Why have we dented the circles

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For $x \in L$, the join of all covers of x is denoted by x^*. If, for all $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$, then L is *meet-semidistributive*. If $\operatorname{Ji} L$ is the union of two chains, then L is *slim*. The next statement is known and gives a good understanding of join-distributive lattices within lattice theory. For further characterizations, see S. P. Avann [8], which is recalled in P. H. Edelman [16, Theorem 1.1], and see also M. Stern [22, Theorem 7.2.27].

Proposition

For a finite lattice L, the following properties are equivalent.

- *L is join-distributive, that is, semimodular and meet-semidistributive.*
- *L has unique meet-irreducible decompositions.*
- *For each $x \in L$, the interval $[x, x^*]$ is distributive.*
- *For each $x \in L$, the interval $[x, x^*]$ is boolean.*
- *The length of each maximal chain of L equals $|\mathrm{Mi}\, L|$.*
- *L is semimodular and diamond-free.*
- *L is semimodular and has no cover-preserving diamond sublattice.*
- *L is a cover-preserving join-subsemilattice of a finite distributive lattice.*

The concept of antimatroids is due to R. E. Jamison-Waldner [18], who was the first to use the term "antimatroid". At the same time, an equivalent complementary concept was introduced by P. H. Edelman [15] under the name "anti-exchange closures". There are several ways to define antimatroids, see D. Armstrong [7, Lemma 2.1]; here we accept the following one. The set of all subsets of a set E is denoted by $P(E)$.

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Definition

A pair $\langle E, \mathfrak{F} \rangle$ is an *antimatroid* if it satisfies the following properties:

- E is a finite set, and $\emptyset \neq \mathfrak{F} \subseteq P(E)$;
- $\mathfrak{F}$ is a *feasible set*, that is, for each nonempty $A \in \mathfrak{F}$, there exists an $x \in A$ such that $A \setminus \{x\} \in \mathfrak{F}$;
- $\mathfrak{F}$ is closed under taking unions;
- $E = \bigcup \{A : A \in \mathfrak{F}\}$.

Now, we define a concept closely related to antimatroids, see P.H. Edelman [15] and K. Adaricheva, V. A. Gorbunov, and V. I. Tumanov [?].

Definition

A pair $\langle E, \Phi \rangle$ is a *convex geometry*, also called *anti-exchange system*, if it satisfies the following properties:

- E is a finite set, and $\Phi \colon P(E) \to P(E)$ is a closure operator.
- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, $x \neq y$, and $x \in \Phi(A \cup \{y\})$, then $y \notin \Phi(A \cup \{x\})$. (This is the so-called *anti-exchange property*.) **Saying it more clearly:** If $x, y \notin P(A) = A$ and $\{x\} \cup A$ and $\{y\} \cup A$ span the same, then $x = y$.
- $\Phi(\emptyset) = \emptyset$.

For a closures system $\mathfrak{G}$ on E with corresponding closure operator Φ, $\langle E, \mathfrak{G} \rangle$ is a *convex geometry* if so is $\langle E, \Phi \rangle$ in the above sense. In what follows, the notations $\langle E, \Phi \rangle$ and $\langle E, \mathfrak{G} \rangle$ can be used interchangeably for the same mathematical object. The members of $\mathfrak{G}$ are called *closed sets*.

Proposition

Let E be a finite set, and let $\emptyset \neq \mathfrak{F} \subseteq P(E)$. Then $\mathfrak{A} = \langle E, \mathfrak{F} \rangle$ is an antimatroid iff $\mathfrak{A}^\delta = \langle E, \{E \setminus X : X \in \mathfrak{F}\} \rangle$ is a convex geometry.

Proposition

If L is a join-distributive lattice and $\mathfrak{A} = \langle E, \mathfrak{F} \rangle$ is an antimatroid, then the following three statements hold.

- $\langle \mathfrak{F}, \subseteq \rangle$ *is a join-distributive lattice; it is denoted by* $\mathrm{LJd}(\mathfrak{A})$.
- $\langle \mathrm{Mi}\, L, \{\mathrm{Mi}\, L \setminus {\uparrow}x : x \in L\} \rangle$ *is an antimatroid; it is denoted by* $\mathrm{Amat}(L)$,
- $\mathrm{LJd}\big(\mathrm{Amat}(L)\big) \cong L$ *and* $\mathrm{Amat}\big(\mathrm{LJd}(\mathfrak{A})\big) \cong \mathfrak{A}$.

For a lattice L of finite length, the set $\big\{[a,b] : a \prec b,\ a,b \in L\big\}$ of prime intervals of L will be denoted by $\mathrm{PrInt}(L)$. For $[a,b],[c,d] \in \mathrm{PrInt}(L)$, we say that $[a,b]$ and $[c,d]$ are *consecutive* if $\{a,b,c,d\}$ is a covering square, that is, a 4-element cover-preserving boolean sublattice of L. The transitive reflexive closure of the consecutiveness relation on $\mathrm{PrInt}(L)$ is an equivalence, and the blocks of this equivalence relation are called the *trajectories* of L; this concept was introduced for some particular semimodular lattices in G. Czédli and E. T. Schmidt [14]. For distinct $[a,b],[c,d] \in \mathrm{PrInt}(L)$, these two prime intervals are *comparable* if either $b \leq c$, or $d \leq a$.

Proposition (Czédli and Adaricheva)

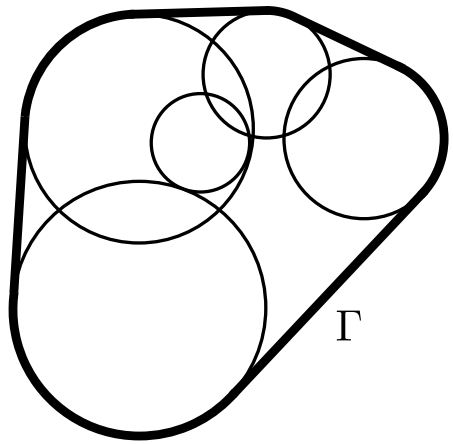
For a semimodular lattice L, the following three conditions are equivalent.

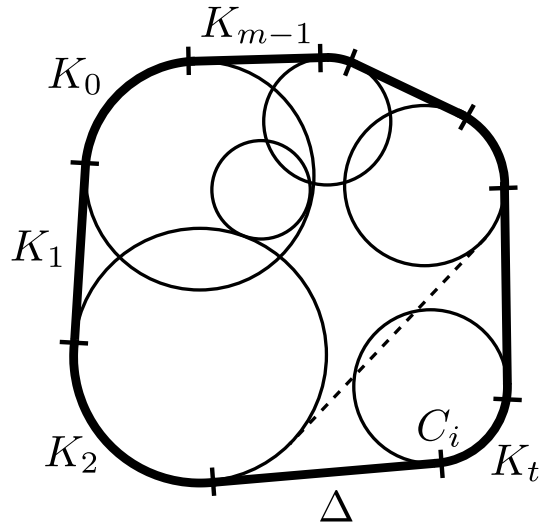
- *L is join-distributive.*
- *L is of finite length, and for every trajectory T of L and every maximal chain C of L, $|\mathrm{PrInt}(C) \cap T| = 1$.*
- *L is of finite length, and no two distinct comparable prime intervals of L belong to the same trajectory.*

Corollary

Abstract convex geometries correspond to the DUAL of my favorite lattices.

$\langle \mathbb{R}^n, \mathrm{Conv}_{\mathbb{R}^n} \rangle$ is a convex geometry. Not every finite convex geometry can be embedded. Hence,





Circles (or disks) form a locally convex geometry (Czédli [11]); see the figure. $\mathbb{R}^n$ also. Still not every finite convex geometry can be embedded (see later: Adaricheva and Bolat).
Richter and Rogers [21, Lemma 3]: polygons in $\mathbb{R}^2$ do. Do they? Not satisfactorily . . .

Today's knowledge: almost-circles give rise to representation (Czédli and Kincses), circles do not do that (Adaricheva and Bolat; shorter proof by Czédli), n-dimensional ellipsoids do (Kincses, few days ago).
Not known whether ellipses do (I am almost sure that they do not), or whether n-dimensial spheres do (I do not have a conjecture).
Czédli recently: the converse of Adaricheva and Bolat also holds (their property characterizes circles); in a sense, even a stronger result holds (congruence is sufficient, no similarity is needed).

Theorem (Czédli and Kincses[13])

There exists a set T_{new} of some subsets of the plane, that is, $T_{\text{new}} \subseteq \mathrm{P}(\mathbb{R}^2)$, with the following properties.

