An association rule motivated by Grätzer and Schmidt

by **Gábor Czédli**

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the best result I will recall is due to
Kalle Kaarli

2006. június 8.

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— Schmidt: my PhD advisor ("Candidate Degree" 1984)

— Schmidt, E. Tamás **Remarks on dependence relations in relational database models. (Hungarian. English summary)** Alkalmaz. Mat. Lapok **8** (1982), no. 1-2, 177–182. 68H05 (68B15)

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K. Kaarli: Finite uniform lattices are congruence permutable: *Acta Sci. Math. (Szeged)*, to appear.

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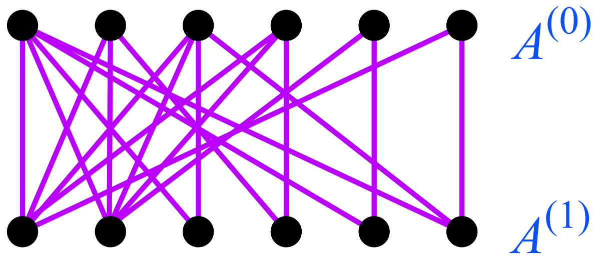
"Association rule" : **Context** (Rudolf Wille, 1982): $(A^{(0)}, A^{(1)}, \rho)$ where $A^{(0)}$ = {objects}, $A^{(1)}$ = {attributes}, $\rho \subseteq A^{(0)} \times A^{(1)}$.

Rows: $A^{(0)}$; columns: $A^{(1)}$

X	X	X		X	X
X	X		X		
X	X	X			X
X	X		X		
	X			X	
X					X

Another possibility
(the same context):



Permanent notations for $(A^{(0)}, A^{(1)}, \rho)$ and $X \in P(A^{(i)})$:

$$\rho_0 = \rho \qquad \rho_1 := \rho^{-1},$$
$$X\rho_i := \{y \in A^{(1-i)} : \forall x \in X,\ (x,y) \in \rho_i\},$$
$$\mathcal{C}_0^{(i)} : P(A^{(i)}) \to P(A^{(i)}),\ \mathcal{C}_0^{(i)}(X) := (X\rho_i)\rho_{1-i},$$

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To express that $b \in \mathcal{C}_0^{(i)}(X)$, **data miners** say that $X \to b$ is a **strong association rule**. *Association rule* is the fuzzy version.

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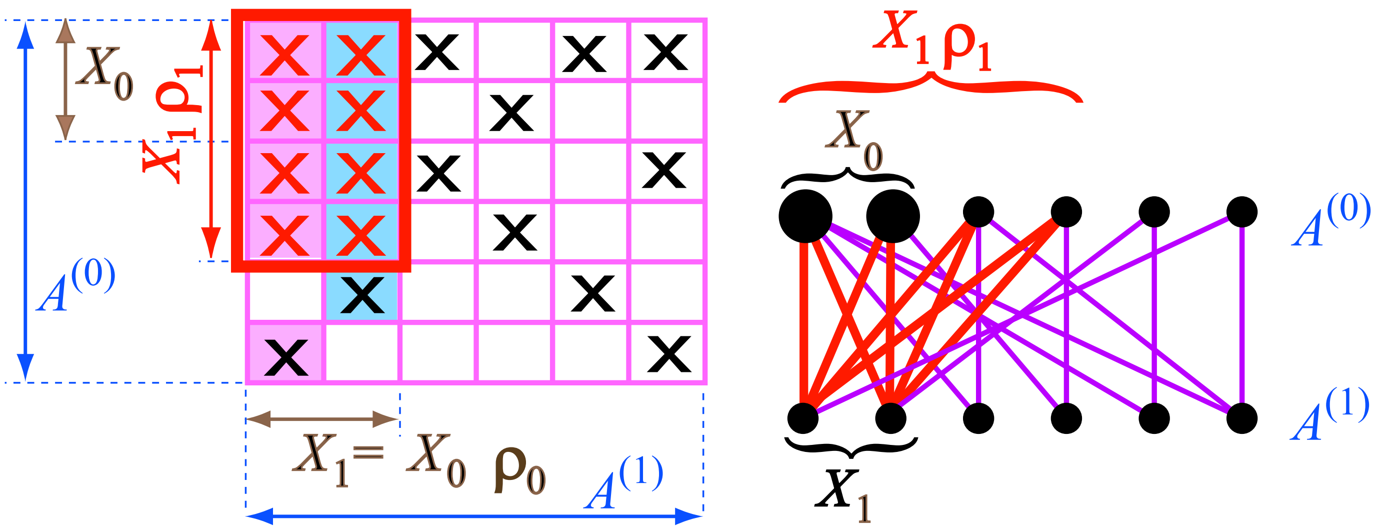
Pictorially, $\mathcal{C}_0^{(i)}(X_0)$, the G–W closure of $X_0 \in P_0(A^{(0)})$, is defined by maximal full rectangles, in other terminology (cf. Wille), by **concepts**:



$$\mathcal{C}_0^{(i)}(X_0) = \bigcap_{X_0 \subseteq S,\ S \text{ is a side of a max. full rectangle}} S\,.$$

Instead of maximal full rectangles we need something else.

