Cometic functors for principal lattice congruences

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Dedicated to the 80th birthday of professor **Tibor Katriňák**

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Slides: `http://www.math.u-szeged.hu/~czedli/`

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Princ(A) $= \langle \mathrm{Princ}(A); \subseteq \rangle$: the poset of principal congruences of an algebra A. Princ(L), if $A = L$ is a *lattice*.

Problem (G. Grätzer AU 70, 95–105, 2013, and The Congruences of a Finite Lattice, A Proof-by-Picture approach, 2nd ed. 2016)

Which posets can be represented as Princ(A) *or* Princ(L)*?*

Theorem (G. Grätzer: AU 70 (2013), 95–105)

Every bounded poset is $\cong$ Princ(L) *such that length*(L) ≤ 5.

Theorem (G. Czédli: Acta Sci. Math. (Szeged) 82 (2016), 3–18)

For $\forall$ bounded non-singleton poset P and any group G can ***simultaneously*** *be represented as $P \cong$ Princ(L) and $G \cong$ Aut(L), where L is a selfdual lattice.*

The non-bounded case is characterized only up to $\aleph_0$:

Theorem (G. Czédli: AU 75 (2016), 351–380)

Let P be a poset with $|P| \leq \aleph_0$. Then there is a lattice L with $P \cong \mathrm{Princ}(L)$ if and only if $0 \in P$ and P is directed.

$\mathrm{Princ}(A)$ is not directed in general. We know almost nothing on $\mathrm{Princ}(A)$-representability. The example below indicates the difficulty.

Example (Corollary 2.1 in Czédli arXiv:1705.10833v3)

The 3-element "V-shaped" poset is not representable as $\mathrm{Princ}(A)$.

Definition (the action of Princ for a homomorphism)

If $f\colon A \to B$ is a homomorphism, then
$\mathrm{Princ}(f)\colon \mathrm{Princ}(A) \to \mathrm{Princ}(B)$ is defined by
$\mathrm{con}_A(x,y) \mapsto \mathrm{con}_B(f(x), f(y))$;
it is a well-defined, monotone, 0-preserving map.

Definition (representing a map by principal lattice congruences)

For a given 0-preserving monotone map $\varphi\colon P_1 \to P_2$, the task is to find lattices L_1 and L_2, poset isomorphisms τ_1 and τ_2, and a lattice homomorphism f such that the following diagram commutes:

$$\begin{tikzcd} P_1 \arrow[r, "\varphi"] \arrow[d, "\tau_1"'] & P_2 \\ \mathrm{Princ}(L_1) \arrow[r, "\mathrm{Princ}(f)"] & \mathrm{Princ}(L_2) \arrow[u, "\tau_2^{-1}"] \end{tikzcd}$$

Theorem (Czédli, 2015)

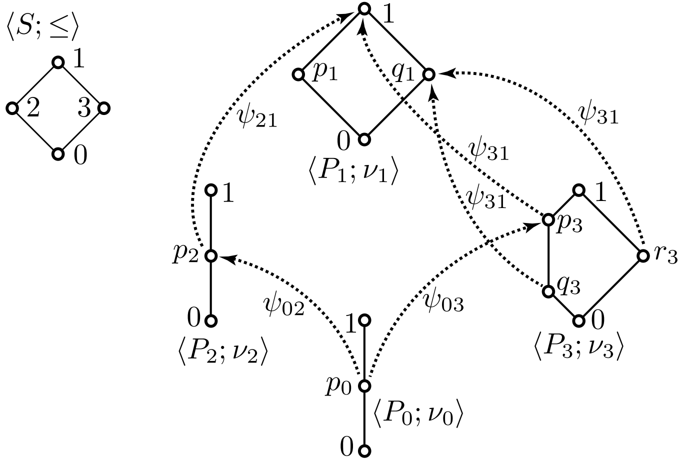
If P_1 and P_2 are bounded posets and $\varphi\colon P_1 \to P_2$ is a $\{0,1\}$-preserving and 0-separating monotone map, then φ is representable by principal lattice congruences.

As mentioned in SSAOS, Stara lesna, 2014,

Theorem roughly saying (Czédli: AU 77 (2017), 51–77)

Every "poset-indexed diagram" of $\{0,1\}$-preserving and 0-separating monotone maps among bounded posets can be represented in simultaneous way similarly'.

To give the precise meaning of this theorem, we need categories and the concept of *lifting a functor*, due to Gillibert and Wehrung [From objects to diagrams for ranges of functors, Springer, 2011].



All the ψ_{ij}, $i \leq j$, are $\{0,1\}$-preserving and, say, $\psi_{21} \circ \psi_{02} = \psi_{01}$.

The index poset S in the previous slide can be considered a *small category*. (Small: the objects form a set.) This motivates:

Definition (Diagram of $\{0,1\}$-preserving monotone maps)

By an **A**-indexed *diagram* (of $\{0,1\}$-preserving monotone maps) we shall mean a functor F from a small category $\mathbf{A}$ to the category $\mathbf{Posets}^{+}_{01}$ of nonsingleton bounded posets with $\{0,1\}$-preserving monotone maps as morphisms.