- *Every non-singleton member of T_{new} is a differentiable convex simple closed planar curve, and for all $\vec{p} \in \mathbb{R}^2$, the singleton $\{\vec{p}\}$ belongs to T_{new}.*
- *$\langle T_{\text{new}}; \mathrm{Conv}_{T_{\text{new}}} \rangle$ is a locally convex geometry.*
- *T_{new} is closed with respect to non-degenerate affine transformations.*
- *For every finite convex geometry $\langle E_0; \Phi_0 \rangle$ and for every (small) positive real number $\varepsilon < 1$, there exist continuum many pairwise affine-disjoint finite subsets E of T_{new} such that $\langle E_0; \Phi_0 \rangle$ is isomorphic to the restriction $\langle T_{\text{new}}; \mathrm{Conv}_{T_{\text{new}}} \rangle \restriction_E = \langle E; \mathrm{Conv}_E \rangle$ and E consists of non-degenerate almost-circles of accuracy $1 - \varepsilon$.*

Lemma ((†) and Theorems 5.1 and 5.2 in Edelman and Jamison [17])

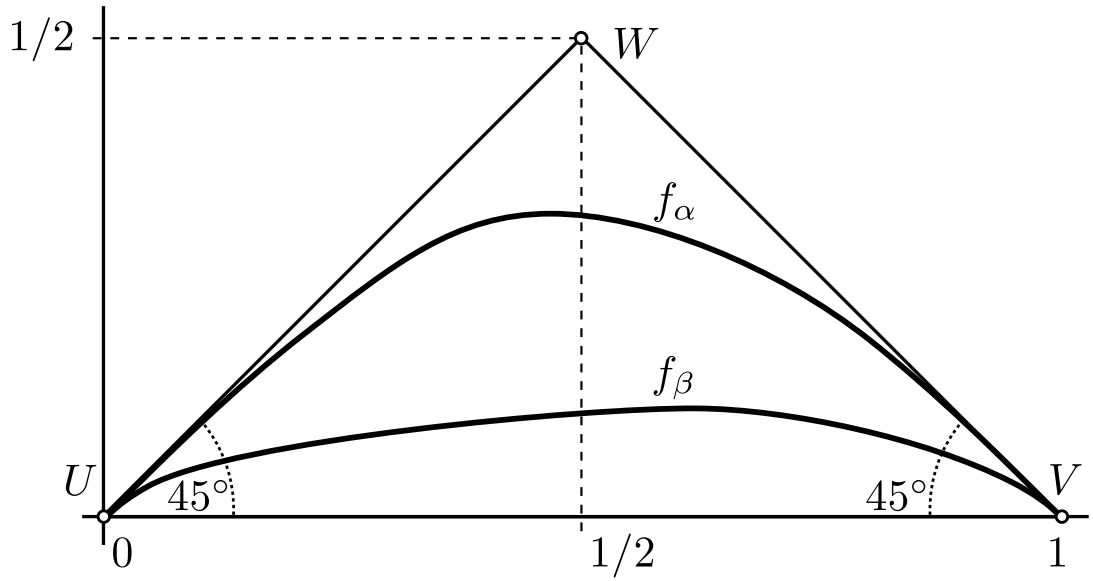
(A) *If* $\leq_1, \ldots, \leq_t$ *are linear orderings on a finite set* $E = \{1, 2, \ldots, n\}$ *and we define* $\mathfrak{G}$ *as*

$$\mathfrak{G} := \{\emptyset\} \cup \{X \in \mathsf{P}(E) : (\forall y \in E \setminus X)\,(\exists i \in \{1, \ldots, t\})\,(X <_i y)\}, \tag{1}$$

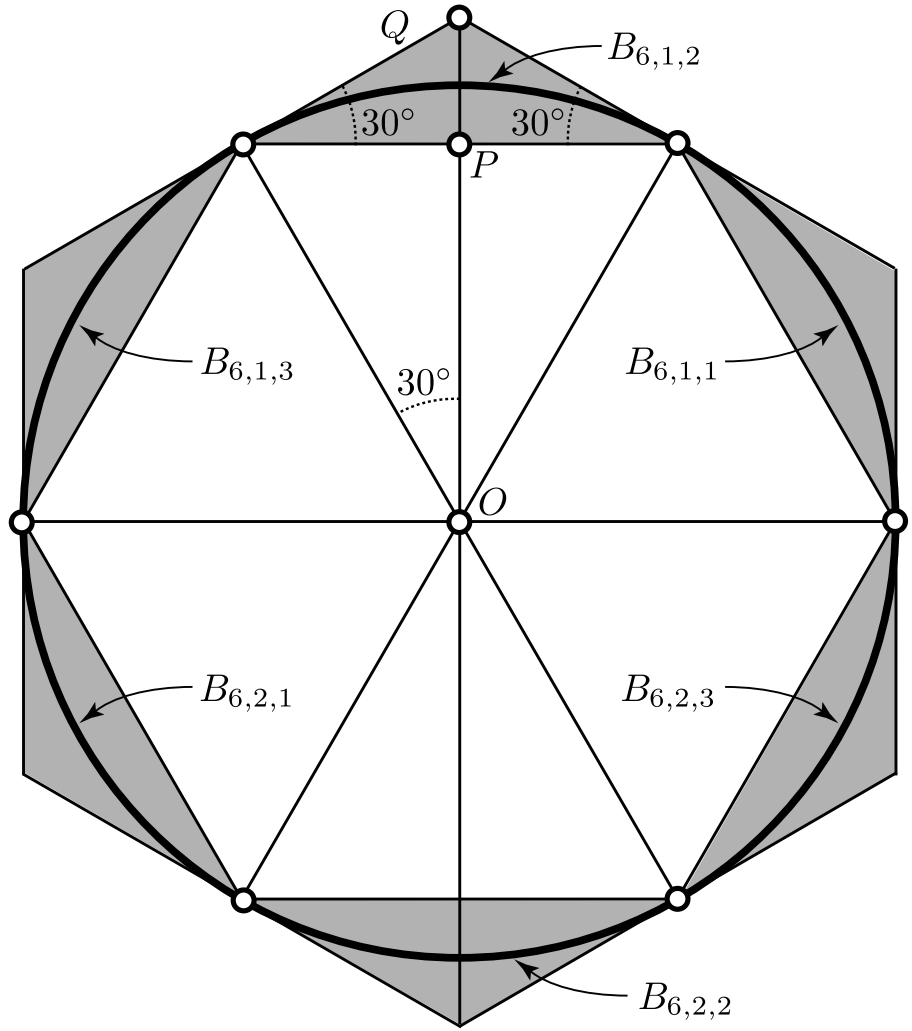
then $\langle E; \mathfrak{G} \rangle$ *is a convex geometry.*
(B) *Every finite convex geometry is isomorphic to some* $\langle E; \mathfrak{G} \rangle$ *such that* $\mathfrak{G}$ *is determined by finitely many linear orderings as in* (1).

Next, we define the following polynomials and consider them functions $[0,1] \to \mathbb{R}$:

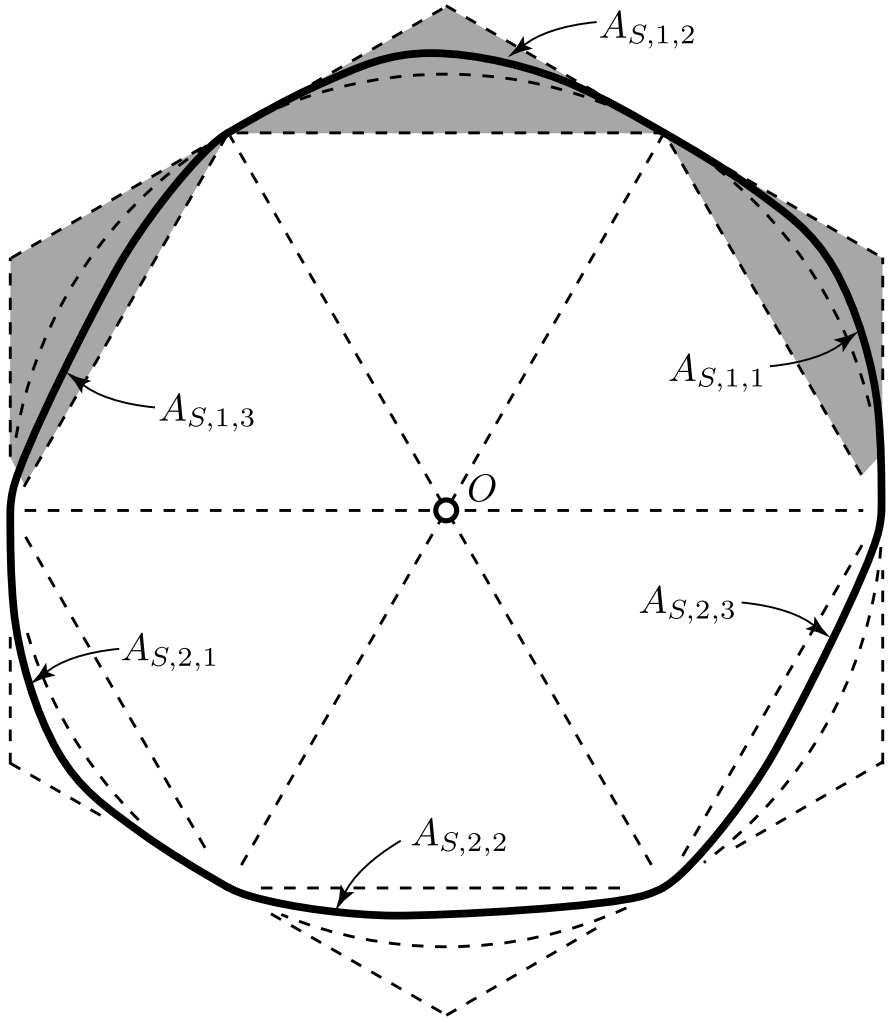
$$p(x) = x(1-x)(x^5 - x^4 + 1) = -x^7 + 2x^6 - x^5 - x^2 + x,$$
$$q(x) = x(1-x) = -x^2 + x, \text{ and, for } \alpha \in (0,1),$$
$$f_\alpha(x) = \alpha p(x) + (1-\alpha)q(x) = -\alpha x^7 + 2\alpha x^6 - \alpha x^5 - x^2 + x.$$



A affine-rigid family of functions



The unit circle with inscribed and circumscribed regular t-gons and $mt = 6$ circular arcs.