For $X \in P(A^{(i)})$ and $Y \in P(A^{(1-i)})$ we say that Y **is a ρ-image of X**

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$$\psi(X) := \{Y \in P(A^{(1-i)}) : Y \text{ is a } \rho\text{-image of } X\} =$$

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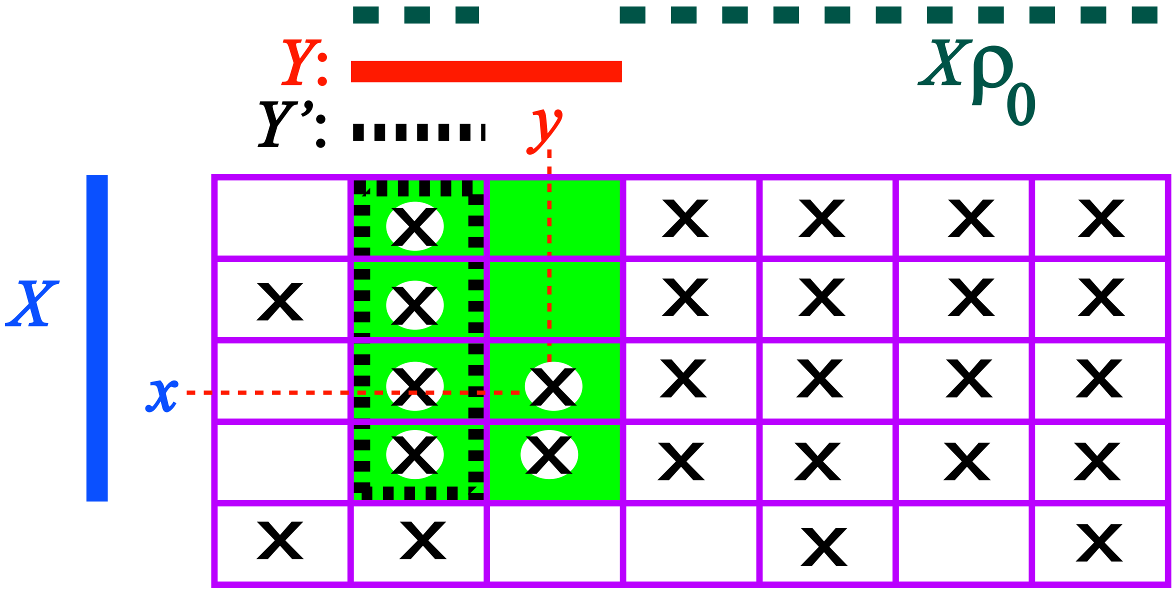
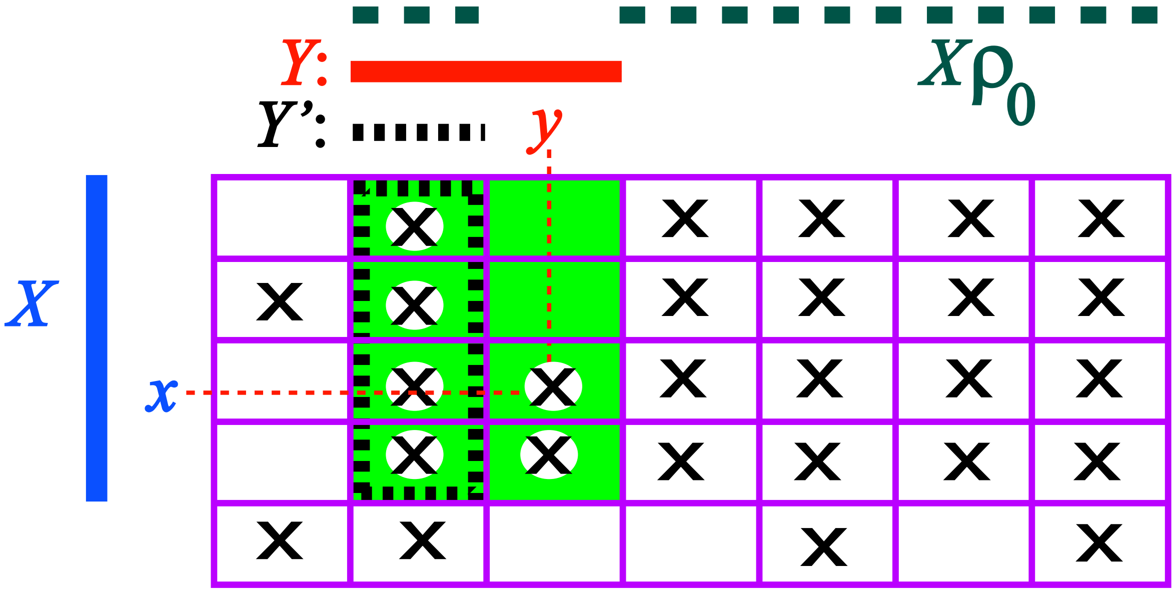
$$\psi_{\text{min}}(X) := \{Y : Y \text{ is minimal in } (\psi(X), \subseteq) \}.$$

Pictorially:



Here $Y \in \psi(X)$, $Y' \in \psi_{\text{min}}(X)$ but



Here $Y \in \psi(X)$, $Y' \in \psi_{\min}(X)$ but $X\rho_0 \notin \psi(X)$. So the terminology „ρ-image" should not be confused with $X\rho_0$!

