja Tepavčević

Definition (Sešelja and Vojvodić, 1988)

Weak congruence: a “congruence without reflexivity”; i.e. a compatible, symmetric, transitive relation on  $A = \langle A, F \rangle$ . In other words: a congruence of a subalgebra or  $\emptyset$ . The lattice they form:  $\text{Cw}(A) = \langle \text{Cw}(A); \subseteq \rangle$ .

Šešelja and Tepavčević: Weak congruences in universal algebra, Institute of Mathematics, Novi Sad, 2001. 150 pp.

Remark

For a group  $G$ ,  $\text{Cw}(G) \cong \mathcal{Nw}(G) := \{ \langle X, S \rangle : X \triangleleft S \leq G \}$

- **Dedekind** group  $\stackrel{\text{def}}{\iff} \forall$  subgroups are normal. (Hamiltonian group  $\stackrel{\text{def}}{\iff}$  Dedekind & non-abelian.)
- If  $\forall x, y \in a$  lattice,  $d \vee x = d \vee y \Rightarrow d \vee x = d \vee (x \wedge y)$ , then  $d$  is a **join-semidistributive element**.
- $\Delta := \{\langle x, x \rangle : x \in A\} \in Cw(A)$ ,  $\Delta := \langle \{1\}, G \rangle \in \mathcal{N}w(G)$ .
- Czédli, Šešelja, Tepavčević: On the semidistributivity of elements in weak congruence lattices of algebras and groups, Algebra Universalis 58 (2008), 349–355 ; [it includes:](#)

### Theorem (Czédli, Šešelja, Tepavčević; AU2008)

For every finite group  $G$ , the following are equivalent

- 1  $G$  is a Dedekind group;
- 2  $\Delta$  is a join-semidistributive element in  $Cw(G)$  ;
- 3  $\Delta$  is a join-semidistributive element in  $\mathcal{N}w(G)$ .

Theorem (Funayama & Nakayama 1942, Dilworth, Wehrung 2007)

For every lattice  $L$ ,  $\text{Con}(L)$  is distributive. Conversely:

- if  $D$  is a finite distributive lattice, then  $\exists L$  s.t.  $D \cong \text{Con}(L)$ .
- F. Wehrung (2007): in general (without finiteness) the converse fails!

Find **nice**  $L$ ! G. Grätzer: *The congruences of a finite lattice. A “proof-by-picture” approach. 2nd ed., 2016.* E.g.:

Definition (G. Grätzer and E. Knapp 2009)

A lattice  $L$  is **rectangular** if the following conditions hold:

- $L$  is planar (and so finite by definition) and  $|L| > 1$ ,
- $L$  is semimodular, i.e.,  $\forall x, y, z$ , if  $x \preceq y$  then  $x \vee z \preceq y \vee z$ ,
- $\exists u, v$  doubly irreducible s.t.  $u \vee v = 1$  and  $u \wedge v = 0$ .

Theorem (G. Grätzer and E. Knapp 2009)

For every finite distributive  $D$  with  $|D| > 1$ ,  $\exists L$  **rectangular** s.t.  
 $D \cong \text{Con}(L)$ .

This is not a mathematical theorem!

For this purpose, there are no nicer lattices than rectangular.

Sketch of "proof".

- $L$  modular  $\Rightarrow$   $\text{Con}(L)$  is boolean.
- $L$  is nonplanar  $\Rightarrow$  diagrams are difficult to read.
- $L$  is only planar and semimodular (G. Grätzer, H. Lakser and E. T. Schmidt, 1998): ragged diagram.

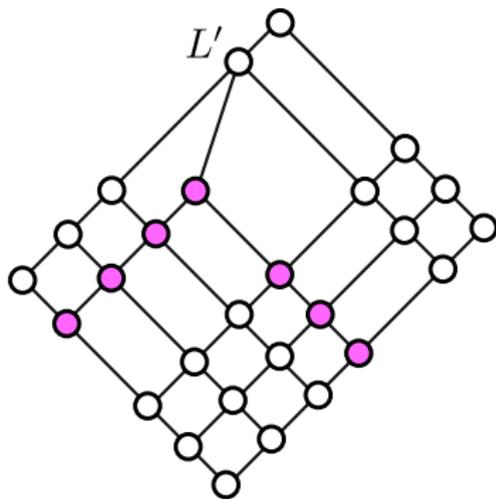
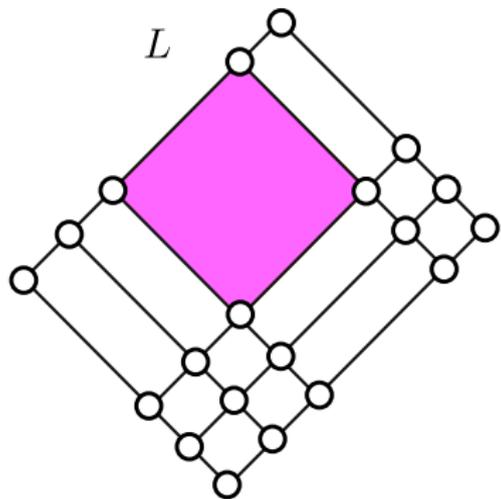


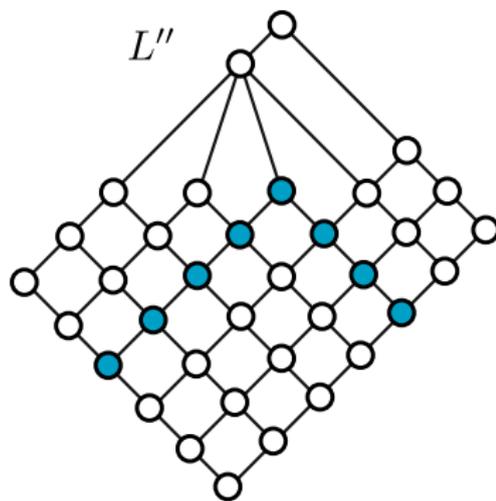
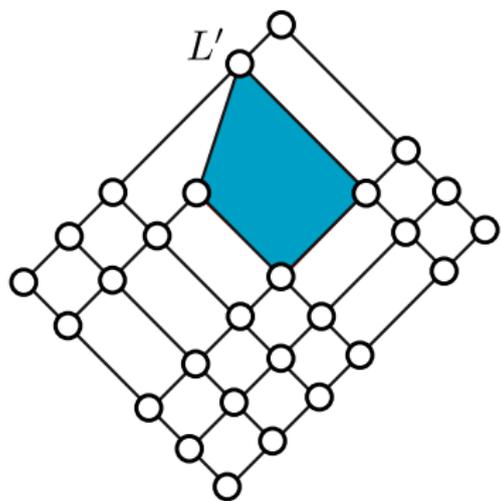
Definition (G. Grätzer and E. Knapp 2007)