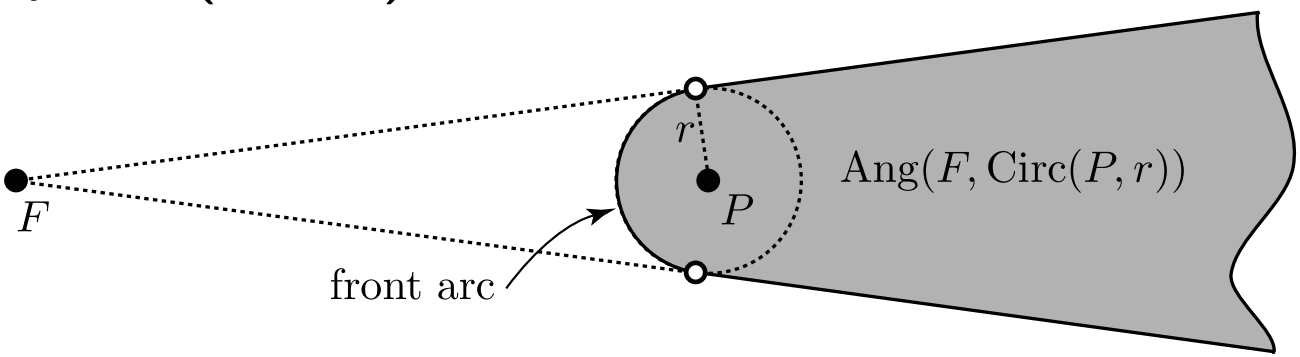
$C({}^{\mathrm{gf}}F, S)$, an almost-circle of accuracy $1 - (\pi/6)^2 \approx 0.7258$

Theorem (Adaricheva and Bolat [3])

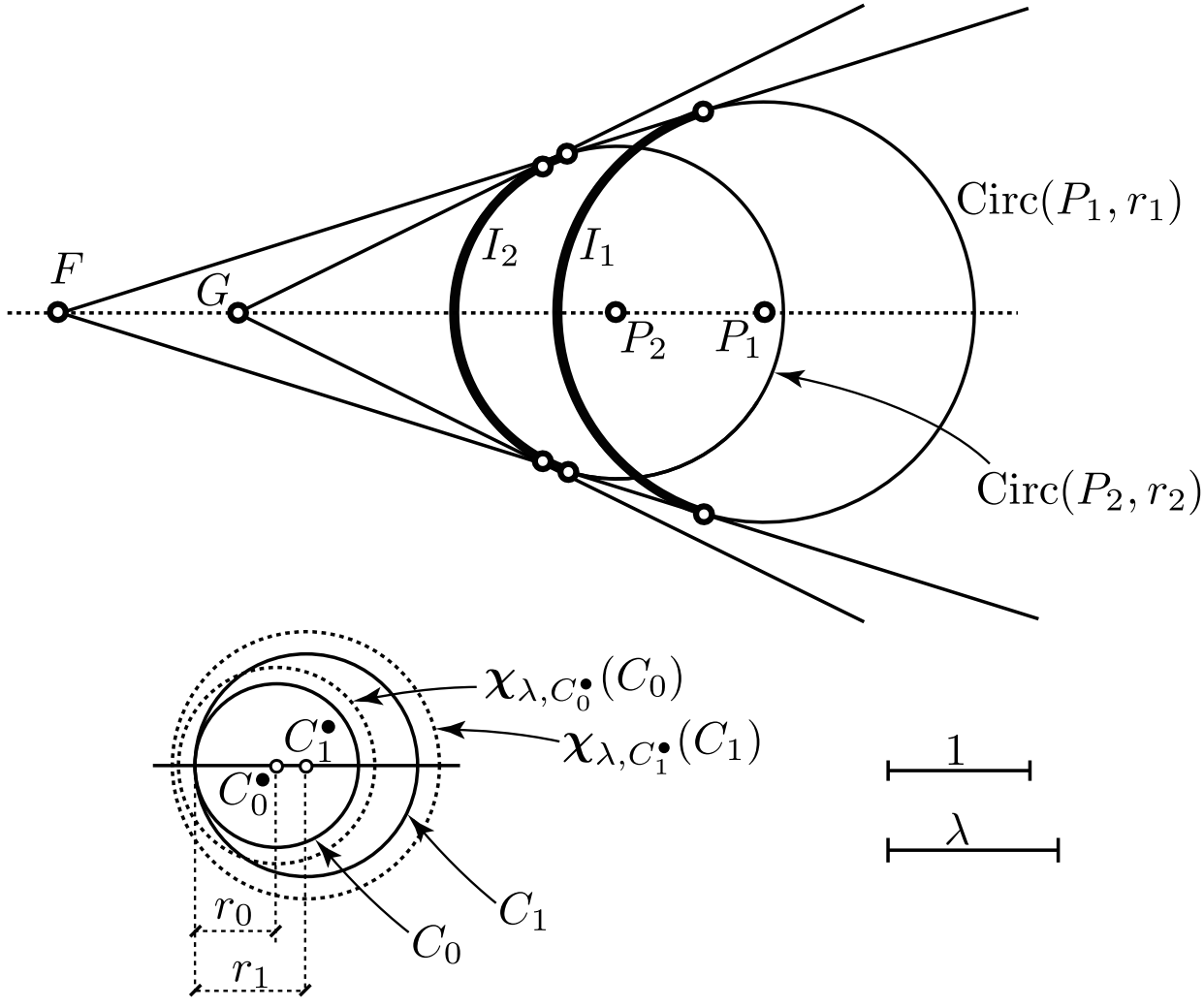
Let A_0, A_1, A_2 be points in the plane. If U_0 and U_1 are circles such that $U_i \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\})$ for $i \in \{0, 1\}$, then there exist subscripts $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$ such that

$$U_{1-k} \subseteq \mathrm{Conv}_{\mathbb{R}^2}\big(U_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big). \tag{2}$$

Second proof (Czédli)



