Geometric constructibility of cyclic polygons and a limit theorem

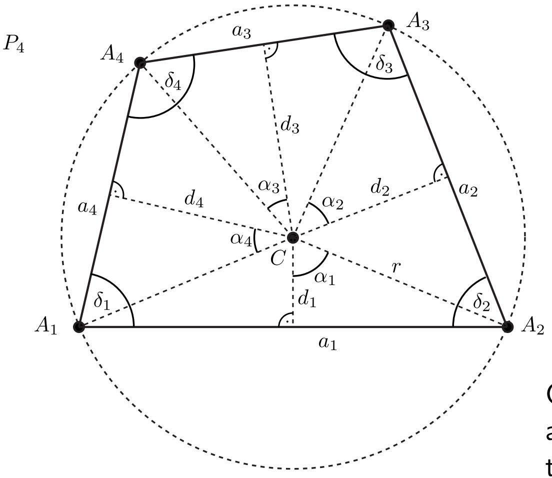
Gábor Czédli and **Ádám Kunos**
(SSAOS, Srní, August 29–September 4, 2015)

!

These slides and the corresponding paper
are available at

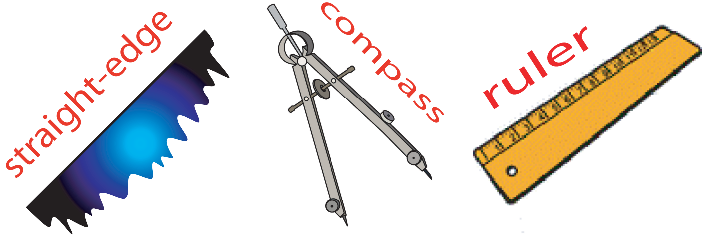
http://www.math.u-szeged.hu/~czedli/

September 3, 2015



Cyclic n-gon $\overset{\text{def}}{\Longleftrightarrow}$ all of its vertices are on the same circle (AKA inscribed polygon)

Constructibility := **geometric constructibility** with straight-edge and compass ("Straight-edge" is more precise than "ruler".)



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Theorem (Schreiber): The cyclic pentagon cannot be constructed from its side lengths.

He also claims: for every $n > 5$, the cyclic n-gon cannot be constructed from its side lengths.

13 line long argument **wrong**

In essence: Suppose P_n is constructible for some $n > 5$. Let $a_n \to 0$, then $P_n \to P_{n-1}$, so P_{n-1} is also constructible. Etc., $P_6 \to P_5$, so the cyclic pentagon is constructible, a contradiction.

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Theorem (Czédli–Kunos, 2015)

- *For $n \geq 5$, the cyclic n-gon is not constructible from its side lengths in general.*
- *Furthermore, for each $n \geq 5$, there exist positive integers $a_1, \ldots, a_n$ such that the cyclic n-gon with these side lengths exists, it is not constructible from $a_1, \ldots, a_n$, and*

$$|\{a_1, \ldots, a_n\}| = \begin{cases} 2, & \text{if } n \neq 6, \\ 3, & \text{if } n = 6. \end{cases}$$

For $n = 5$: Schreiber, 1993. For $n \in \{6, \ldots, 100\}$: Czédli and Ágnes Szendrei, 1997.

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$K = K_0 \subset K_1 \subset \cdots \subset K_m$ quadratic tower

$K^\Box$ is the smallest subfield of $\mathbb{R}$ such that includes K and is closed with respect to real square roots. real quadratic closure

Known: b is constructible from $a_1, \ldots, a_n$ iff b is a real quadratic number over $\mathbb{Q}(a_1, \ldots, a_n)$, that is, iff $b \in \mathbb{Q}(a_1, \ldots, a_n)^\Box$.

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quadratic tower of length k over $F(x)$

Its elements, like f, are just expressions (=strings).

Always, $F \subseteq \mathbb{R}$. ${}^{\star}f$: the real function determined by $f \in F_k\langle x\rangle$ with the largest possible domain (induction!), $\mathrm{Dom}({}^{\star}f) \subseteq \mathbb{R}$.

