Cometic functors for principal lattice congruences

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Dedicated to the 80th birthday of professor **Tibor Katriňák**

Gábor Czédli (University of Szeged)

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Slides: `http://www.math.u-szeged.hu/~czedli/`

August 11, 2017

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Which posets can be represented as $\mathrm{Princ}(A)$ *or* $\mathrm{Princ}(L)$*?*

Theorem (G. Grätzer: AU 70 (2013), 95–105)

Every bounded poset is $\cong \mathrm{Princ}(L)$ *such that* $\mathrm{length}(L) \leq 5$.

Theorem (G. Czédli: Acta Sci. Math. (Szeged) 82 (2016), 3–18)

For $\forall$ *bounded non-singleton poset* P *and any group* G *can* ***simultaneously*** *be represented as* $P \cong \mathrm{Princ}(L)$ *and* $G \cong \mathrm{Aut}(L)$*, where* L *is a selfdual lattice.*

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The non-bounded case is characterized only up to $\aleph_0$:

Theorem (G. Czédli: AU 75 (2016), 351–380)

Let P be a poset with $|P| \leq \aleph_0$. Then there is a lattice L with $P \cong \mathrm{Princ}(L)$ if and only if $0 \in P$ and P is directed.

$\mathrm{Princ}(A)$ is not directed in general. We know almost nothing on $\mathrm{Princ}(A)$-representability. The example below indicates the difficulty.

Example (Corollary 2.1 in Czédli arXiv:1705.10833v3)

The 3-element "V-shaped" poset is not representable as $\mathrm{Princ}(A)$.

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Definition (the action of Princ for a homomorphism)

If $f\colon A \to B$ is a homomorphism, then
$\mathrm{Princ}(f)\colon \mathrm{Princ}(A) \to \mathrm{Princ}(B)$ is defined by
$\mathrm{con}_A(x, y) \mapsto \mathrm{con}_B(f(x), f(y))$;
it is a well-defined, monotone, 0-preserving map.

Definition (representing a map by principal lattice congruences)

For a given 0-preserving monotone map $\varphi\colon P_1 \to P_2$, the task is to find lattices L_1 and L_2, poset isomorphisms τ_1 and τ_2, and a lattice homomorphism f such that the following diagram commutes:

$$\begin{array}{ccc} P_1 & \xrightarrow{\ \varphi\ } & P_2 \\ {\scriptstyle\tau_1}\big\downarrow & & \big\uparrow{\scriptstyle\tau_2^{-1}} \\ \mathrm{Princ}(L_1) & \xrightarrow{\ \mathrm{Princ}(f)\ } & \mathrm{Princ}(L_2) \end{array}$$

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Representing 0-separating maps

Theorem (Czédli, 2015)

If P_1 and P_2 are bounded posets and $\varphi\colon P_1 \to P_2$ is a $\{0,1\}$-preserving and 0-separating monotone map, then φ is representable by principal lattice congruences.

As mentioned in SSAOS, Stara lesna, 2014,

Theorem roughly saying (Czédli: AU 77 (2017), 51–77)

Every "poset-indexed diagram" of $\{0,1\}$-preserving and 0-separating monotone maps among bounded posets can be represented in simultaneous way similarly'.

To give the precise meaning of this theorem, we need categories and the concept of *lifting a functor*, due to Gillibert and Wehrung [From objects to diagrams for ranges of functors, Springer, 2011].

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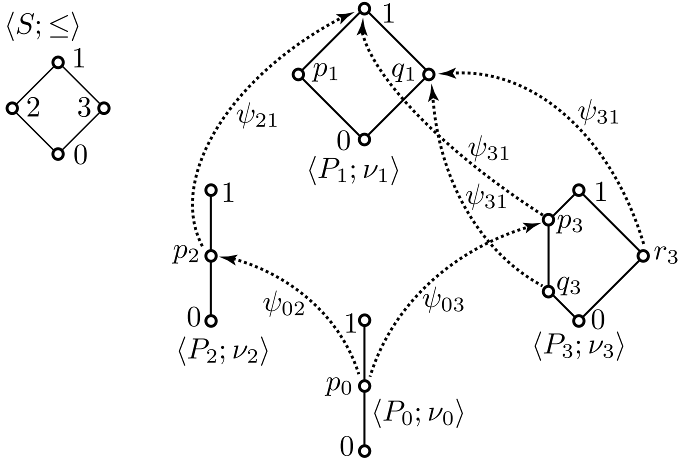

All the ψ_{ij}, $i \leq j$, are $\{0,1\}$-preserving and, say, $\psi_{21} \circ \psi_{02} = \psi_{01}$.