Round-edged angle


F
G
I_2
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$\mathrm{Circ}(P_2, r_2)$
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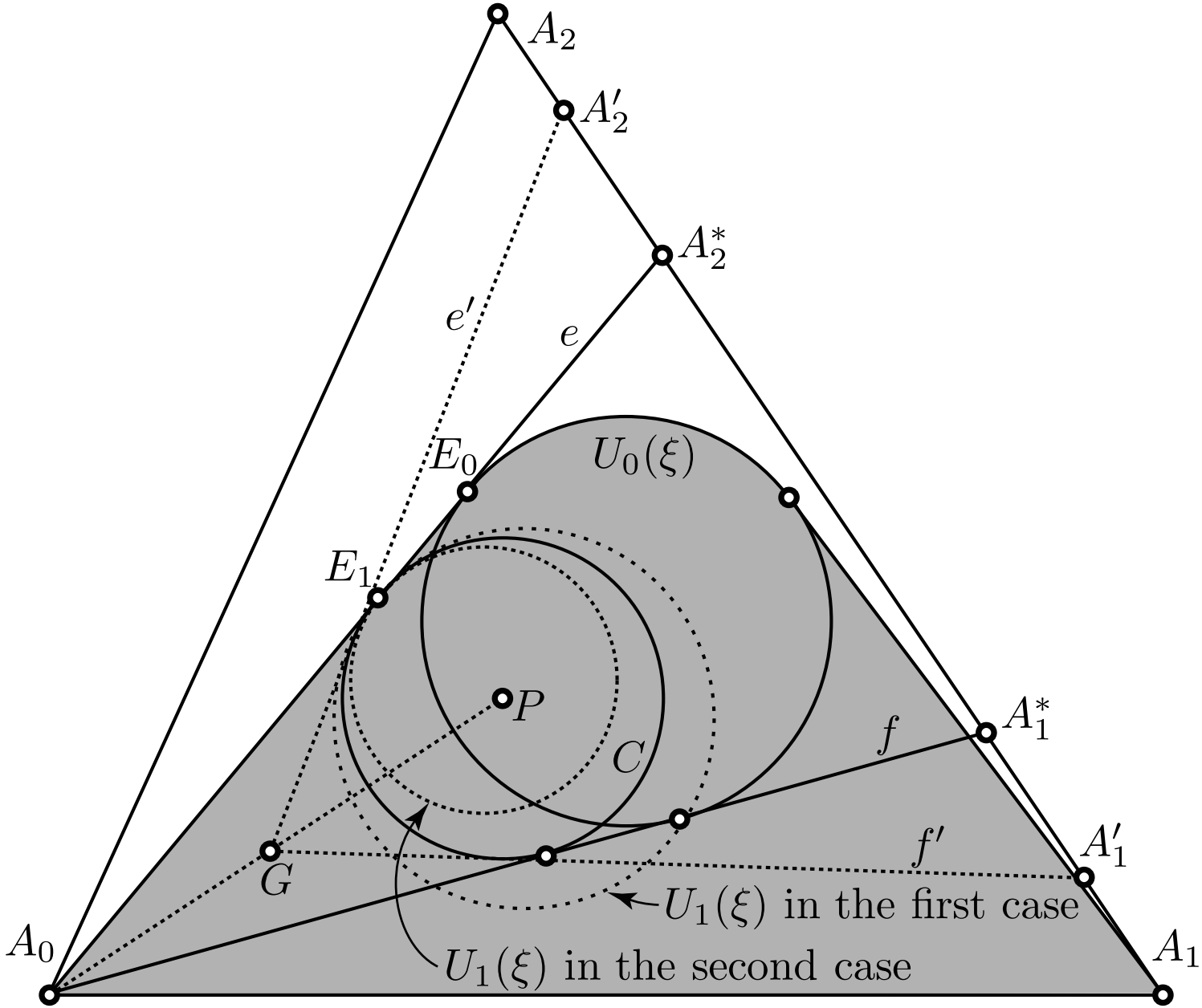
A_2
A_2'
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C
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f'
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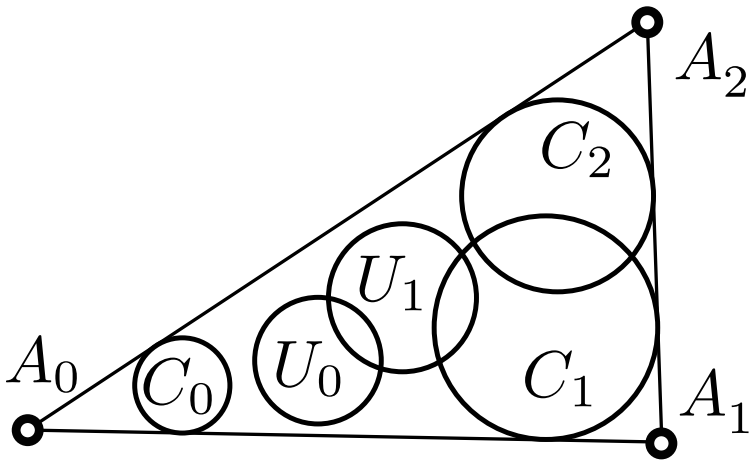
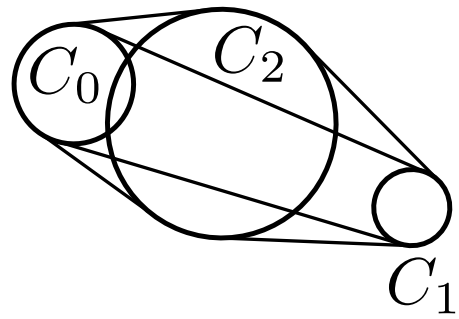
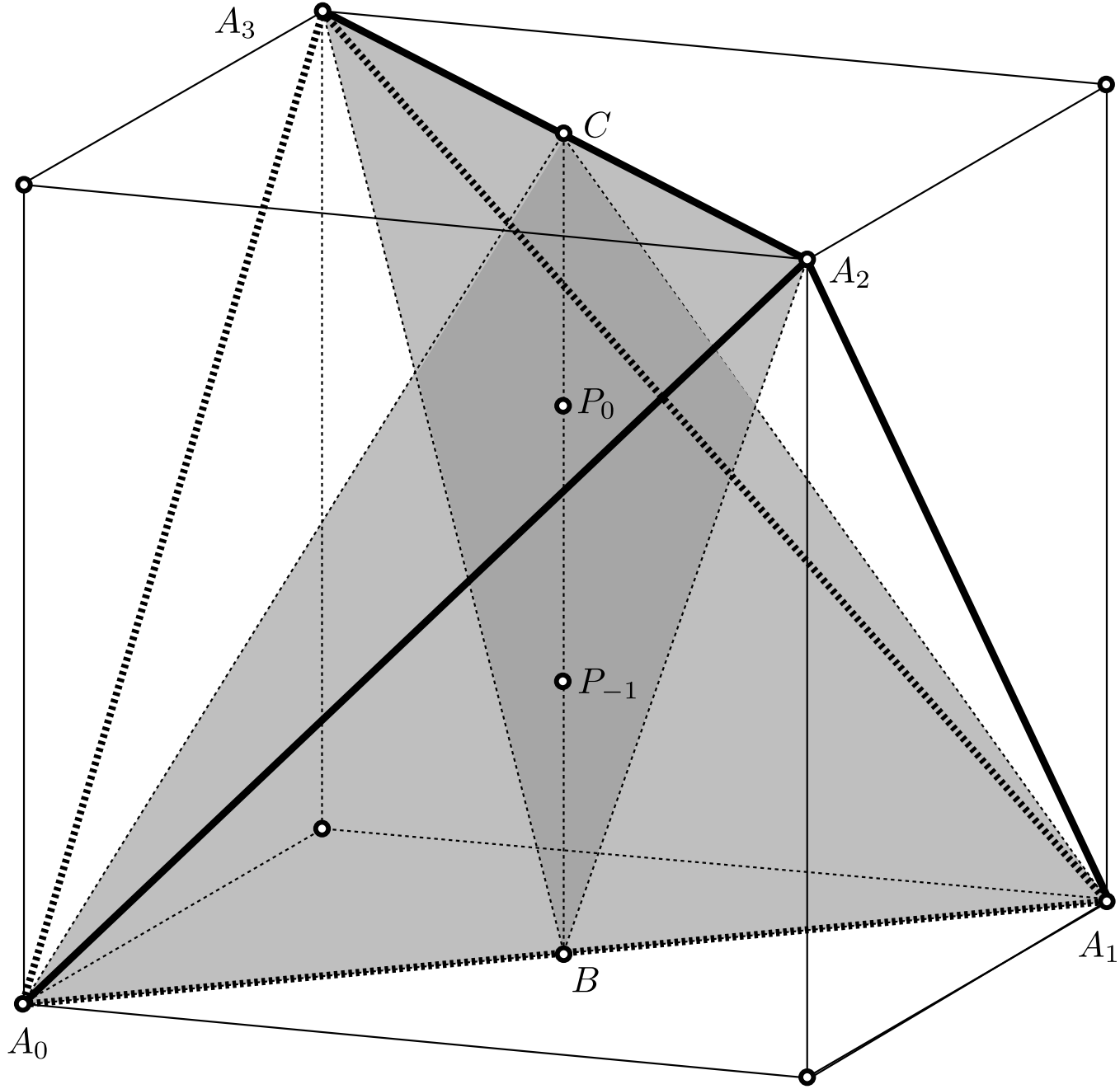
$U_1(\xi)$ in the first case
$U_1(\xi)$ in the second case
A_0
A_1