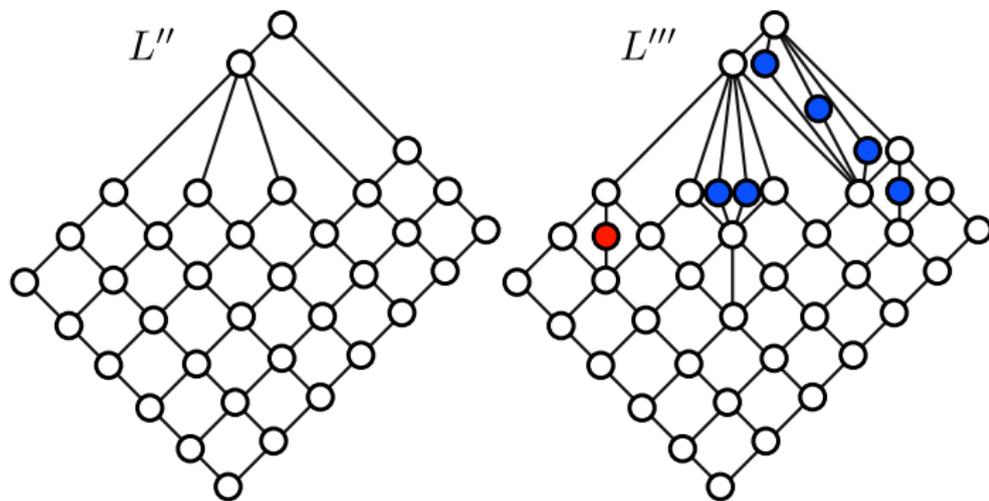
A planar semimodular lattice  $L$  is **slim** if it contains no  $M_3$   $\iff$  it contains no cover-preserving  $M_3$ .

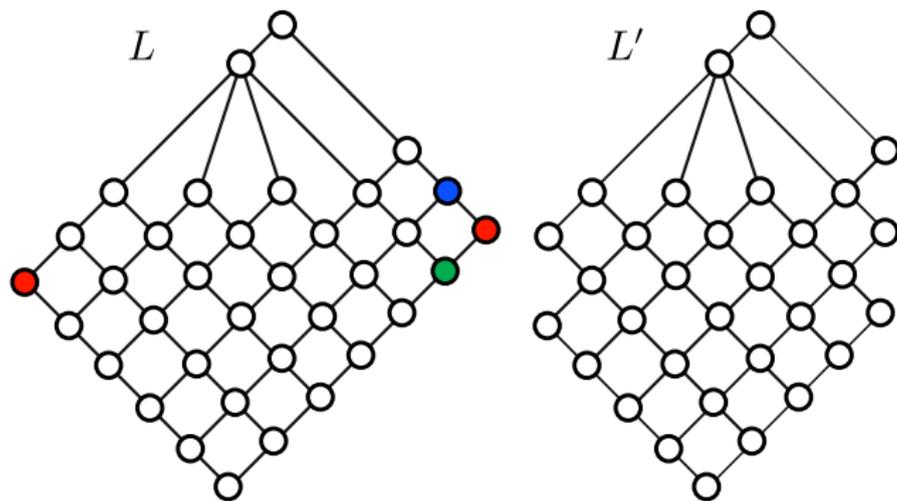
**Grid**: direct product of two finite chains (with the usual diagram).

Before formulating a structure theorem, we need to define three constructing steps.









**Theorem (Czédli and E.T. Schmidt, 2012)**

*A lattice  $L$  is rectangular iff it can be obtained from a grid by*

- 1 *inserting forks (finitely many, one by one) first, and*
- 2 *inserting eyes (finitely many, one by one)*

Note: (1) and (2) do not commute (it would not make sense) !

**Theorem (Czédli and E.T. Schmidt, 2012)**

*Planar and semimodular  $\iff$  obtained from a grid by*

- 1 *inserting forks (finitely many, one by one),*
- 2 *deleting corners (finitely many, one by one), and*
- 3 *inserting eyes (finitely many, one by one)*

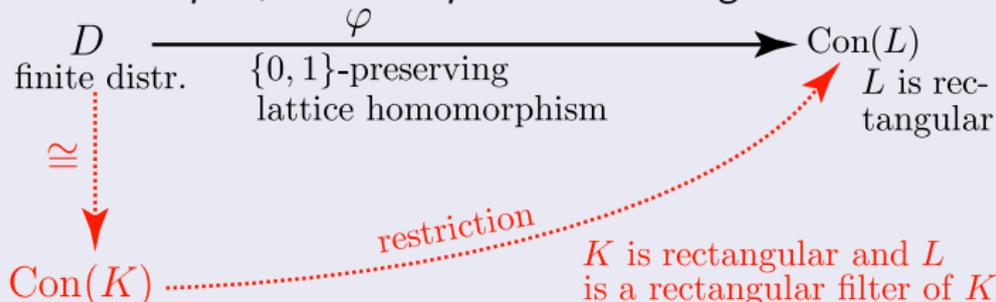
*in this order.*

Note: we cannot change the order of (1), (2), and (3) in general.

Grätzer–Knapp 2009:  $(\forall D \text{ with } |D| > 1) (\exists K \text{ rect}) (D \cong \text{Con}(K))$ .

Theorem (Czédli, AU Algebra 67, 313–345, 2012)

$\forall$  BLACK part,  $\exists$  a RED part s.t. the diagram commutes:

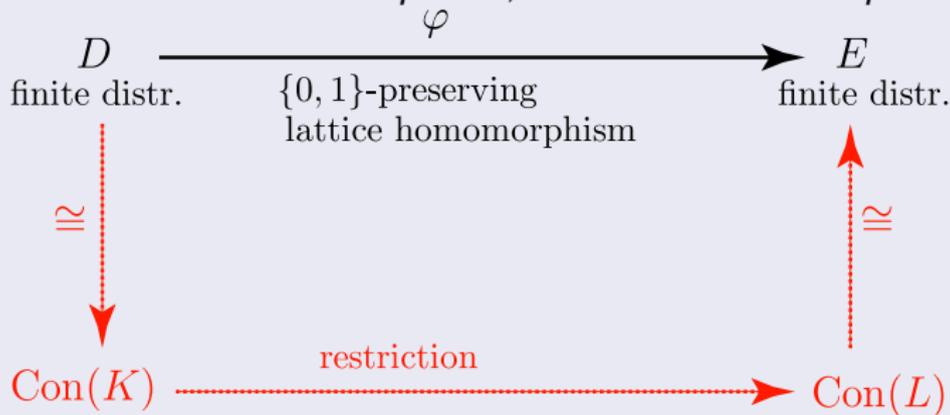


The 2012 Thm. implies the 2009 Thm. as follows.

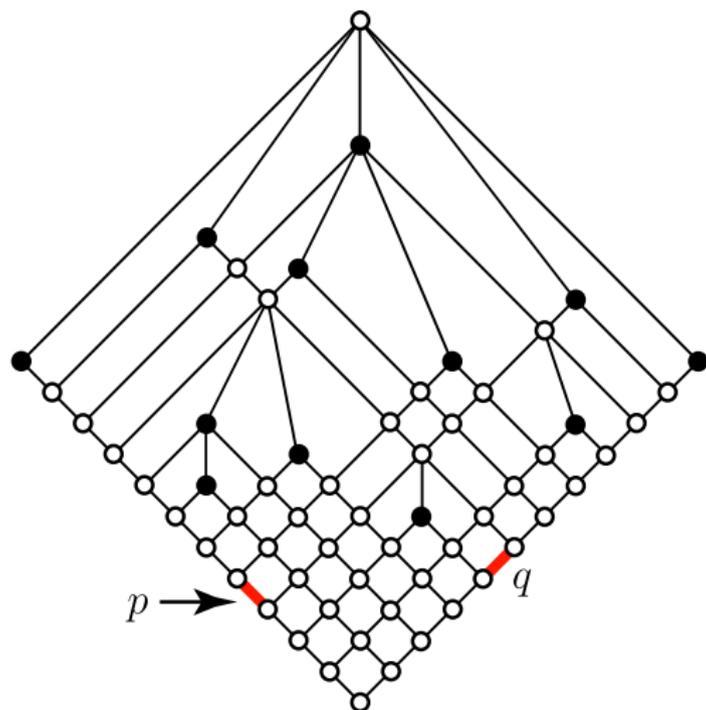
$|\text{Con}(M_3)| = 2$ . The Prime Ideal Theorem yields a  $\varphi: D \rightarrow \text{Con}(M_3)$ .  $M_3$  is rectangular. The 2012 Thm. gives a  $K$ . Thus, the 2009 Thm. follows.  $\square$