The aim is to represent **A**-indexed diagrams F by *lifting*.
We always assume that F is *faithful*, i.e., it is injective on every hom-set.

$\mathbf{Lattices}_5^{\mathrm{sd}}$ will denote the category of selfdual lattices of length 5 with $\{0,1\}$-preserving lattice homomorphisms. Our purpose is *to lift* $\mathbf{A}$-indexed diagrams (that is, a functor $F\colon \mathbf{A} \to \mathbf{Posets}_{01}^{+}$) with respect to Princ in the following sense

Definition (Gillibert and Wehrung, particular case)

Let $F\colon \mathbf{A} \to \mathbf{Posets}_{01}^{+}$ be a functor. We say that a functor $E\colon \mathbf{A} \to \mathbf{Lattices}_5^{\mathrm{sd}}$ ***lifts*** the functor F with respect to the functor Princ if F is naturally equivalent to the composite functor $\mathrm{Princ} \circ E$. If there is such an E, then we say that ***F can be lifted with respect to* Princ**.

In a plane language, we look for a functor $E\colon \mathbf{A} \to \mathbf{Lattices}_5^{\mathrm{sd}}$ and poset isomorphisms τ_i such that for each morphism $f\colon i \to j$ in the index category $\mathbf{A}$, the following diagram commutes:

$$\begin{tikzcd} \text{poset } F(i) \arrow[rr, "\text{monotone map } F(f)"] \arrow[d, "\tau_i"'] & & \text{poset } F(j) \\ (\mathrm{Princ}\circ E)(i) = \mathrm{Princ}(\text{lattice } E(i)) \arrow[rr, "\mathrm{Princ}(\text{lat. homo. } E(f))"] & & (\mathrm{Princ}\circ E)(i) = \mathrm{Princ}(\text{lattice } E(j)) \arrow[u, "\tau_j^{-1}"] \end{tikzcd}$$

Wehrung's section in the monograph Lattice Theory Special Topics and Appl. I, Sect. 7-4.5 + anonymous referee lead to:

Remark (see Ex.6.4 in Czédli: AU 77 (2017) 51–77)

Not every $F\colon \mathbf{A} \to \mathbf{Posets}_{01}^{+}$ *can be lifted to* Princ.

Thus, we need some assumption on the index category $\mathbf{A}$ in the following theorem.

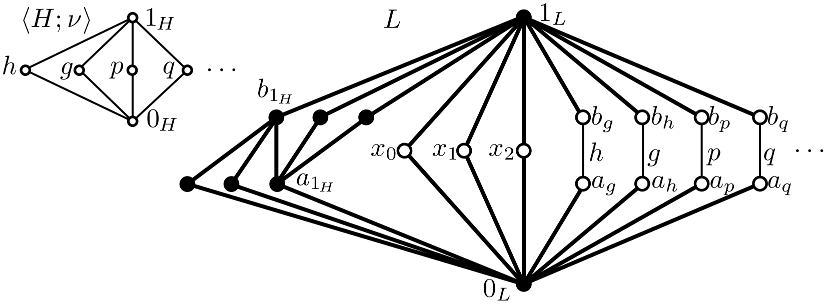
A morphism f in a category $\mathbf{A}$ is *mono* if $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$ whenever both products make sense. A category is *concrete* if its objects are sets (with possible structures on them) and its morphisms are (certain) maps.

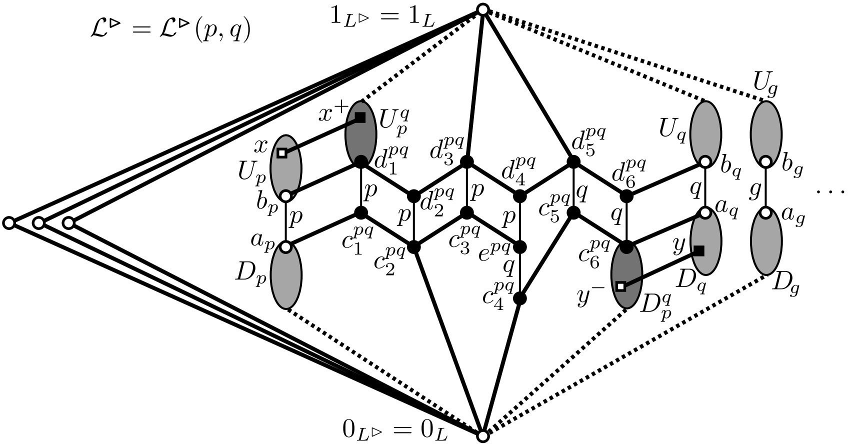
Theorem 1 (Czédli, in "Cometic functors and representing order-preserving maps by principal lattice congruences")

Let $\mathbf{A}$ be a small category such that every morphism in $\mathbf{A}$ is mono. Then every faithful functor (i.e., $\mathbf{A}$-indexed diagram) $F\colon \mathbf{A} \to \mathbf{Posets}_{01}^{+}$ can be lifted to the functor $\mathrm{Princ}\colon \mathbf{Lattices}_5^{\mathrm{sd}} \to \mathbf{Posets}_{01}^{+}$ by means of a faithful functor $E\colon \mathbf{A} \to \mathbf{Lattices}_5^{\mathrm{sd}}$.