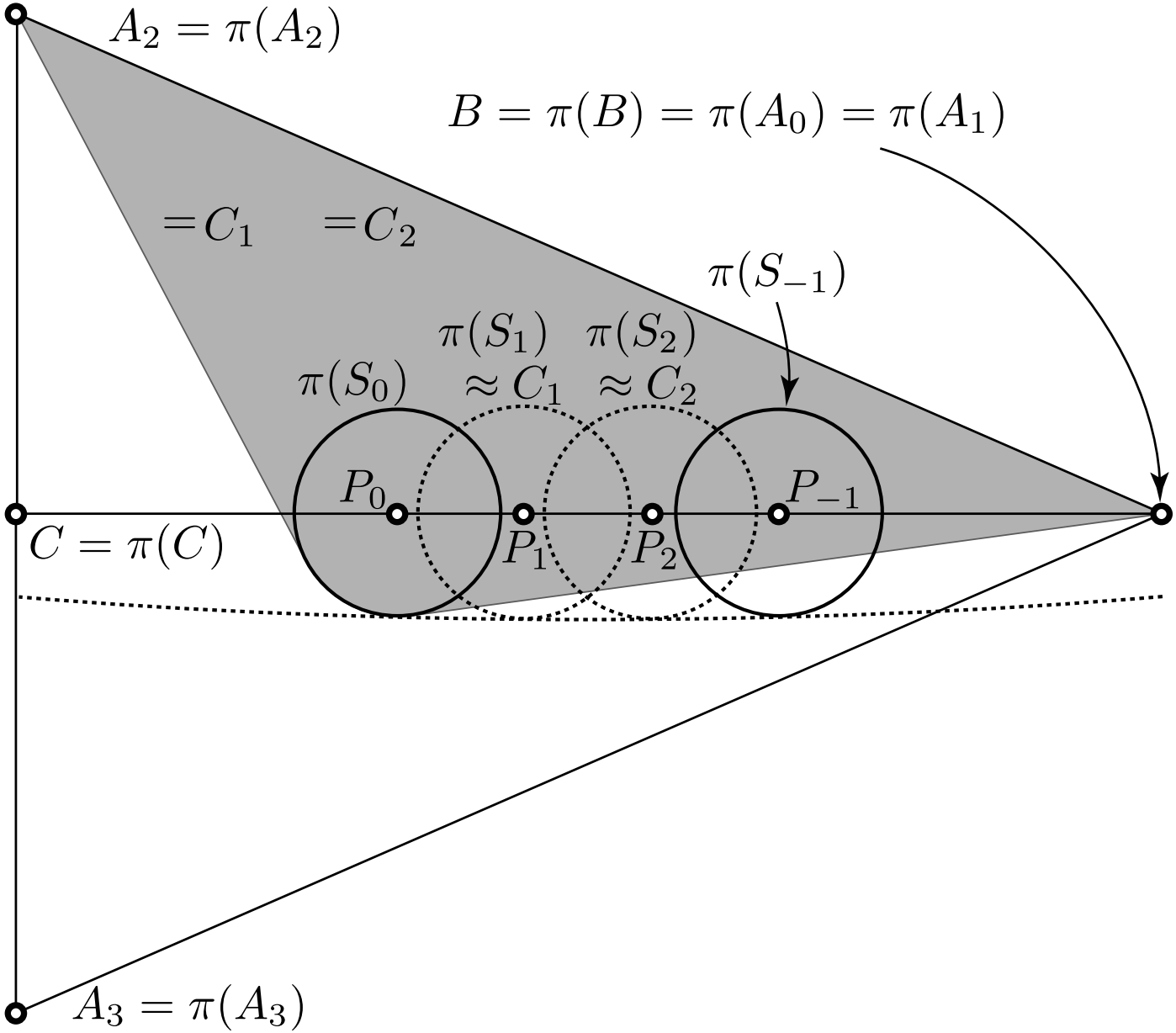




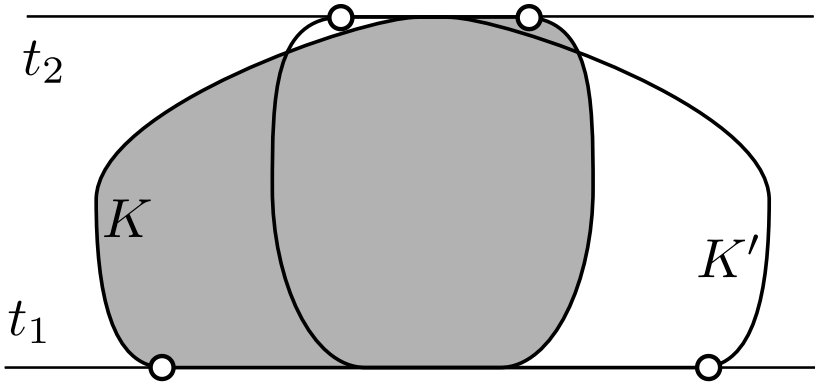
Circles instead the corners of the triangle

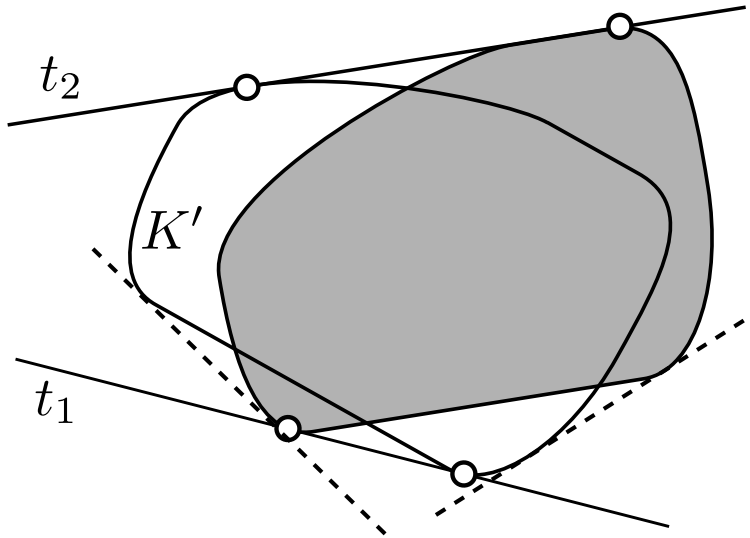


Spheres for which the property fails, because

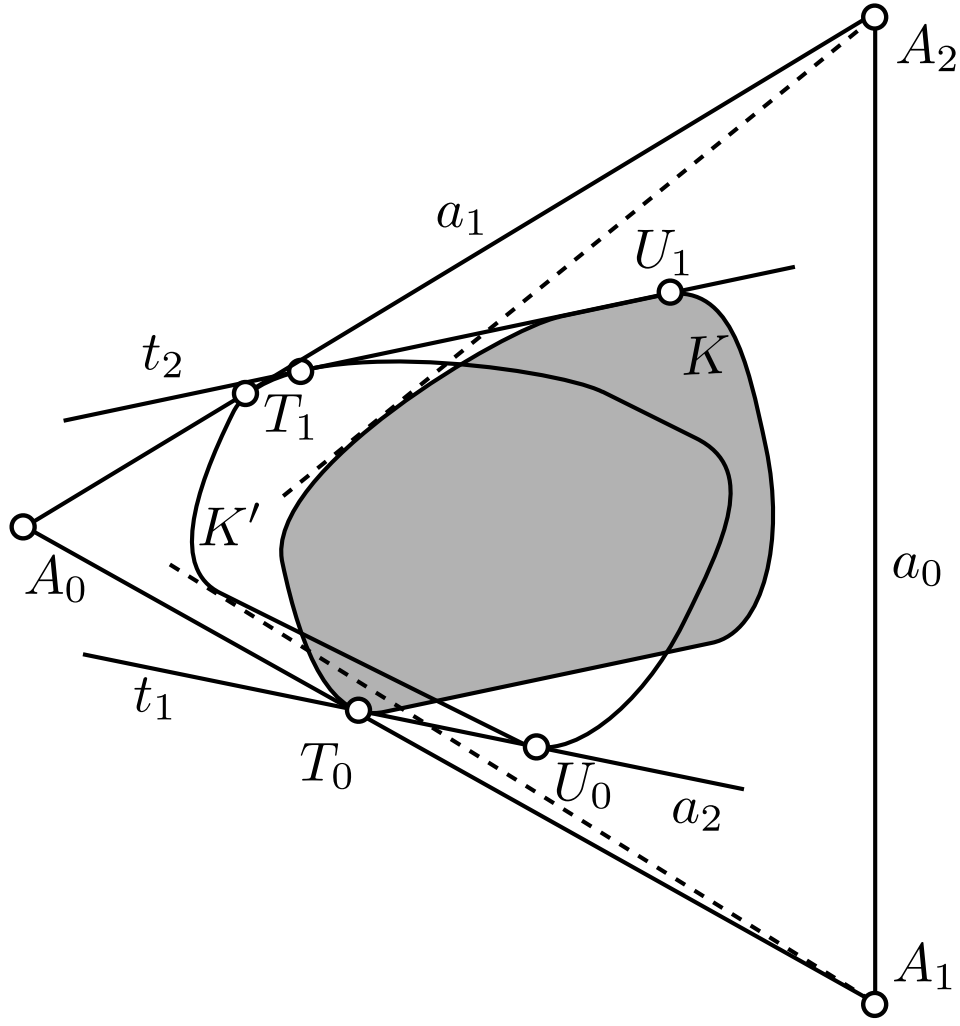


it fails for their projected planar images



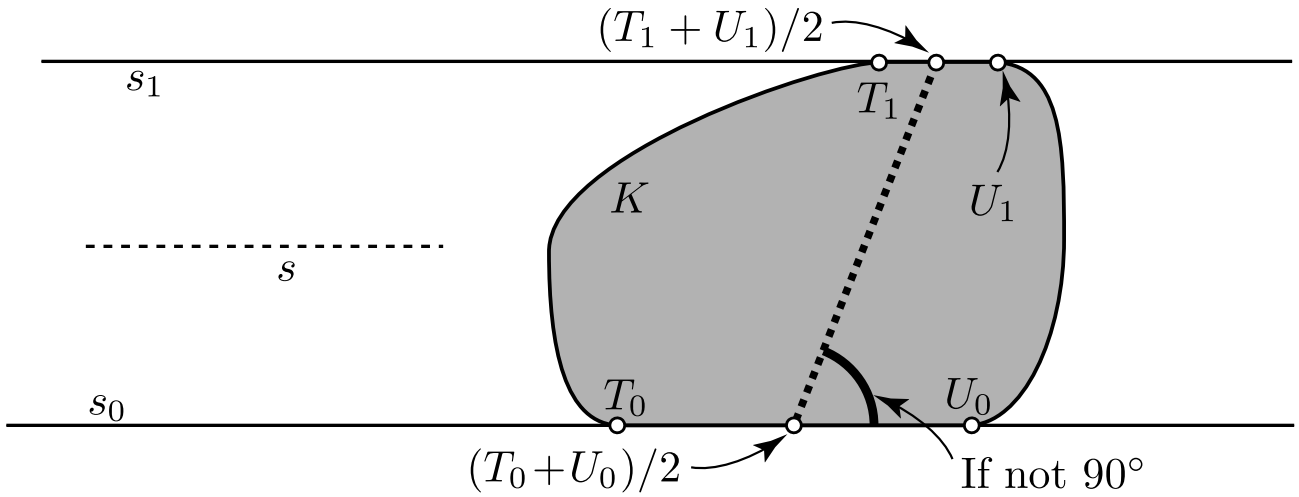


Cross Lemma: if the tangents are parallel, both directions are fine.