Corollary (Czédli, AU Algebra 67, 313–345, 2012)

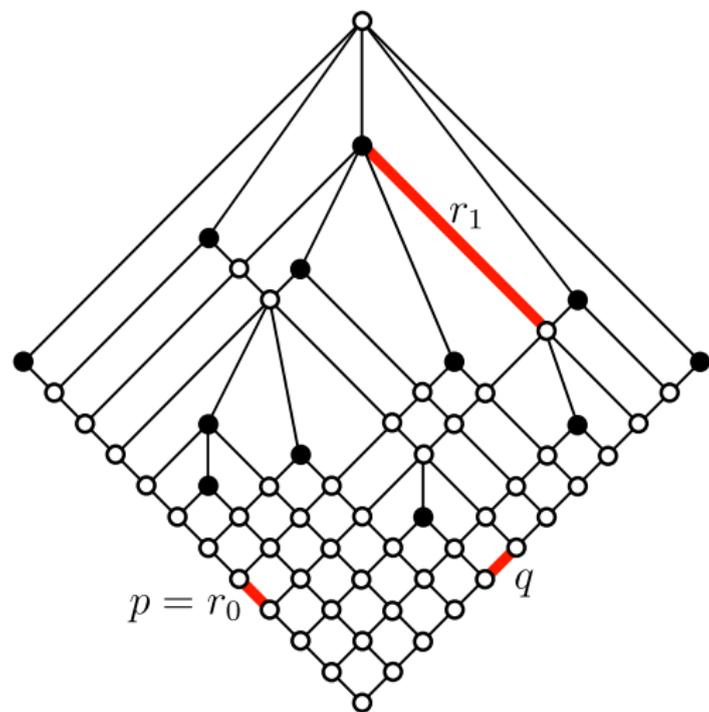
Whatever the **BLACK** part is, there exists a **RED** part:

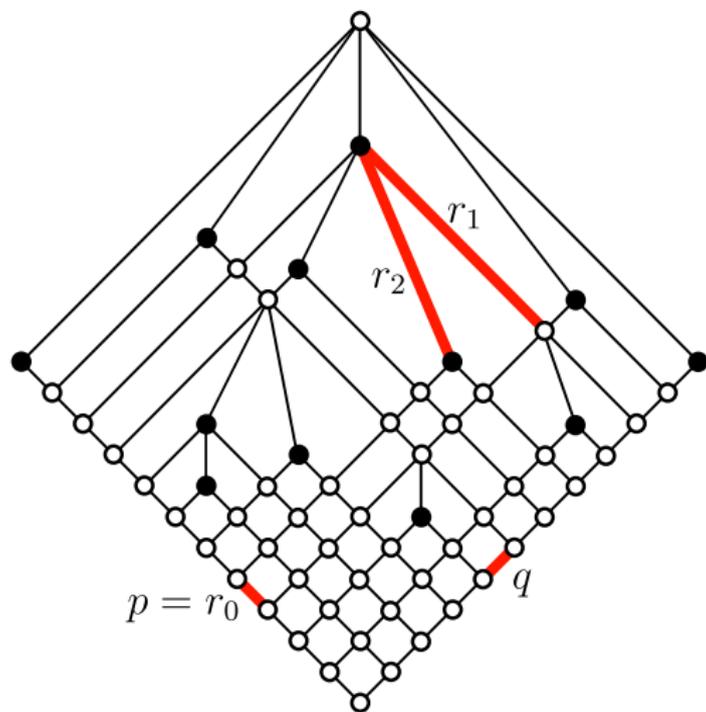


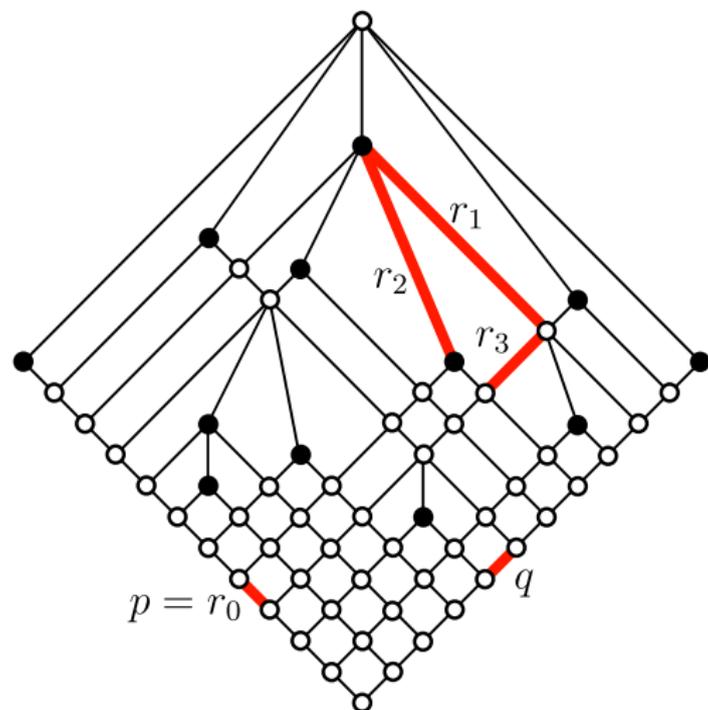
$K$  is rectangular and  $L$  is a rectangular filter of  $K$  such that the diagram commutes.