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$$\mathcal{C}^{(i)}_{n+1}(X) := \mathcal{C}^{(i)}_{n}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}^{(1-i)}_{n}(Y)} \{y\}\rho_{1-i} \, .$$

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In the finite case $\psi(X)$ can be replaced by $\psi_{\mathsf{min}}(X)$.

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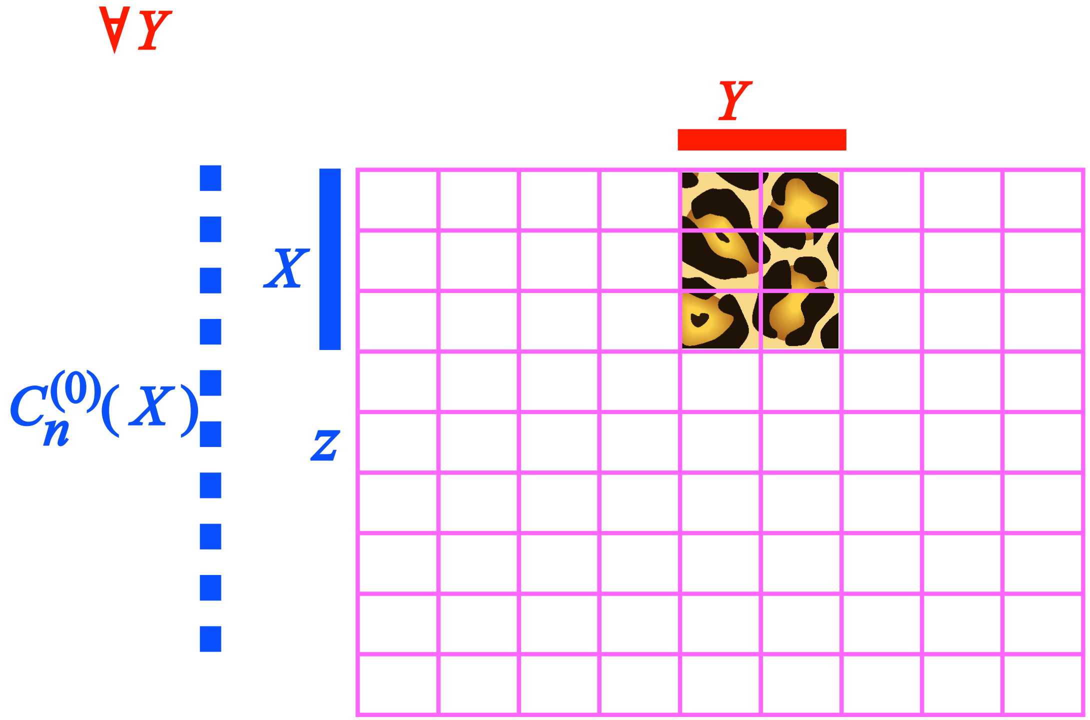
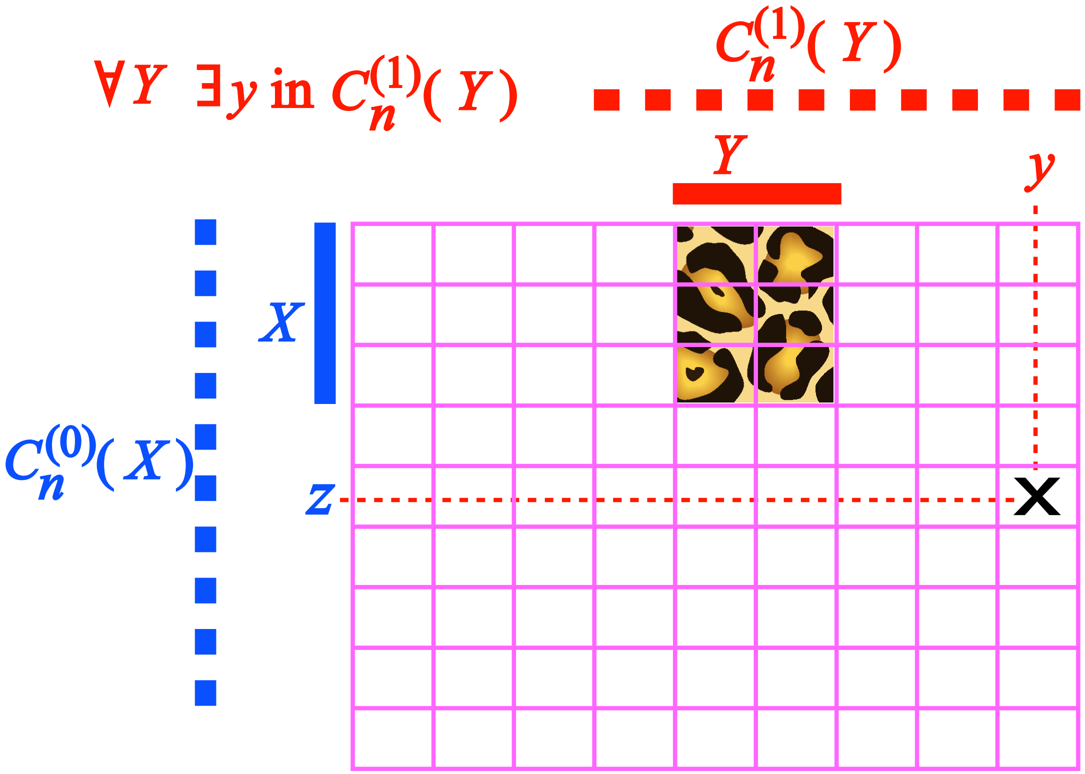
For $i = 0$ we explain this formula pictorially for the finite case. (Not all the crosses will be indicated.)