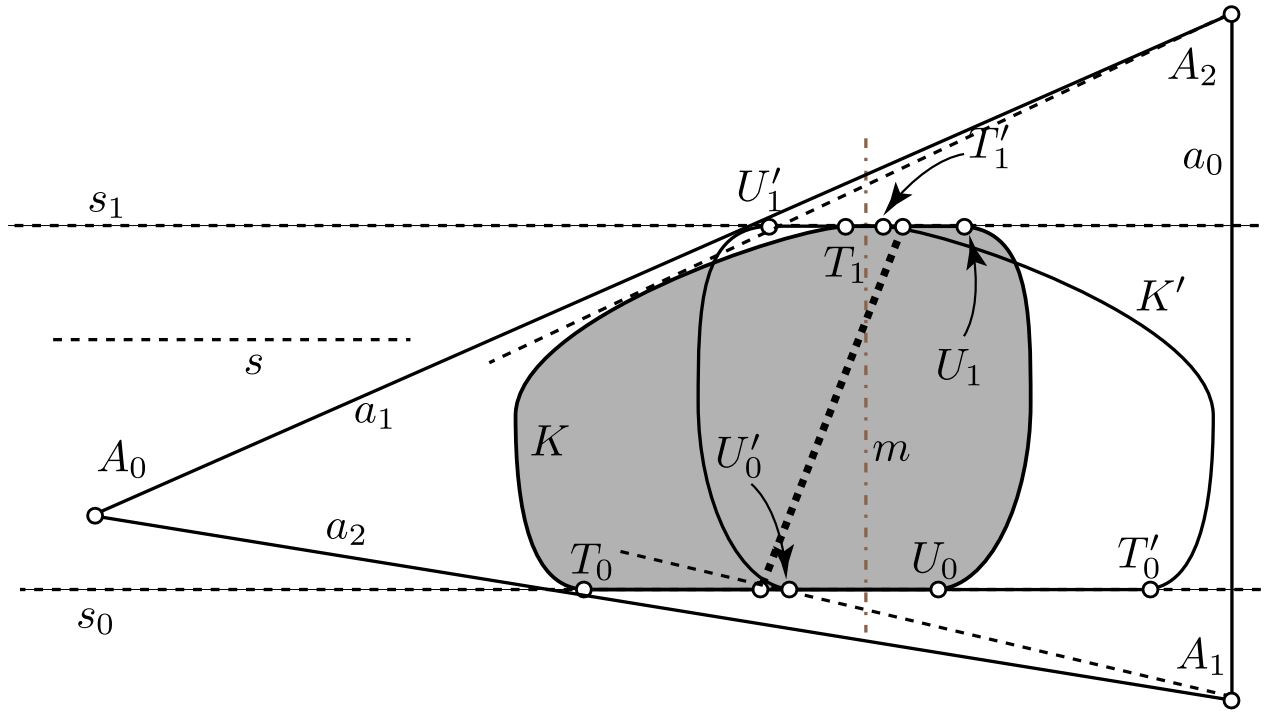


Cross Lemma Proof: tangently to K or K', rotate the tangent lines slightly; clock-wise the lower one and anti-clockwise the upper one.

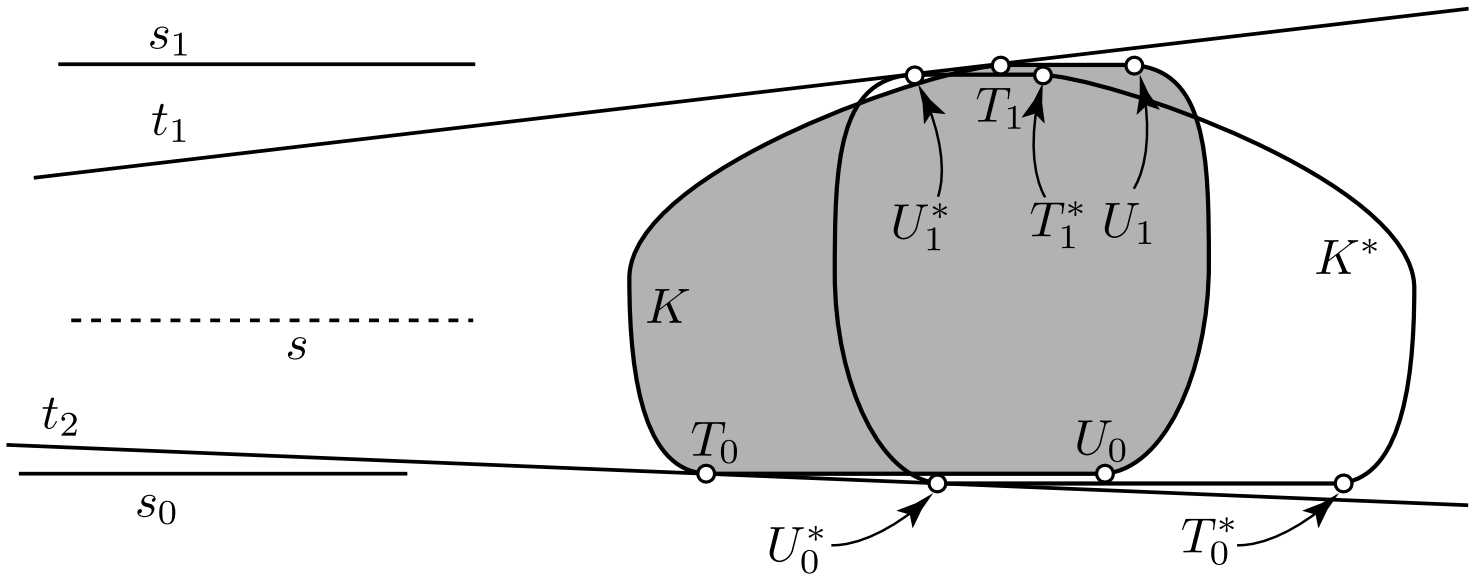
$T_1 \in K' \not\subseteq \mathrm{Conv}(K \cup \{A_0, A_1\})$, $\quad U_0 \in K' \not\subseteq \mathrm{Conv}(K \cup \{A_0, A_2\})$,
$T_1 \in K' \not\subseteq \mathrm{Conv}(K \cup \{A_1, A_2\})$, symmetric roles. Done.



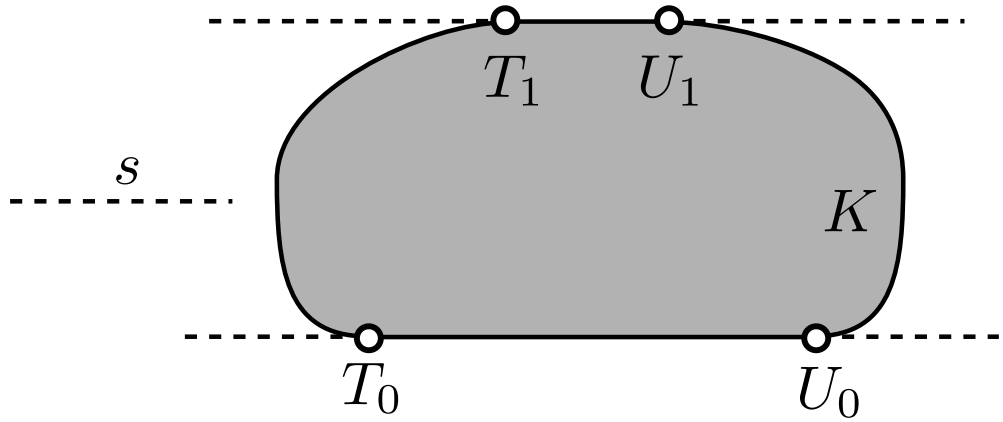
Perpendicularity Lemma: this angle must be 90°.