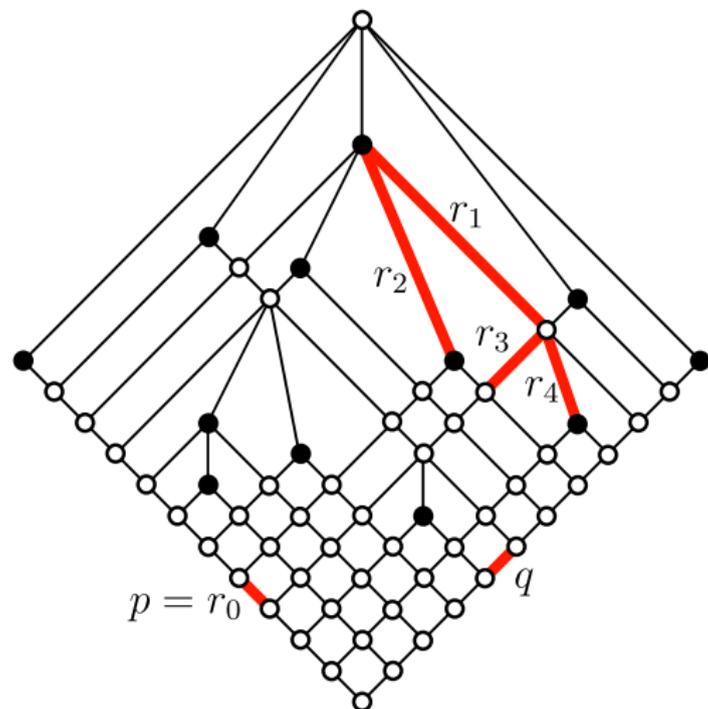


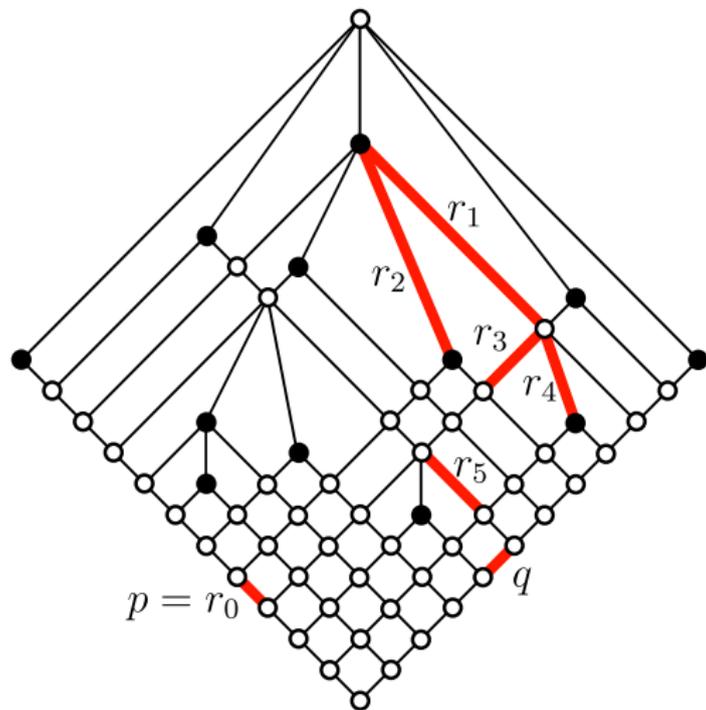
**Swing Lemma** (Grätzer, 2015): Let  $L$  be slim semimodular.  
 $\text{con}(p) \geq \text{con}(q)$  iff  $\exists$  a sequence of edges as shown in the figures.

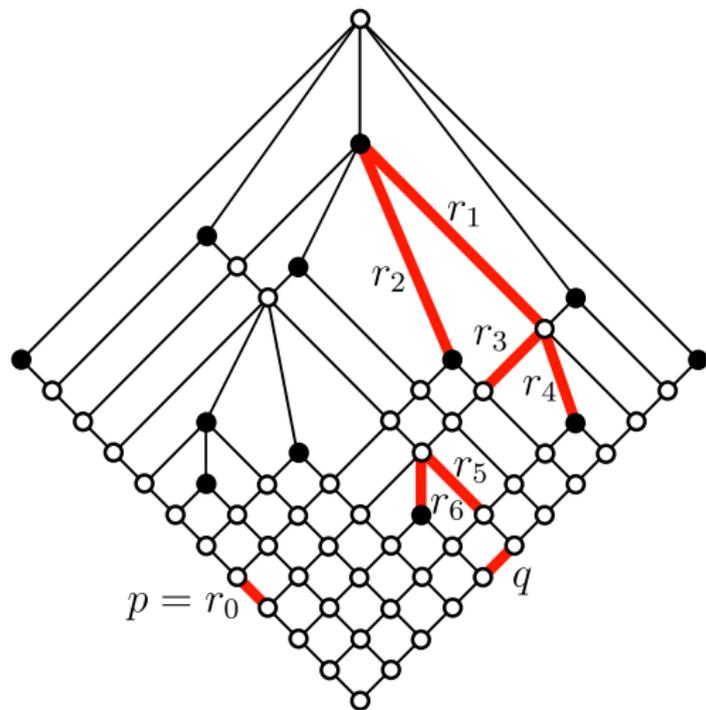


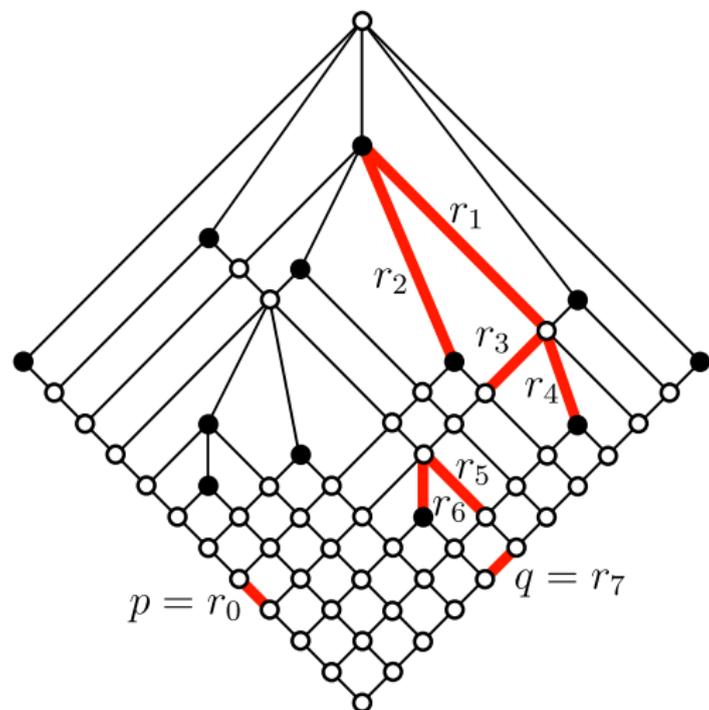








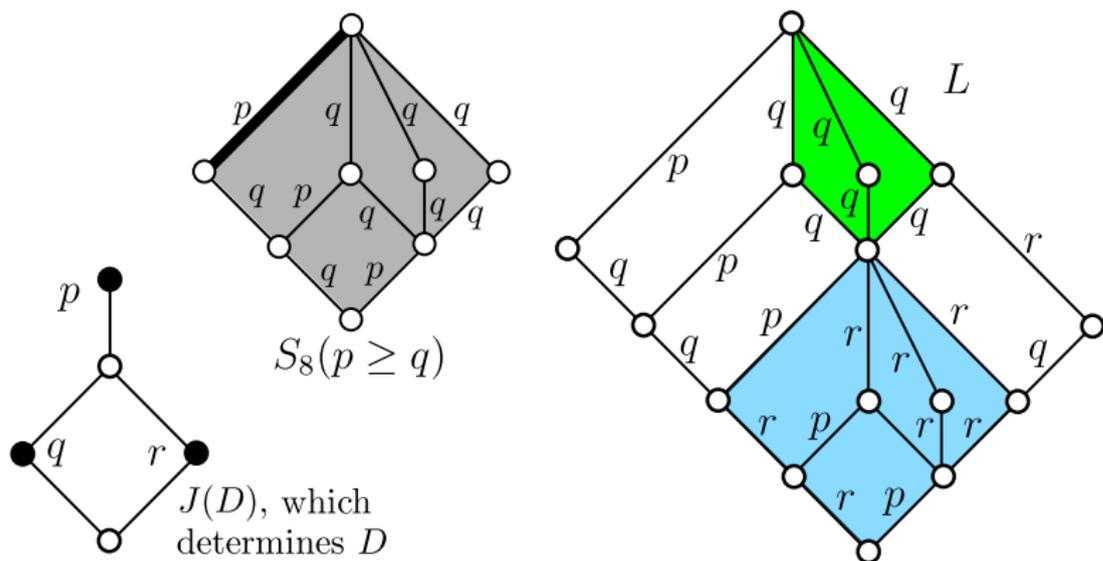




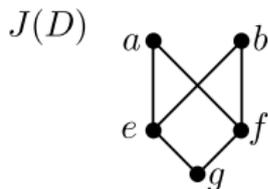
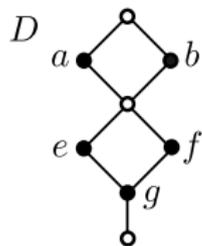
## Swing Lattice Game (Czédli and Géza Makay, 2016)

See <http://www.math.u-szeged.hu/~makay/swinglattice/>

or my web-site; instructions are given there. In addition of jumping and swinging, tilting (the dual of swinging) is also allowed.  $L$  is almost slim and semimodular; only one eye is added and its position is controlled by the player.