The index poset S in the previous slide can be considered a *small category*. (Small: the objects form a set.) This motivates:

Definition (Diagram of $\{0,1\}$-preserving monotone maps)

By an **A**-indexed *diagram* (of $\{0,1\}$-preserving monotone maps) we shall mean a functor F from a small category **A** to the category $\mathbf{Posets}^+_{01}$ of nonsingleton bounded posets with $\{0,1\}$-preserving monotone maps as morphisms.

The aim is to represent **A**-indexed diagrams F by *lifting*.
We always assume that F is *faithful*, i.e., it is injective on every hom-set.

$\mathbf{Lattices}_5^{\mathrm{sd}}$ will denote the category of selfdual lattices of length 5 with $\{0,1\}$-preserving lattice homomorphisms. Our purpose is *to lift* $\mathbf{A}$-indexed diagrams (that is, a functor $F\colon \mathbf{A} \to \mathbf{Posets}_{01}^{+}$) with respect to Princ in the following sense

Definition (Gillibert and Wehrung, particular case)

Let $F\colon \mathbf{A} \to \mathbf{Posets}_{01}^{+}$ be a functor. We say that a functor $E\colon \mathbf{A} \to \mathbf{Lattices}_5^{\mathrm{sd}}$ ***lifts*** the functor F with respect to the functor Princ if F is naturally equivalent to the composite functor $\mathrm{Princ} \circ E$. If there is such an E, then we say that ***F can be lifted with respect to Princ***.

In a plane language, we look for a functor $E\colon \mathbf{A} \to \mathbf{Lattices}_5^{\mathrm{sd}}$ and poset isomorphisms τ_i such that for each morphism $f\colon i \to j$ in the index category $\mathbf{A}$, the following diagram commutes:

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$$\begin{array}{ccc} \text{poset } F(i) & \xrightarrow{\text{monotone map } F(f)} & \text{poset } F(j) \\ \tau_i \big\downarrow & & \big\uparrow \tau_j^{-1} \\ (\text{Princ} \circ E)(i) = \text{Princ}(\text{lattice } E(i)) & \xrightarrow{\text{Princ(lat. homo. } E(f))} & (\text{Princ} \circ E)(i) = \text{Princ}(\text{lattice } E(j)) \end{array}$$

Wehrung's section in the monograph Lattice Theory Special Topics and Appl. I, Sect. 7-4.5 + anonymous referee lead to:

Remark (see Ex.6.4 in Czédli: AU 77 (2017) 51–77)

Not every $F\colon \mathbf{A} \to \mathbf{Posets}^{+}_{01}$ *can be lifted to* Princ.

Thus, we need some assumption on the index category **A** in the following theorem.

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$\tau_i \downarrow$ $\quad\quad$ $\tau_j^{-1} \uparrow$

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A morphism f in a category **A** is *mono* if $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$ whenever both products make sense. A category is *concrete* if its objects are sets (with possible structures on them) and its morphisms are (certain) maps.

Theorem 1 (Czédli, in "Cometic functors and representing order-preserving maps by principal lattice congruences")

Let **A** *be a small category such that every morphism in* **A** *is mono. Then every faithful functor (i.e.,* **A***-indexed diagram)* $F\colon \mathbf{A} \to \mathbf{Posets}^{+}_{01}$ *can be lifted to the functor* $\mathrm{Princ}\colon \mathbf{Lattices}^{\mathrm{sd}}_{5} \to \mathbf{Posets}^{+}_{01}$ *by means of a faithful functor* $E\colon \mathbf{A} \to \mathbf{Lattices}^{\mathrm{sd}}_{5}$.