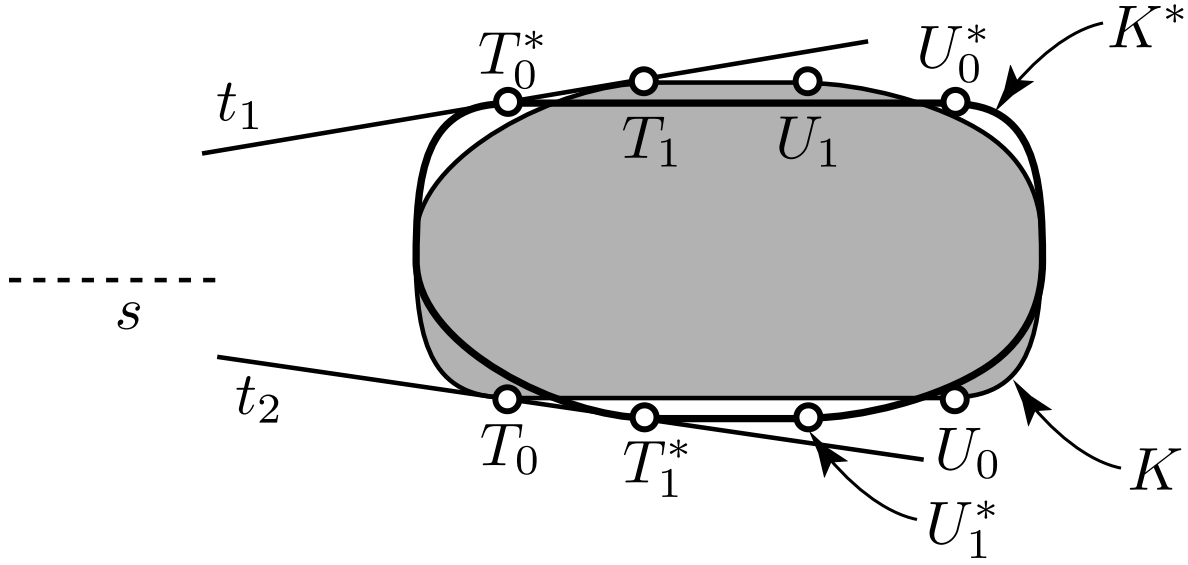
Perpendicularity Lemma Proof: reflect and Cross Lemma.



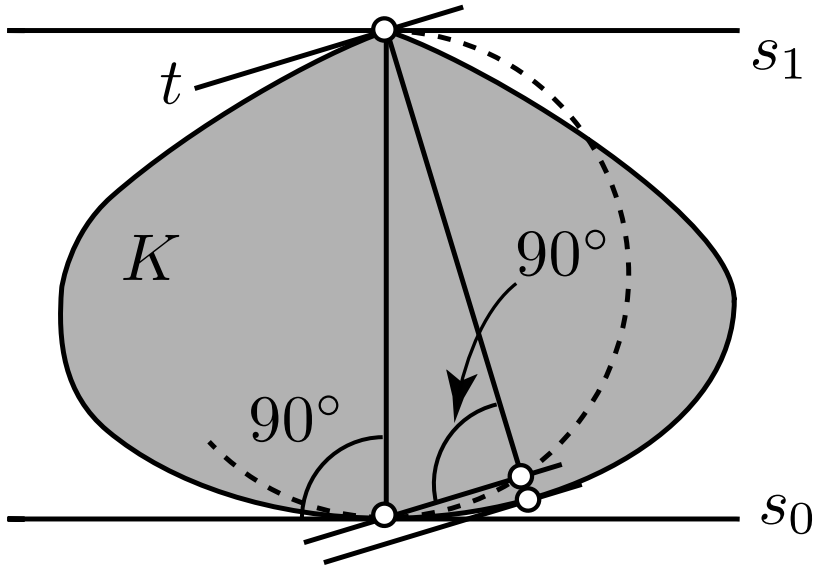
Perpendicularity Lemma 2nd Proof (also, exemplifies the "parallel" case of the Cross Lemma): see the figure above.



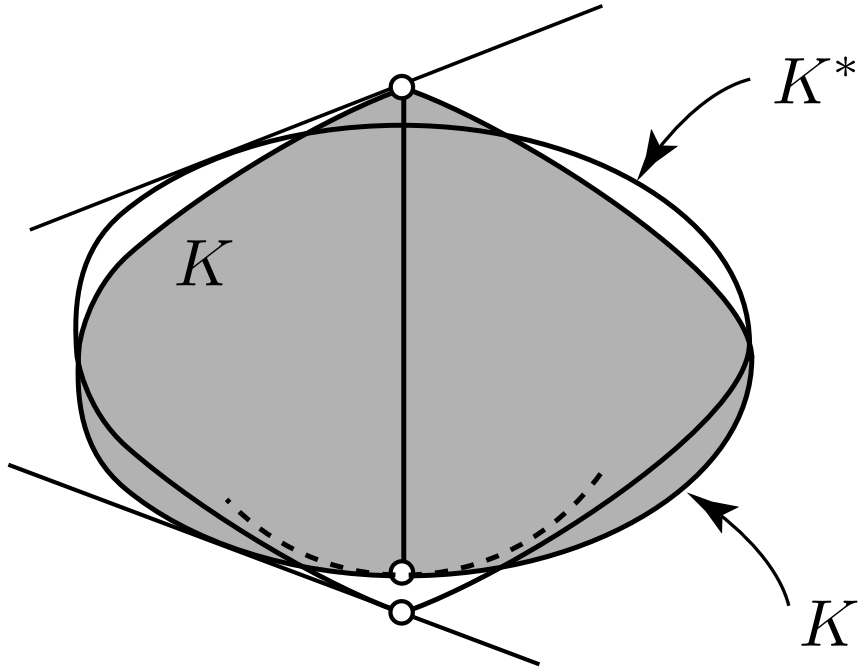
Same Opposite Sides Lemma: $\mathsf{dist}(T_0, U_0) = \mathsf{dist}(T_1, U_1)$.



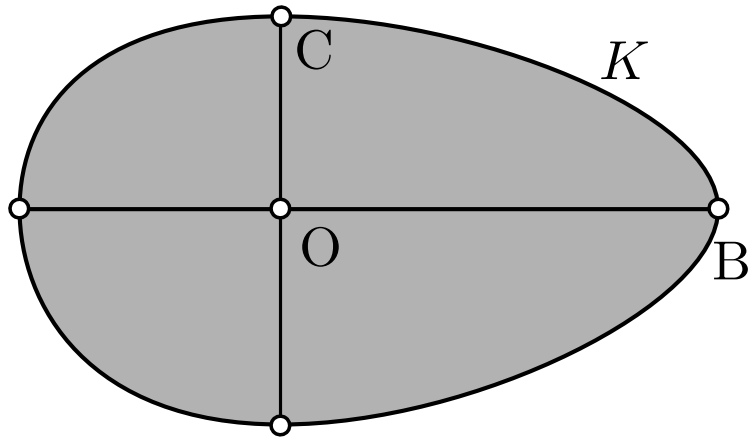
Same Opposite Sides Lemma Proof: reflect, shift slightly down, and Cross Lemma.



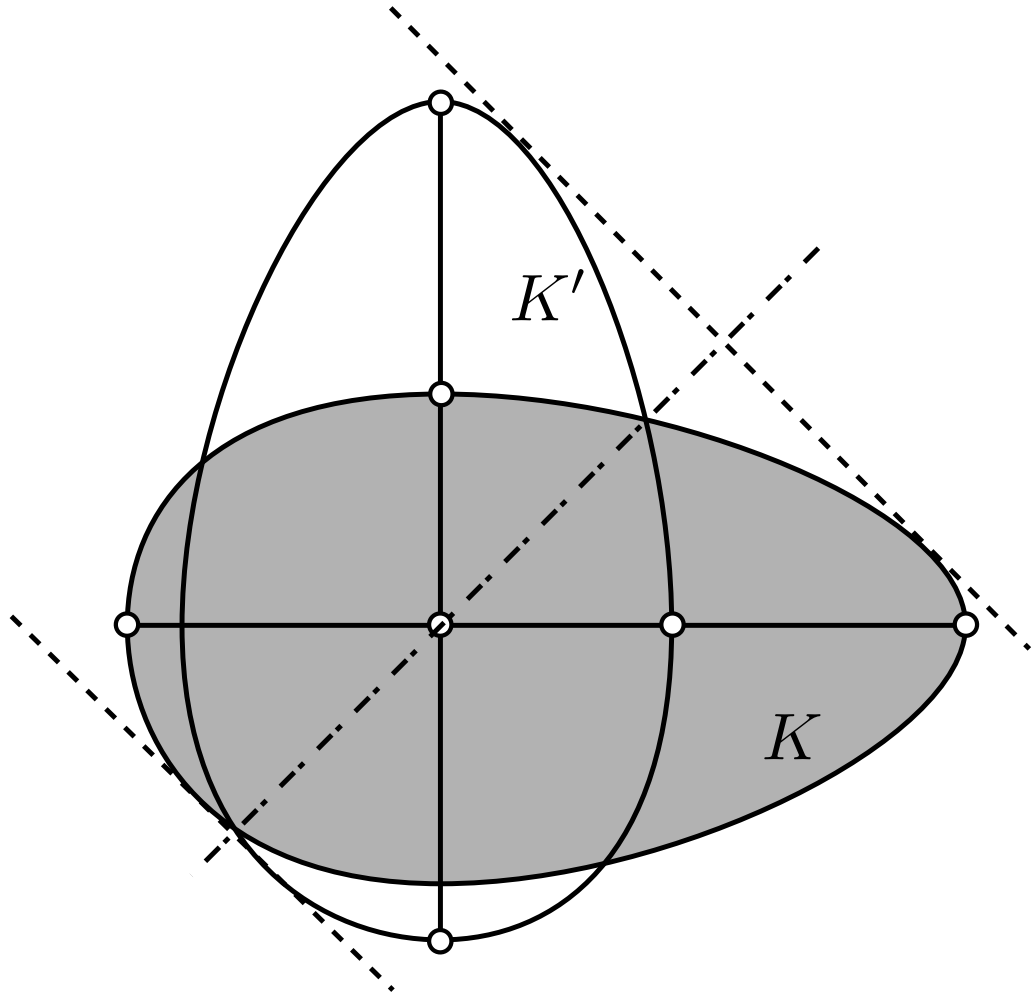
No Corner Lemma Proof: by the Perpendicular Lemma, the opposite of a corner cannot be inside a circular arc.