Note that $f \mapsto {}^{\star}f$ is not injective! For example, $\mathrm{Dom}({}^{\star}(\sqrt{-1-x^2})) = \emptyset = \mathrm{Dom}({}^{\star}(\sqrt{-1-x^4}))$ and ${}^{\star}(\sqrt{-1-x^2}) = {}^{\star}(\sqrt{-1-x^4})$, but $\sqrt{-1-x^2} \neq \sqrt{-1-x^4}$.

Corollary (Corollary 6.4 in the paper)

If $|\mathrm{Dom}({}^{\star}f_1) \cap \mathrm{Dom}({}^{\star}f_2)| = \infty$ *and* $f_1 \neq f_2$, *then* ${}^{\star}f_1 \neq {}^{\star}f_2$.

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Theorem (on "quadratic" Puiseux series (the heart of the paper))

With $F \subseteq \mathbb{R}$, closed w.r.t. to real quadratic roots, and f as before (and above), assume $\exists(0,\varepsilon) \subseteq \mathrm{Dom}({}^{\star}f)$. Then there exist an integer $t \in \mathbb{Z}$ and elements $b_t, b_{t+1}, b_{t+2}, \dots \in F$ such that

$$ {}^{\star}f(x) = \sum_{j=t}^{\infty} b_j \cdot x^{j \cdot 2^{-k}} \tag{1} $$

holds in some strict right neighborhood of 0. Furthermore, if $f \neq 0$, then $b_t \neq 0$ and $t, b_t, b_{t+1}, b_{t+2}, \dots$ are uniquely determined.

We obtain (1) from a Laurant series with finitely many powers of negative exponents, by substituting the complex variable z with $\sqrt[2^k]{x}$. Using an arbitrary $\ell \in \mathbb{N}$ rather than 2^k, we get the concept of a *Puiseux series*, which goes back to Isaac Newton.

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Proof of the "quadratic version of the Newton–Puiseux theorem".

Induction on k (looking for the b_j as unknowns) gives that $\exists$ $b_j \in F$. (The problem reduces to square rooting, because $+$ and $\cdot$ create no problem. At present, only in the ring of formal Puiseux series.)

Substituting $x := y^{2^k}$ and using the uniqueness of Laurent expansions $\Rightarrow$ if we have convergence, then the b_j are uniquely determined.

For convergence: some arguments on holomorphic functions and using the binomial series, namely, for $z \in (-1, 1)$, we have

$$\sum_{j=0}^{\infty} \binom{1/2}{j} x^j = 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$

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Proving part •: for $n \geq 5$, the cyclic n-gon is not constructible.

Take the smallest $n > 5$ such that the cyclic n-gon is constructible. Pick $a_1, \dots, a_{n-1} \in \mathbb{R}$ such that the cyclic $(n-1)$-gon is non-constructible. Let $x = a_n \to 0$ through irrationals numbers. $F := \mathbb{Q}(a_1, \dots, a_{n-1})^{\square}$. Since the cyclic n-gon is constructible "over $F(x)$" and so is its radius, this radius is of the form ${}^{*}\!f(x)$ for some $f \in F_k\langle x \rangle$. (Pigeon hole principle $\Rightarrow$ k and f do not depend on x.) By our "quadratic Newton–Puiseux", ${}^{*}\!f = \sum_{j=t}^{\infty} b_j \cdot x^{j \cdot 2^{-k}}$; with the $b_j \in F$. Since ${}^{*}\!f(x) = {}^{*}\!f(a_n)$ is the radius of the n-gon that tends to the nonzero radius of the $(n-1)$-gon, $\lim_{x \to 0+0} {}^{*}\!f(x)$ exists (and it is positive). Hence, $t = 0$, and this limit is $b_t = b_0 \in F$. But this limit is the radius of the $(n-1)$-gon. Thus, this radius and the cyclic $(n-1)$-gon is constructible, a contradiction. □

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