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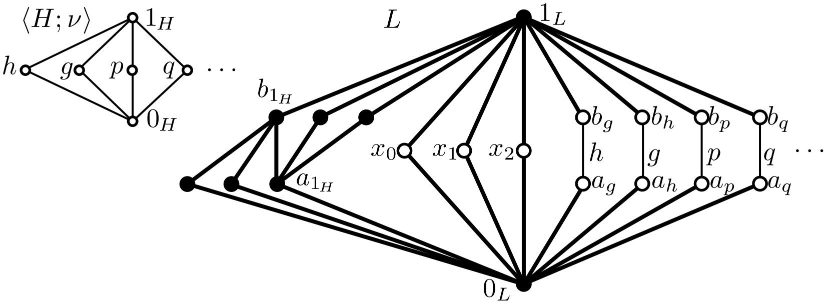
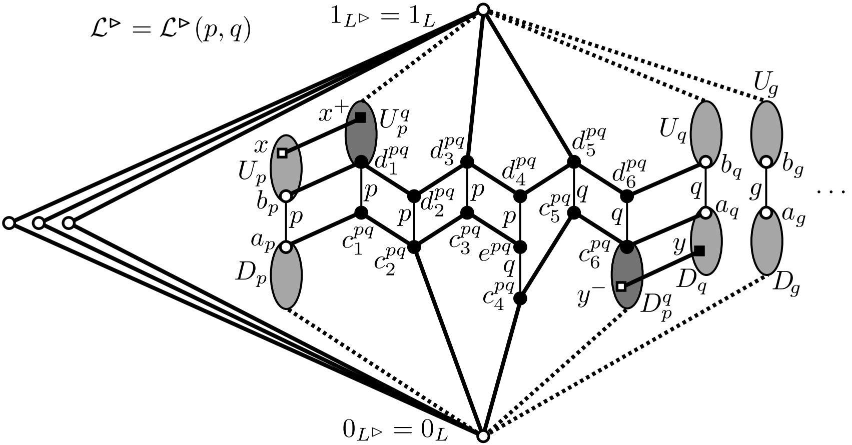
The Stara Lesna 2014 construction **does not work for non-injective maps**.





$\mathcal{L}^{\triangleright} = \mathcal{L}^{\triangleright}(p,q)$
$1_{L^{\triangleright}} = 1_L$
$0_{L^{\triangleright}} = 0_L$
x^+
U_p^q
x
U_p
b_p
a_p
D_p
d_1^{pq}
d_2^{pq}
d_3^{pq}
d_4^{pq}
d_5^{pq}
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c_1^{pq}
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p
q
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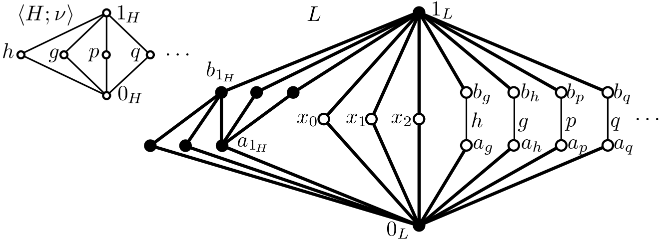
$\ldots$



Because: if a map $f\colon H \to$ somewhere collapses, say, $p, q \in H$, then it collapses a_p and a_q in the corresponding lattice. But then, since $\mathrm{con}(a_p, a_q)$ is the largest congruence, it collapses the hole lattice, which should not happen.

$\Rightarrow$ **we need a tool** that turns monomorphism into injective maps.

For a concrete category $\mathbf{A}$, $\iota_{\mathbf{A},\mathbf{Set}}$ denotes the forgetful functor.

Theorem 2 (Czédli: "Cometic functors ...")

For every small concrete category **A**, *there exists a so-called* ***cometic functor*** $F_{\text{com}}\colon \mathbf{A} \to \mathbf{Set}$ *and a natural transformation* $\boldsymbol{\pi}^{\text{com}}\colon F_{\text{com}} \to \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}$ *such that the components of* $\boldsymbol{\pi}^{\text{com}}$ *are surjective maps and the* F_{com}*-image of every* ***mono****morphisms of* **A** *is an injective map.*

Since $\iota_{\mathbf{A},\mathbf{Set}}$ does nothing, the meaning of the theorem is this:

$$\begin{array}{ccc} F_{\text{com}}(X) & \xrightarrow[\text{injective if } f \text{ is mono in } \mathbf{A}]{F_{\text{com}}(g),} & F_{\text{com}}(Y) \\ \text{surj. } \pi_X^{\text{com}} \downarrow & & \downarrow \text{surj. } \pi_Y^{\text{com}} \\ X & \xrightarrow{g} & Y \end{array}$$

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$F_{\text{com}}(Y) := \{$"mapsto arrows" $x \overset{f}{\mapsto} y \ : \ y \in Y\}$. Thus, $F_{\text{com}}(Y)$ has a comet-like picture. For another explanation of the name "cometic", see below:

$$X \xrightarrow{\ g\ } Y$$

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Where did the name "cometic" come from?

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