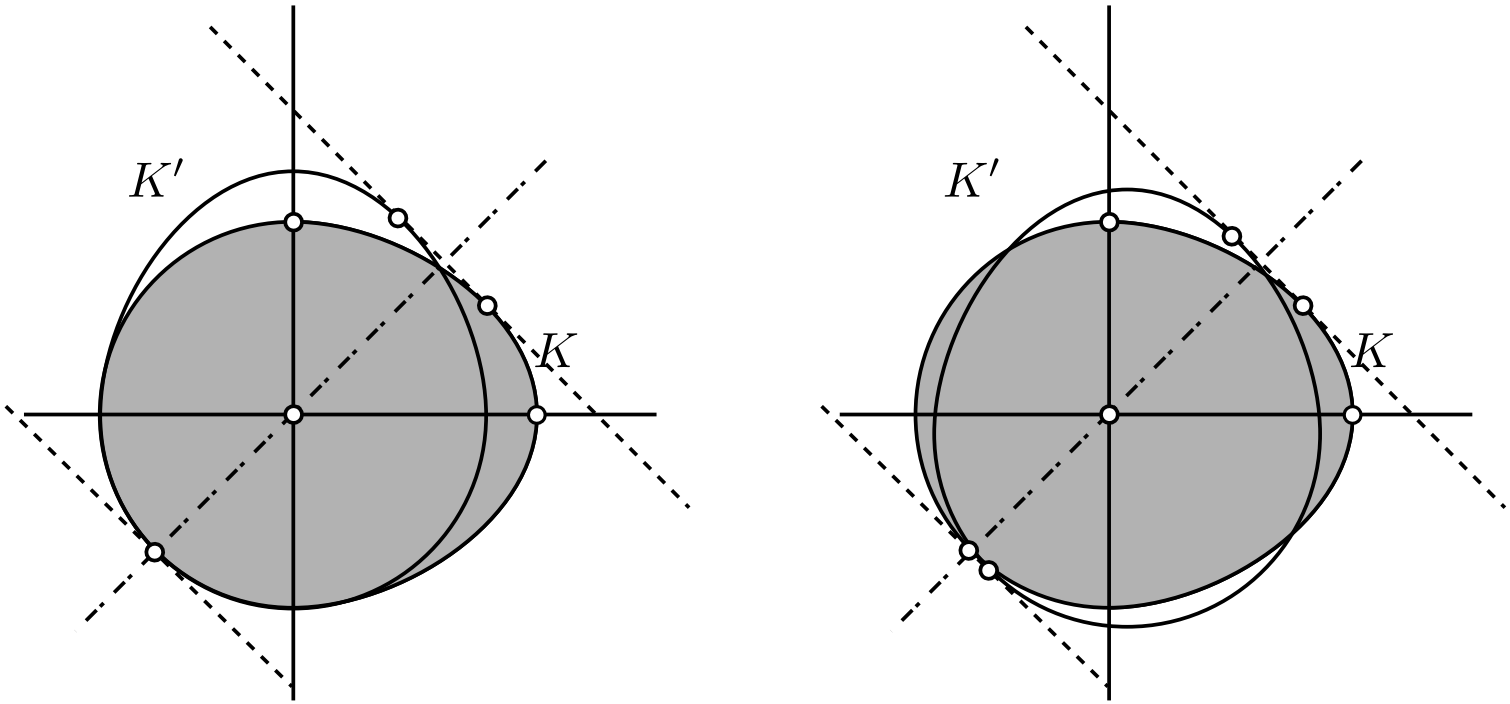
No Corner Lemma Proof cont'd: reflect across a horizontal axis, shift down a little, and apply the Cross Lemma. Q.e.d
Diagonal (def): line segment between two opposite tangent points.



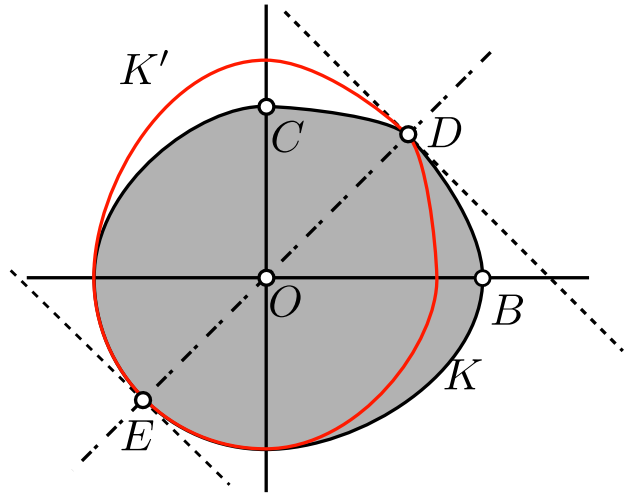
Two Diagonals Lemma: If O is the intersection of two diagonals, then $\text{dist}(O, B) = \text{dist}(O, C)$.



Two Diagonals Lemma Proof/1: if not, then reflect; in lucky case, use Cross Lemma. (Note that this works even if the two diagonals in question are not perpendicular.)



Two Diagonals Lemma Proof/2: A less easy case: excluded by the Perpendicularity Lemma.



Two Diagonals Lemma Proof/3: A difficult case: no similar shift can help: Then the tangents at D and E are kept fixed be the reflecion, DE is also a diagonal. If $\text{dist}(O, E) \neq \text{dist}(O, B)$, then repeat the procedure with the diagonals through OE and OB, and also the diagonals through OE and OC. And so on, keep halving the angles. Either we get a good (or easy) case and the indirect proof becomes complete, or we got a dense set of diagonals through O. Since K is perpendicular to it, we have a slope field (direction field). The circle is a solution of the correspondind ODE, and by a classical uniqueness theorem, there is no other solution.

Adaricheva, K.: Representing finite convex geometries by relatively convex sets. European J. of Combinatorics **37**, 68–78 (2014)

Adaricheva, K.; Bolat, M.: Representation of convex geometries by circles on the plane. arXiv:1609.00092v1

Adaricheva, K.; Czédli, G.: Note on the description of join-distributive lattices by permutations. Algebra Universalis **72**, 155–162 (2014)

Adaricheva, K.V; Gorbunov, V.A., Tumanov, V.I.: Join semidistributive lattices and convex geometries. Advances in Mathematics **173**, 1–49 (2003)

Adaricheva, K.; Nation, J.B.: Convex geometries. In Lattice Theory: Special Topics and Applications, volume 2, G. Grätzer and F. Wehrung, eds., Birkhäuser, 2015.

Adaricheva, K.; Nation, J.B.: A class if infinite convex geometries. arXiv:1501.04174

Armstrong, D.: The Sorting Order on a Coxeter Group. Journal of Combinatorial Theory Series A **116**, 1285–1305 (2009)

Avann, S.P.: Application of the join-irreducible excess function to semimodular lattices. Math. Ann. **142**, 345–354 (1961)

Avann, S.P.: Increases in the join-excess function in a lattice. Math. Ann. **154**, 420–426 (1964)

Czédli, G.: Coordinatization of join-distributive lattices. Algebra Universalis **71**, 385–404 (2014)

Czédli, G.: Finite convex geometries of circles. Discrete Math. **330**, 61–75 (2014)

Czédli, G.: An easy way to a theorem of Kira Adaricheva and Madina Bolat on convexity and circles. http://arxiv.org/abs/1610.02540

Czédli, G.; Kincses, J.: Representing convex geometries by almost-circles. arXiv:1608.06550

Czédli, G., Schmidt, E.T.: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices. Algebra Universalis **66**, 69–79 (2011)

Edelman, P.H.: Meet-distributive lattices and the anti-exchange closure. Algebra Universalis **10**, 290–299 (1980)

Edelman, P.H.: Abstract convexity and meet-distributive lattices. In: Proc. Conf. Combinatorics and ordered sets (Arcata, Calif., 1985). Contemp. Math. vol. **57**, pp. 127–150. Amer. Math. Soc., Providence, RI (1986)

P. H. Edelman; R. E. Jamison: The theory of convex geometries. Geom. Dedicata **19**, 247–271 (1985)

Jamison-Waldner, R.E.: Copoints in antimatroids. Proceedings of the 11th Southeast. Conference, Boca Raton/Florida 1980. Combinatorics, graph theory and computing, vol. II, Congr. Numerantium **29**, pp. 535–544 (1980)

Kashiwabara, Kenji; Nakamura, Masataka; Okamoto, Yoshio: The affine representation theorem for abstract convex geometries. Comput. Geom. **30** 129–144 (2005)

B. Monjardet: A use for frequently rediscovering a concept. Order **1**, 415–417 (1985)

Richter, Michael; Rogers, Luke G.: Embedding convex geometries and a bound on convex dimension. Discrete Mathematics; accepted subject to minor changes; arXiv:1502.01941

Stern, M.: Semimodular Lattices. Theory and Applications. Encyclopedia of Mathematics and its Applications vol. **73**. Cambridge University Press (1999)