Let $z \in \mathcal{C}^{(0)}_n(X)$. Then $z \in \mathcal{C}^{(0)}_{n+1}(X)$ iff



for **each** $Y \in \psi_{\text{min}}(X)$



there is a $y \in \mathcal{C}_n^{(1)}(Y)$ with the cross indicated (i.e., $(z, y) \in \rho$).

This

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$$\mathcal{C} = (\mathcal{C}^{(0)}, \mathcal{C}^{(1)}) := (\bigcap_{n=0}^{\infty} \mathcal{C}^{(0)}_n, \bigcap_{n=0}^{\infty} \mathcal{C}^{(1)}_n),$$

which means that, for all $X \in P(A^{(i)})$,

$$\mathcal{C}^{(i)}(X) = \bigcap_{n=0}^{\infty} \mathcal{C}^{(i)}_n(X).$$

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Sketch of the proof. Pictorially. Suppose $\mathcal{C}_n$ is already a pair of closure operators and we already know that $\mathcal{C}_{n+1}^{(0)}$ and $\mathcal{C}_{n+1}^{(1)}$ are **monotone** and **extensive** mappings.

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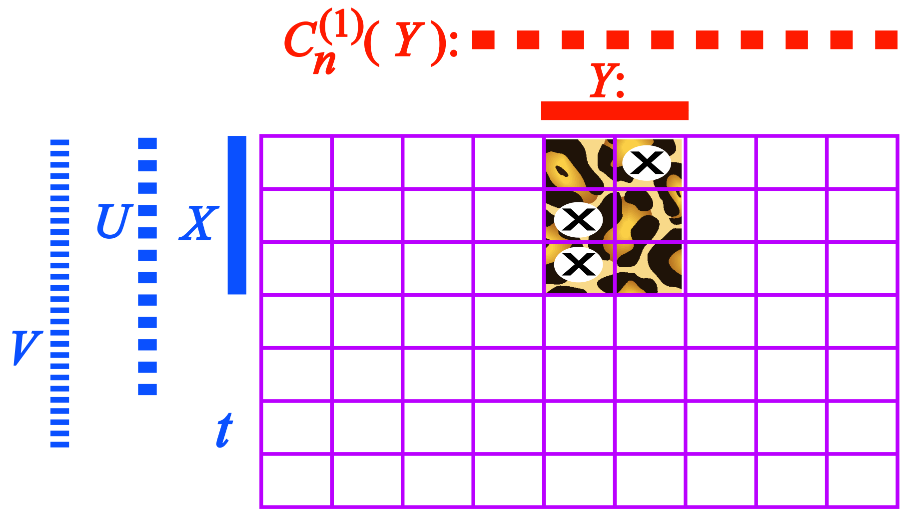
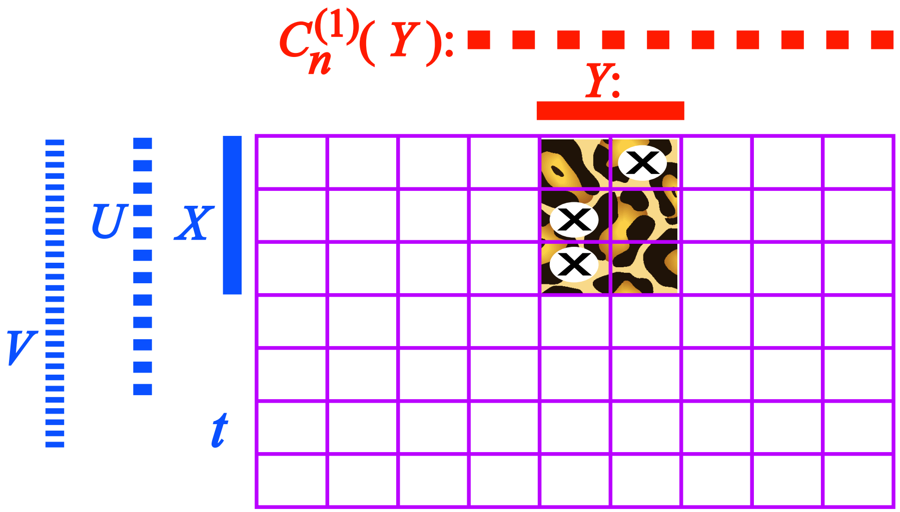
Sketch of the proof. Pictorially. Suppose $\mathcal{C}_n$ is already a pair of closure operators and we already know that $\mathcal{C}_{n+1}^{(0)}$ and $\mathcal{C}_{n+1}^{(1)}$ are **monotone** and **extensive** mappings.