Representing  $D$  with a rectangular  $L$  such that  $D \cong \text{Con}(L)$ .

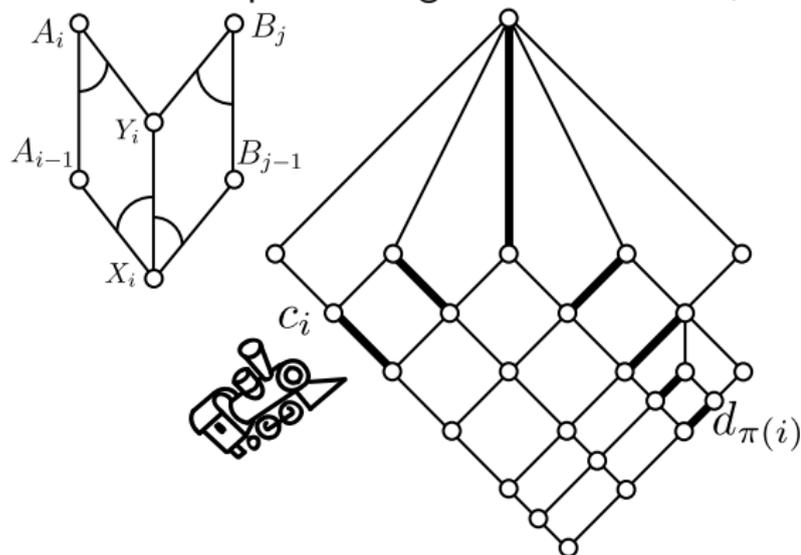


Theorem (Czédli, AU 72 225–230, 2014)

This  $D$  **cannot** be represented as  $\text{Con}(L)$  with a slim planar semimodular  $L$ .

That is, eyes are really needed in general.

$\exists$  down-and-up matching: Grätzer–Nation, AU 64 309-311, 2011.



**Theorem (Czédli and Schmidt, AU 66, 69-79 (2011))**

*Between any two composition series of a finite group, there is exactly one down-and-up matching.*

Remember: **slim rectangular** = rectangular without  $M_3$  = what we can obtain from a grid by inserting forks.

Theorem (Czédli, Dékány, Gyenizse, and Kulin, AU 75 33-50, 2016)

*The number of slim rectangular lattices of length  $n$  is asymptotically  $(n - 2)! \cdot e^2/2$ , where*

$e = \lim((1 + 1/n)^n) \approx 2.718\ 281\ 828\ 459\ 045\ 235\ 360\ 287\ 471\ 35 \dots$

Similar results for “slim semimodular” instead of “slim rectangular”:

- Czédli, Ozsvárt, Udvari (Discr. Math. **312** 3523–3536, 2012),
- Czédli, Dékány, Ozsvárt, Szakács, Udvari (Math. Slovaca **66** 2016, 5–18),
- Czédli (Order **33** 2016, 231–237),

without  $e$ . (Yet another aspect of rectangular  $\Rightarrow$  nice.)

Definition (Edelman 1980; Adaricheva, Gorbunov, Tumanov 2003)

A pair  $\langle E, \Phi \rangle$  is a *convex geometry*, also called *anti-exchange system*, if  $\Phi: P(E) \rightarrow P(E)$  is a closure operator,  $\Phi(\emptyset) = \emptyset$ , and

- If  $\Phi(A) = A \in P(E)$ ,  $x, y \in E$ ,  $x \notin A$ ,  $y \notin A$ , and we have  $\Phi(A \cup \{x\}) = \Phi(A \cup \{y\})$ , then  $x = y$ . (Anti-exchange property;  $x$  cannot be exchanged to  $y$ .)

Theorem (Czédli, Discrete Mathematics 330, 61-75, 2014)

*Each slim semimodular  $L \cong_{\text{dual}} \text{Sub}(a \text{ conv. geom. of circles})$ .*

Are circles sufficient to represent all convex geometries in this way?

Thm (Adaricheva–Bolot 2016, <https://arxiv.org/abs/1609.00092>)

*No, because circles satisfy property (3) of the next slide!*

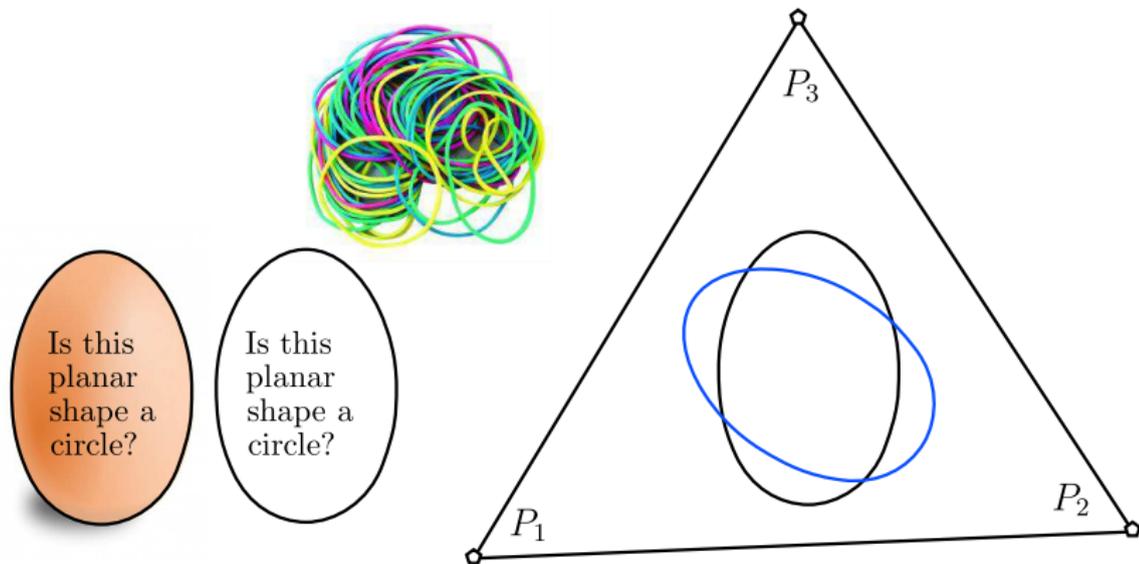
Theorem (K. Adaricheva and M. Bolat, 2016)

*Circles in the plane satisfy the convex combinatorial property described below in (3) (even if  $A_0, A_1, A_2$  are circles.)*