Note that there are many small concrete categories in wich all morphisms are mono but not all of them are injective.

The Stara Lesna 2014 construction **does not work for non-injective maps**.




$\mathcal{L}^\triangleright = \mathcal{L}^\triangleright(p,q)$
$1_{L^\triangleright} = 1_L$
$0_{L^\triangleright} = 0_L$
x^+
U_p^q
x
U_p
b_p
a_p
D_p
d_1^{pq}
d_2^{pq}
d_3^{pq}
d_4^{pq}
d_5^{pq}
d_6^{pq}
c_1^{pq}
c_2^{pq}
c_3^{pq}
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e^{pq}
p
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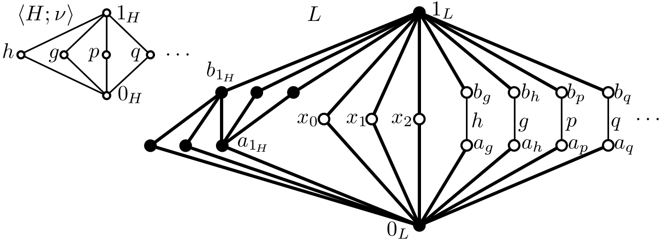

Because: if a map $f\colon H \to$ somewhere collapses, say, $p, q \in H$, then it collapses a_p and a_q in the corresponding lattice. But then, since $\mathrm{con}(a_p, a_q)$ is the largest congruence, it collapses the hole lattice, which should not happen.

$\Rightarrow$ **we need a tool** that turns monomorphism into injective maps.

For a concrete category $\mathbf{A}$, $\iota_{\mathbf{A},\mathbf{Set}}$ denotes the forgetful functor.

Theorem 2 (Czédli: "Cometic functors ...")

For every small concrete category $\mathbf{A}$, *there exists a so-called* ***cometic functor*** $F_{\mathrm{com}}\colon \mathbf{A} \to \mathbf{Set}$ *and a natural transformation* $\boldsymbol{\pi}^{\mathrm{com}}\colon F_{\mathrm{com}} \to \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}$ *such that the components of* $\boldsymbol{\pi}^{\mathrm{com}}$ *are surjective maps and the* F_{com}*-image of every* ***mono****morphisms of* $\mathbf{A}$ *is an injective map.*

Since $\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}$ does nothing, the meaning of the theorem is this:

$$\begin{tikzcd}
F_{\mathrm{com}}(X) \arrow[rr, "{\begin{array}{c}F_{\mathrm{com}}(g),\\ \text{injective if } f \text{ is mono in } \mathbf{A}\end{array}}"] \arrow[d, "\text{surj. } \boldsymbol{\pi}_X^{\mathrm{com}}"'] & & F_{\mathrm{com}}(Y) \arrow[d, "\text{surj. } \boldsymbol{\pi}_Y^{\mathrm{com}}"'] \\
X & \xrightarrow{g} & Y
\end{tikzcd}$$

$F_{\text{com}}(Y) := \{$"mapsto arrows" $x \overset{f}{\mapsto} y \;:\; y \in Y\}$. Thus, $F_{\text{com}}(Y)$ has a comet-like picture. For another explanation of the name "cometic", see below:

$$\begin{tikzcd}
X \arrow[r, "g"] & Y \\
F_{\mathrm{com}}(X) \arrow[rr, "F_{\mathrm{com}}(g)"'] \arrow[d, "\boldsymbol{\pi}^{\mathrm{com}}_X"] & & F_{\mathrm{com}}(Y) \arrow[d, "\boldsymbol{\pi}^{\mathrm{com}}_Y"'] \\
\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(X) \arrow[rr, "\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(g)"] & & \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(Y)
\end{tikzcd}$$

These slides and my corresponding papers are available at

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`http://www.math.u-szeged.hu/~czedli/conference-talks/conflectures0.html`

Thank you for your attention!

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Partially, the name "cometic", see , comes from:

$$X \xrightarrow{g} Y$$

$$\begin{tikzcd}[column sep=huge] F_{\mathrm{com}}(X) \arrow[r, "F_{\mathrm{com}}(g)"'] \arrow[d, "\boldsymbol{\pi}_X^{\mathrm{com}}"] & F_{\mathrm{com}}(Y) \arrow[d, "\boldsymbol{\pi}_Y^{\mathrm{com}}"'] \\ \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(X) \arrow[r, "\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(g)"] & \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(Y) \end{tikzcd}$$

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