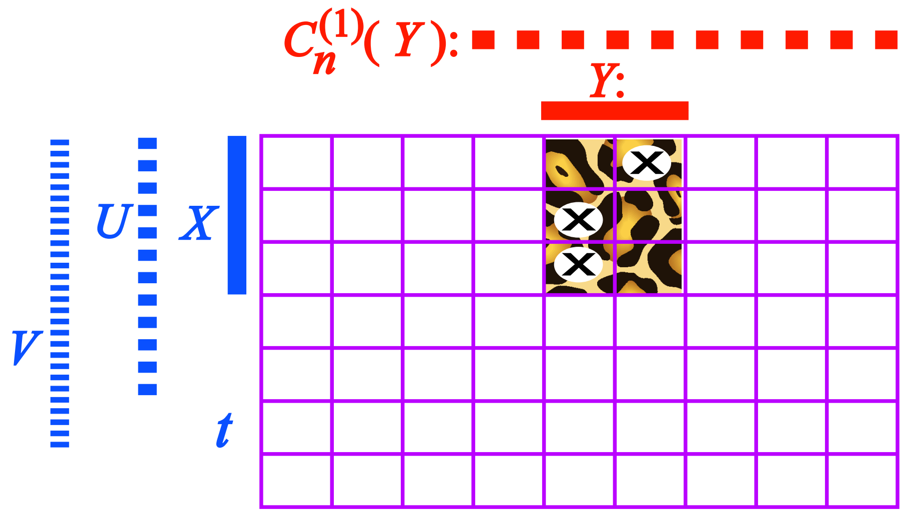
We want: $\mathcal{C}_{n+1}^{(i)}$ is **idempotent**. Let $i = 0$.



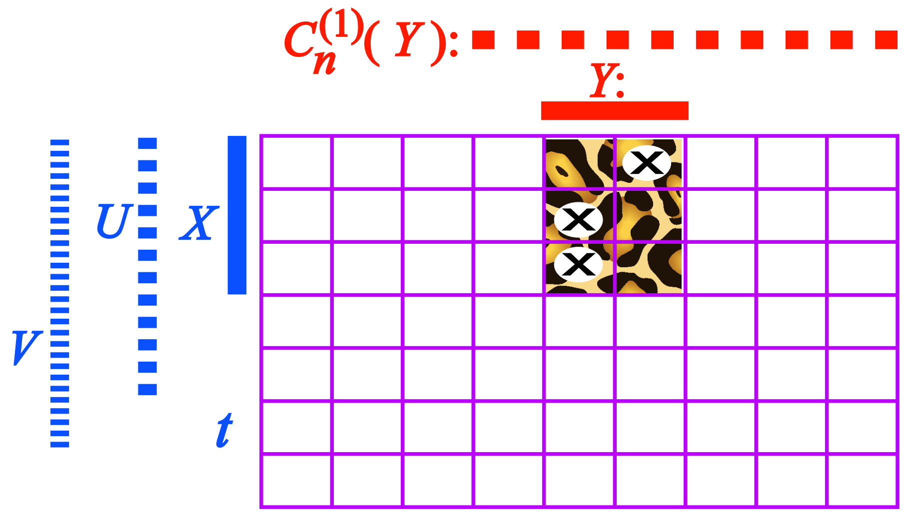
Let $X \in P(A^{(0)})$, $U = \mathcal{C}_{n+1}^{(0)}(X)$ and $V = \mathcal{C}_{n+1}^{(0)}(U)$.



Let $X \in P(A^{(0)})$, $U = \mathcal{C}^{(0)}_{n+1}(X)$ and $V = \mathcal{C}^{(0)}_{n+1}(U)$. We want to show $V \subseteq U$.



Let $X \in P(A^{(0)})$, $U = \mathcal{C}_{n+1}^{(0)}(X)$ and $V = \mathcal{C}_{n+1}^{(0)}(U)$. We want to show $V \subseteq U$. Suppose this is false. Then $\exists\, t \in V \setminus U$.