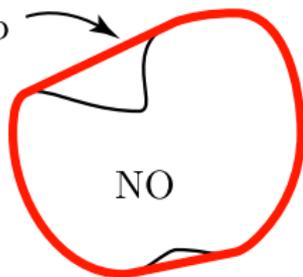
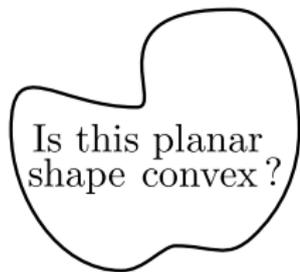
Theorem (Czédli 2016, <http://arxiv.org/abs/1611.09331> )

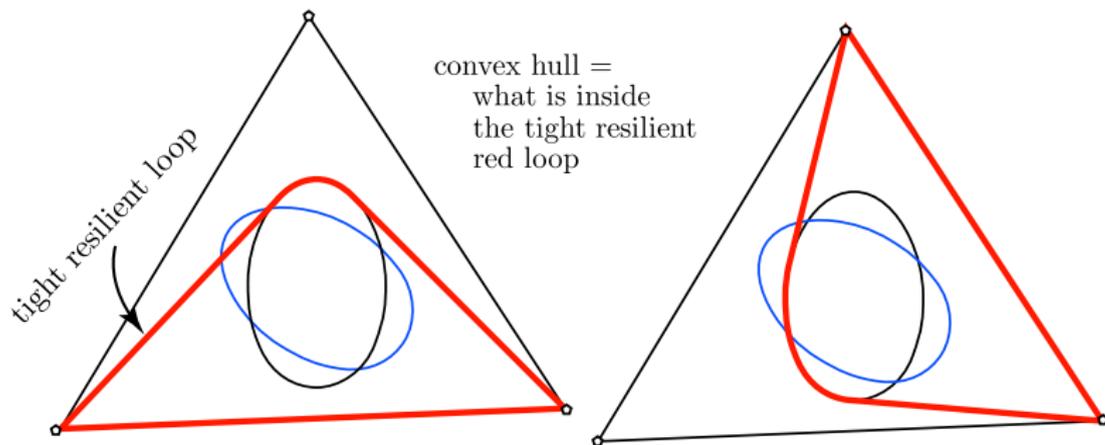
*If  $K_0$  is a compact convex subset of the plane  $\mathbb{R}^2$ , then tfae:*

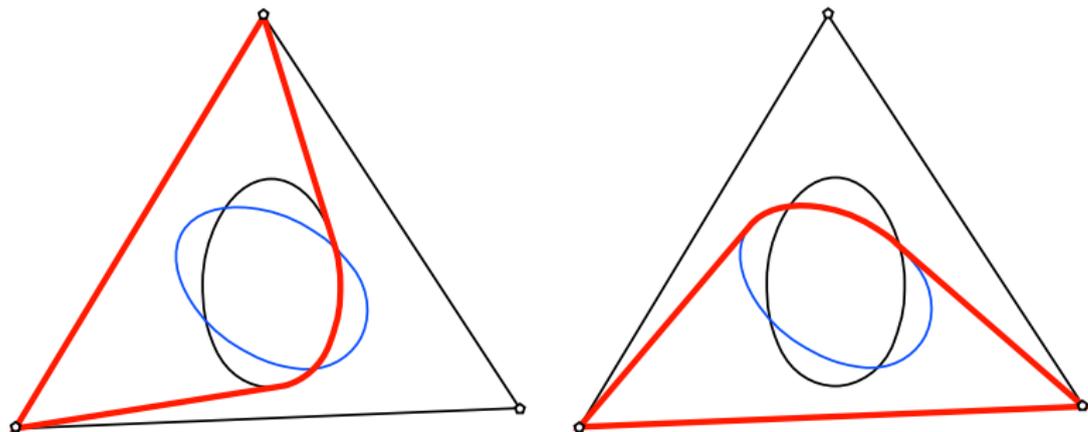
- 1  $K_0$  is a disk.
- 2 For every  $K_1 \subseteq \mathbb{R}^2$  and for arbitrary points  $A_0, A_1, A_2 \in \mathbb{R}^2$ , if  $K_1$  is **isometric** to  $K_0$  and both  $K_0$  and  $K_1$  are subsets of the triangle  $\text{Conv}(\{A_0, A_1, A_2\})$ , then there exist a  $j \in \{0, 1, 2\}$  and an  $i \in \{0, 1\}$  such that  $K_{1-i}$  is a subset of  $\text{Conv}(K_i \cup (\{A_0, A_1, A_2\} \setminus \{A_j\}))$ .
- 3 The same as the second condition but “isometric” is replaced by “similar”. (This is the **Adaricheva–Bolat property**.)

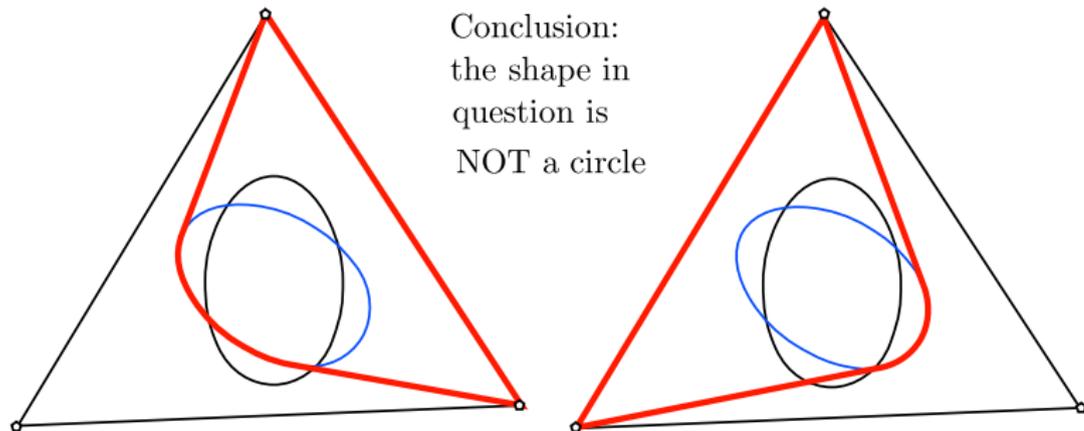


convex hull = what is inside  
the tight resilient red loop









Conclusion:  
the shape in  
question is  
NOT a circle

A complete lattice is *strongly atomic* if each of its non-singleton intervals contains an atom (with respect to the interval).

Definition ( A. P. Huhn's  $n$ -distributivity)

$$x \wedge \bigvee_{0 \leq i \leq n} y_i = \bigvee_{0 \leq j \leq n} \left( x \wedge \bigvee_{0 \leq i \leq n, i \neq j} y_i \right)$$

(Note: distributivity = 1-distributivity.)