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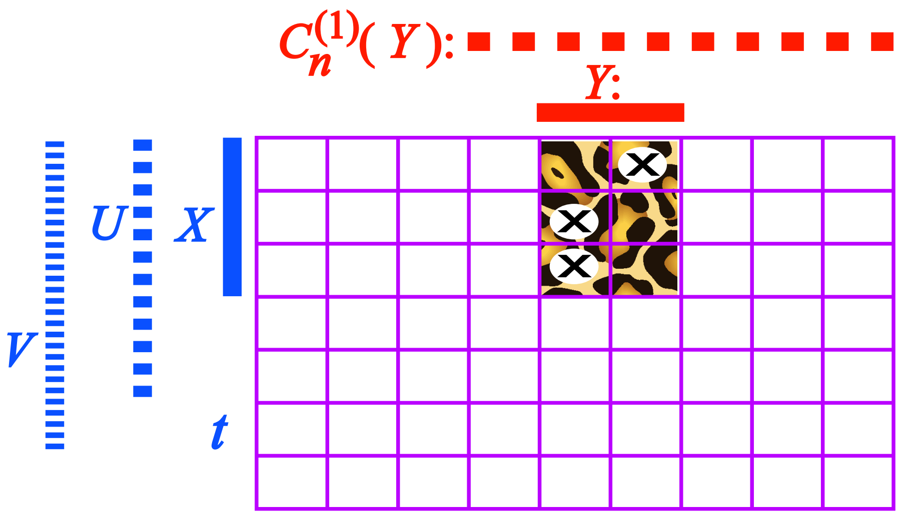
$$U = \mathcal{C}^{(0)}_{n+1}(X) := \mathcal{C}^{(0)}_n(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}^{(1)}_n(Y)} \{y\}\rho_1$$



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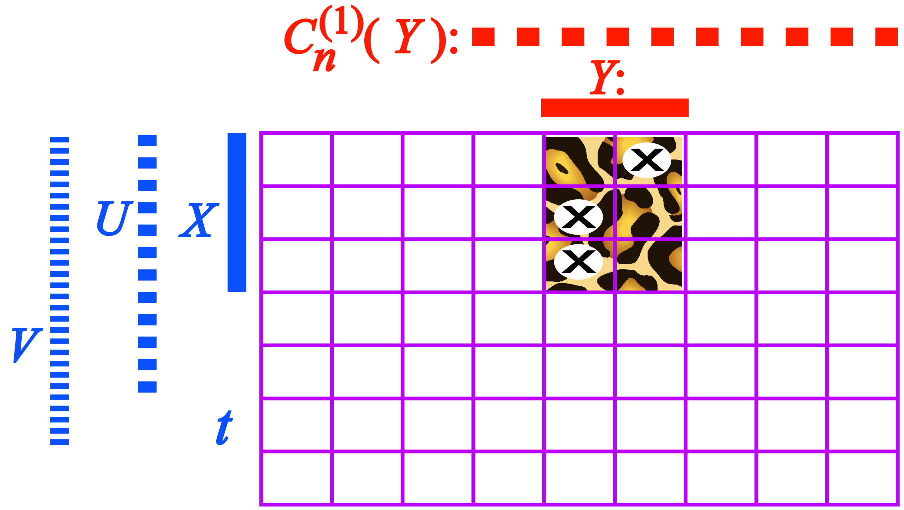

For any $Y \in \psi(X)$ we need a cross in $\{t\} \times \mathcal{C}^{(1)}_n(Y)$.

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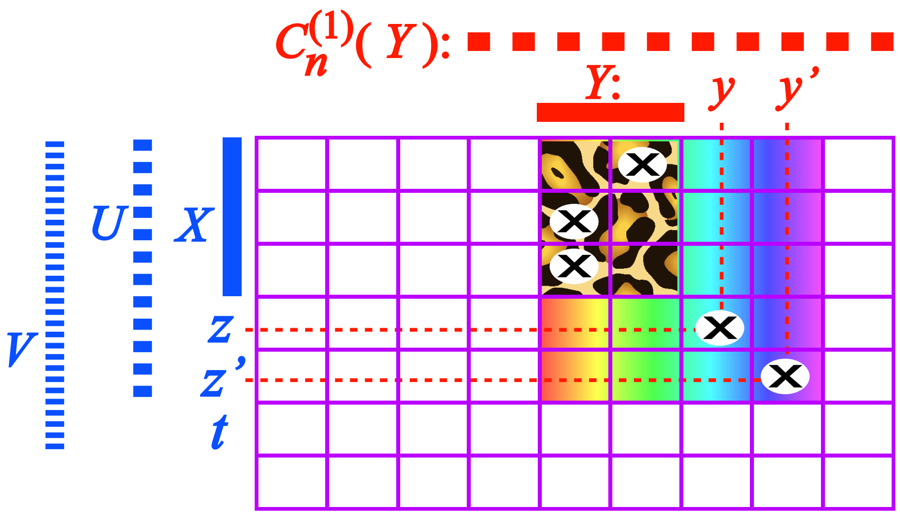

$$\forall z, z', \ldots \in U \setminus X$$

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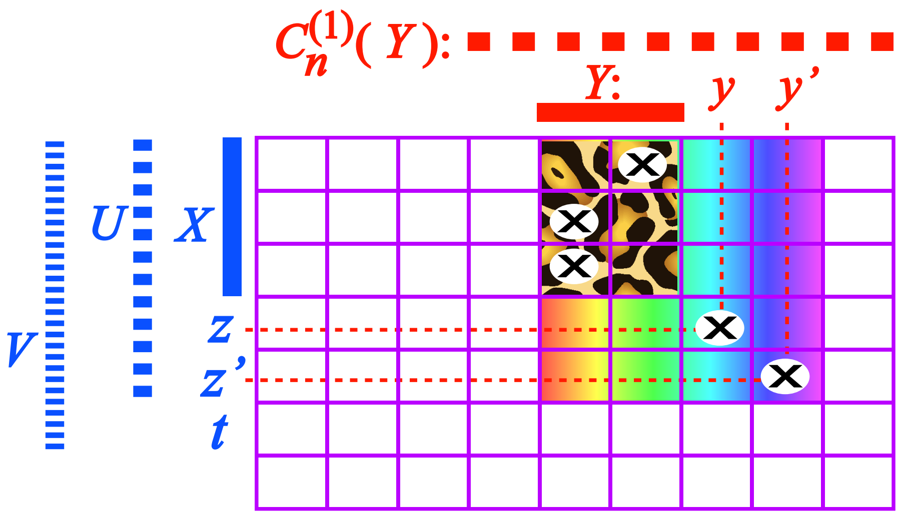

$\forall z, z', \ldots \in U \setminus X \ \exists y, y', \ldots \in \mathcal{C}^{(1)}_{n}(Y)$ by the def. of $U = \mathcal{C}^{(0)}_{n+1}(X)$.

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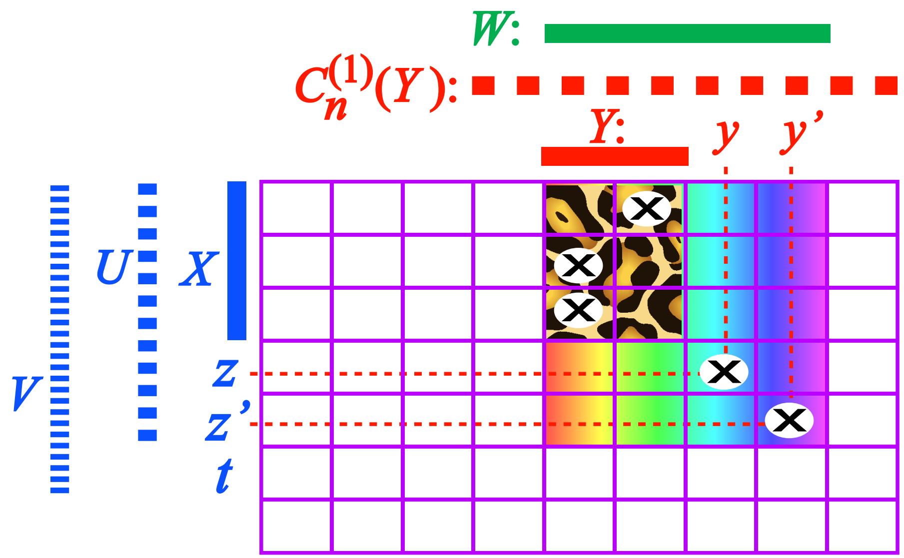

$W :=$

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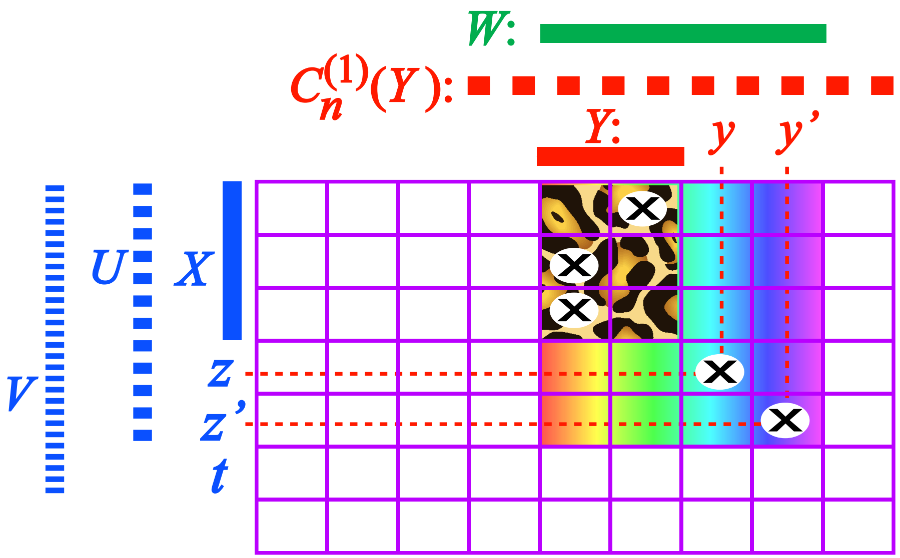