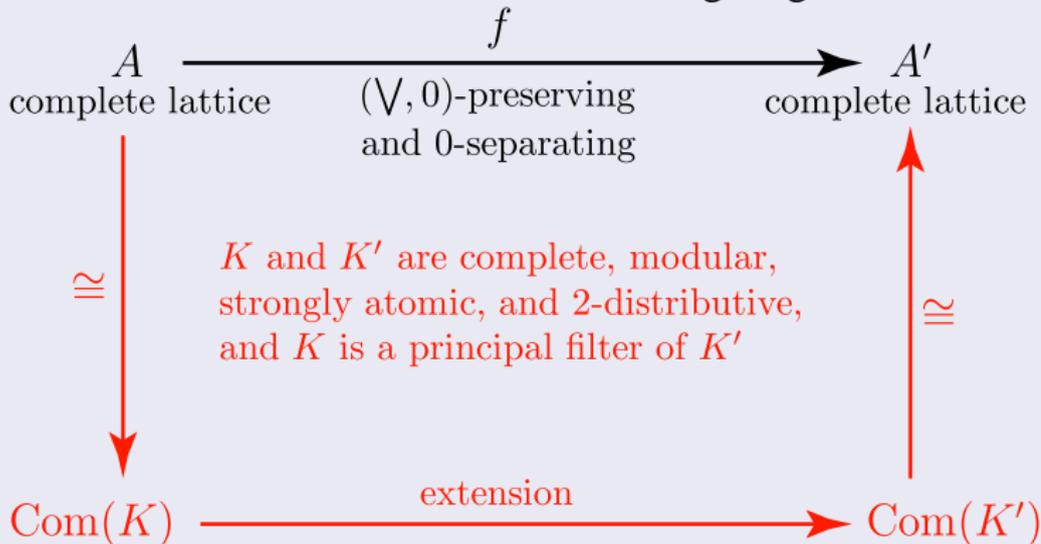
Theorem (R. Freese, G. Grätzer, E.T. Schmidt 1991; Grätzer and E.T. Schmidt 2001)

*Every complete lattice  $A$  is isomorphic to  $\text{Com}(K)$  for a suitable strongly atomic, 3-distributive, complete modular lattice  $K$  (1991). The same holds with "2-distributive" (2001).*

$3 \searrow 2$ : ten years.  $2 \searrow 1$ ?: we know that this is impossible

## Main Theorem (1st part) (Czédli, AU, to appear)

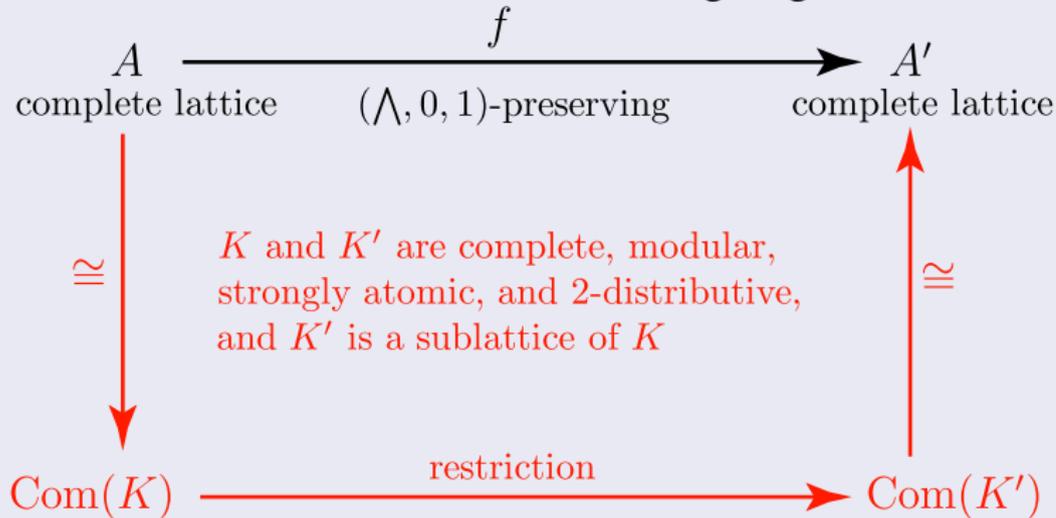
$\forall$  BLACK,  $\exists$  RED such that the following diagram commutes:



Of course, we could say “ideal” instead of “filter”

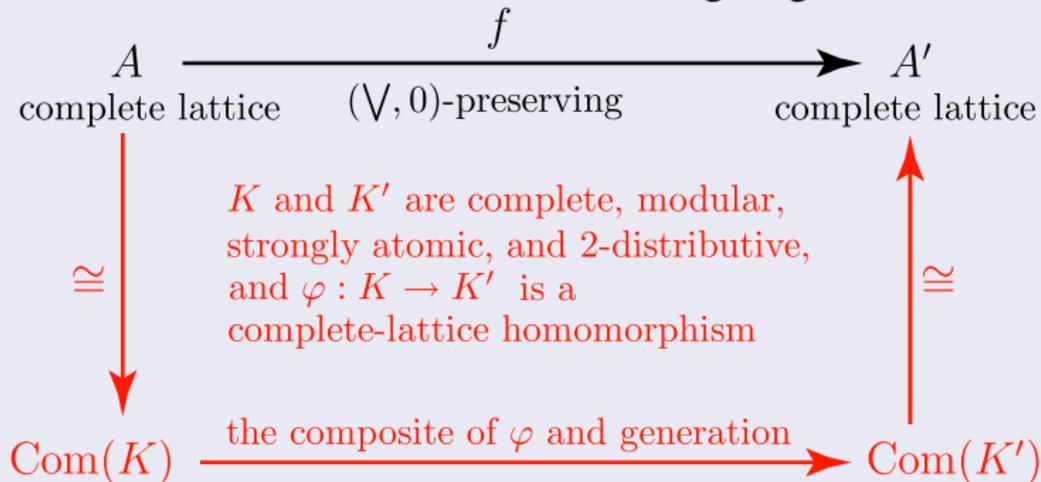
## Main Theorem (2nd part) (Czédli, AU, to appear)

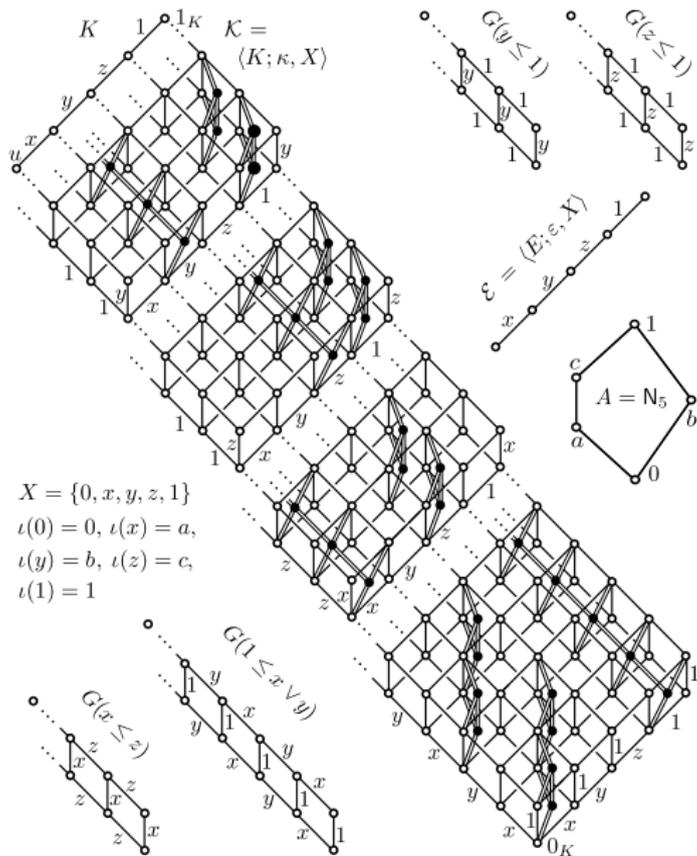
$\forall$  BLACK,  $\exists$  RED such that the following diagram commutes:



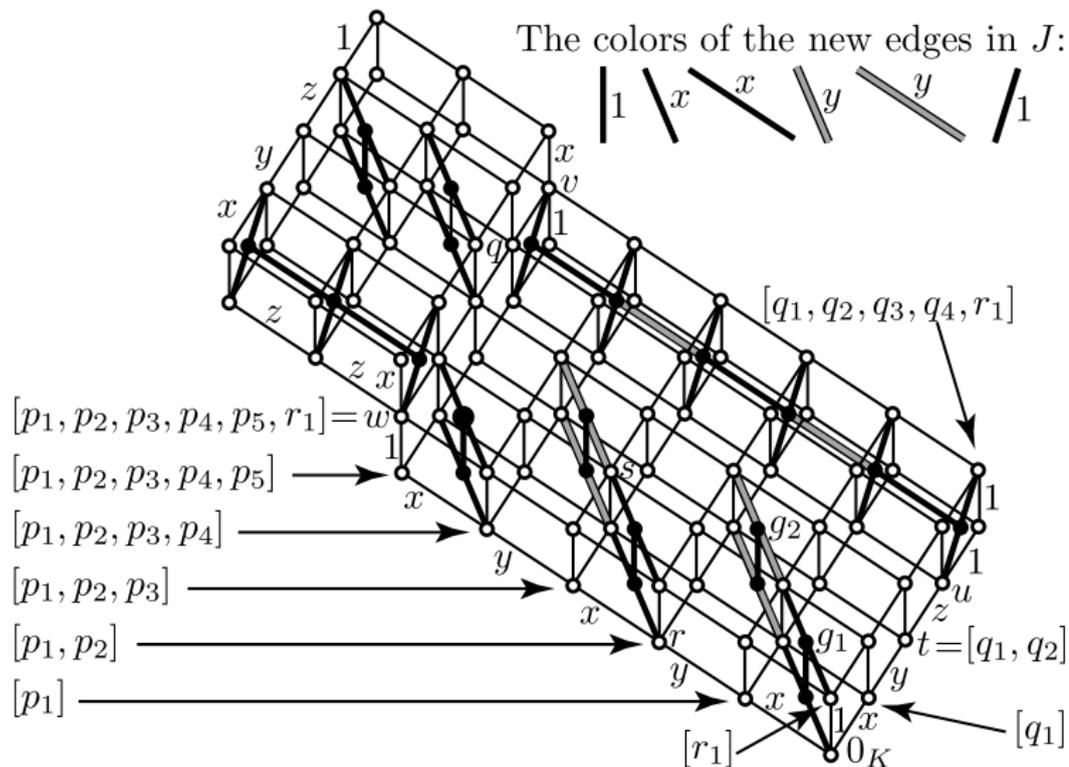
## Main Theorem (3rd part) (Czédli, AU, to appear)

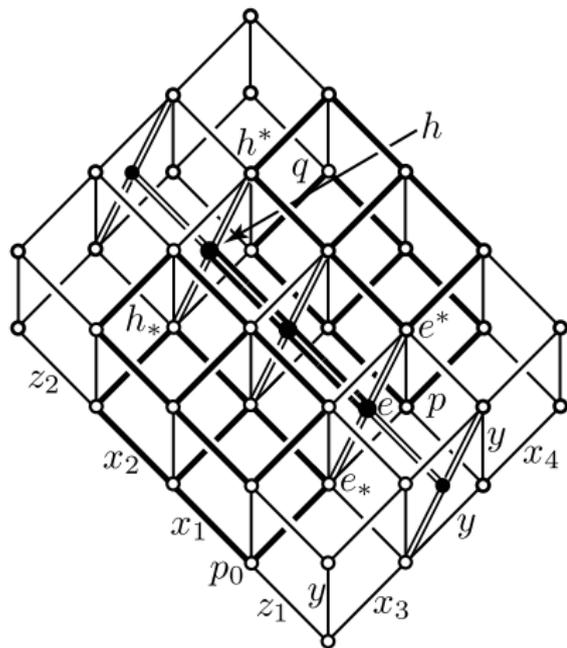
$\forall$  BLACK,  $\exists$  RED such that the following diagram commutes:





$X = \{0, x, y, z, 1\}$   
 $\iota(0) = 0, \iota(x) = a,$   
 $\iota(y) = b, \iota(z) = c,$   
 $\iota(1) = 1$





**Theorem (Grätzer, 2013)**

*For every poset  $Q$  with 0 and 1, there is a lattice  $L$  such that  $\text{Princ}(L)$ , the poset of principal congruences of  $L$  with respect to  $\subseteq$ , is isomorphic to  $Q$ .*

The  $1 \notin Q$  case is much harder and it is still unsolved. The problem is solved only up to  $\aleph_0$ . Namely:

**Theorem (Czédli, AU 75 (2016), 351–380)**

*Let  $Q$  be a poset with  $|Q| \leq \aleph_0$ . Then  $Q$  is representable as  $\text{Princ}(L)$  for some lattice  $L$  iff  $0 \in Q$  and  $Q$  is directed.*

Every map, say,  $f: A \rightarrow B$ , can be decomposed as a composite of a surjective map and an injective one. Namely,  $f = f_2 \circ f_1$ , where  $f_1: A \rightarrow A/\text{Ker}(f_1)$  is surjective, and  $f_2$  is injective. But can we do this **naturally** for a whole family of maps? In general, **no, we cannot**. But sometimes we can do something similar.

A category is **concrete** if its objects are sets, possibly with structures on them, its morphisms are maps, and the composition in the category is that of maps in the usual sense. The category of sets with all maps is denoted by **Set**.

For a concrete category **A**, the **forgetful functor** from **A** to **Set** will be denoted by  $\iota_{\mathbf{A}, \mathbf{Set}}$ .

Theorem (See my home-page: "Cometic functors . . .", 2016)

For every small concrete category  $\mathbf{A}$ , there exists a so-called **cometic functor**  $F_{\text{com}}: \mathbf{A} \rightarrow \mathbf{Set}$  and a natural transformation  $\pi^{\text{com}}: F_{\text{com}} \rightarrow \iota_{\mathbf{A}, \mathbf{Set}}$  such that the components of  $\pi^{\text{com}}$  are surjective maps and the  $F_{\text{com}}$ -image of every **monomorphism** of  $\mathbf{A}$  is an injective map.

$$\begin{array}{ccc}
 X & \xrightarrow{g} & Y \\
 \\
 F_{\text{com}}(X) & \xrightarrow{F_{\text{com}}(g)} & F_{\text{com}}(Y) \\
 \downarrow \pi_X^{\text{com}} & & \downarrow \pi_Y^{\text{com}} \\
 \iota_{\mathbf{A}, \text{Set}}(X) & \xrightarrow{\iota_{\mathbf{A}, \text{Set}}(g)} & \iota_{\mathbf{A}, \text{Set}}(Y)
 \end{array}$$


Here  $X, Y$  are objects and  $g$  is a morphism of  $\mathbf{A}$ . Their  $F_{\text{com}}$ -images are sets and a map, respectively. While there can be non-injective monomorphisms in  $\mathbf{A}$ , this cannot happen in  $F_{\text{com}}(\mathbf{A})$ . Thus,  $F_{\text{com}}(\mathbf{A})$  is *nice* in the sense that injective  $\Leftrightarrow$  mono. Furthermore, the vertical arrows are onto maps.

These slides are available

<http://www.math.u-szeged.hu/~czedli/>

<http://www.math.u-szeged.hu/~czedli/conference-talks/conflectures0.html>