$$W := Y \cup \{y, y', \ldots\}$$

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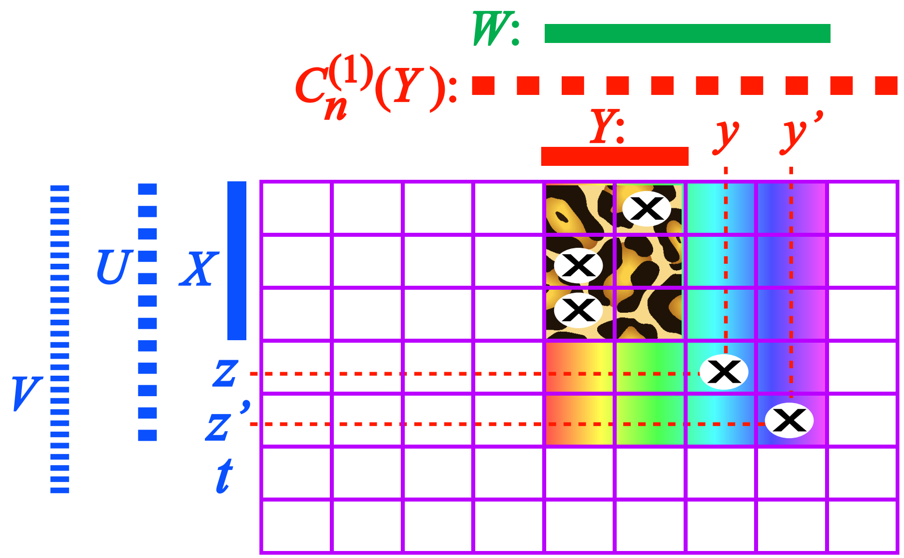
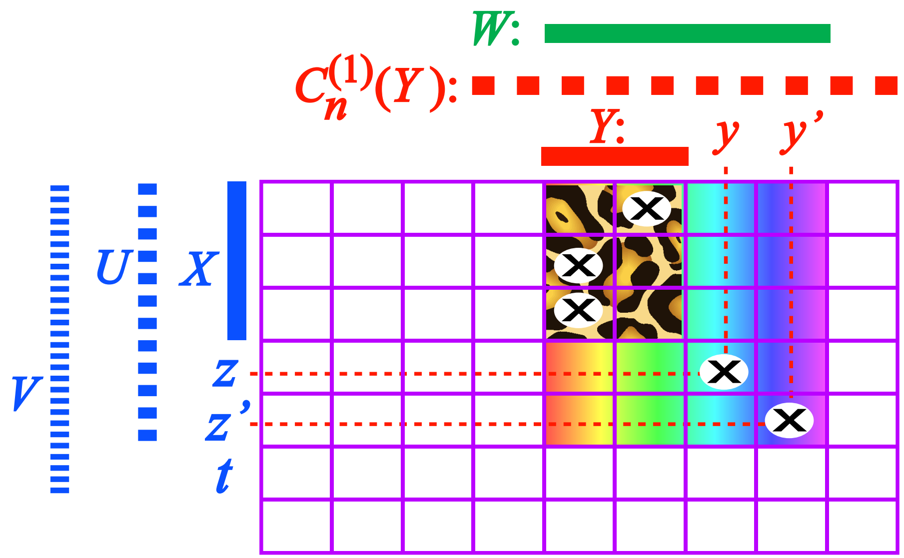


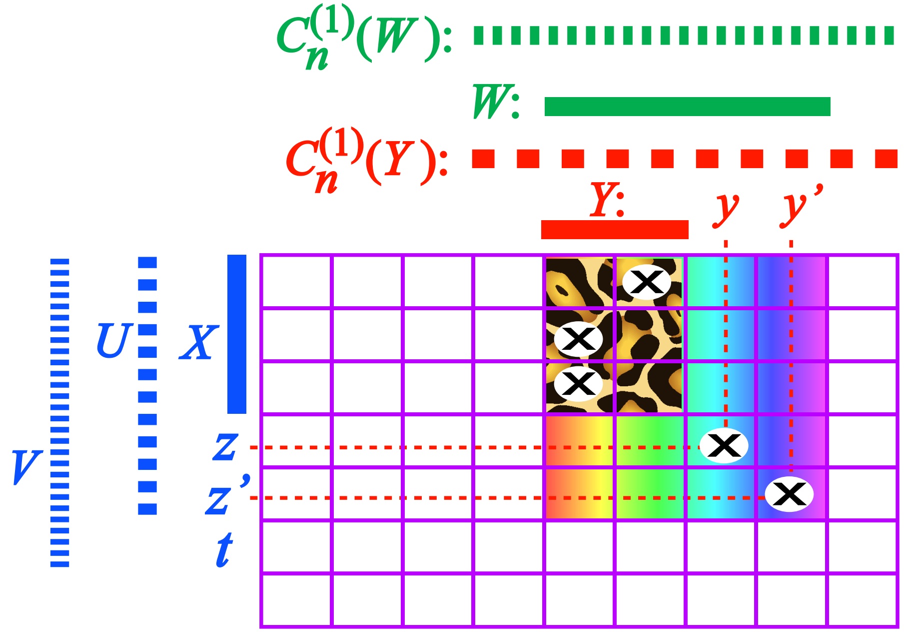
$$W := Y \cup \{y, y', \ldots\} \in \psi(U).$$

$$\mathcal{C}_{n+1}^{(0)}(X) := \mathcal{C}_n^{(0)}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}_n^{(1)}(Y)} \{y\}\rho_1$$



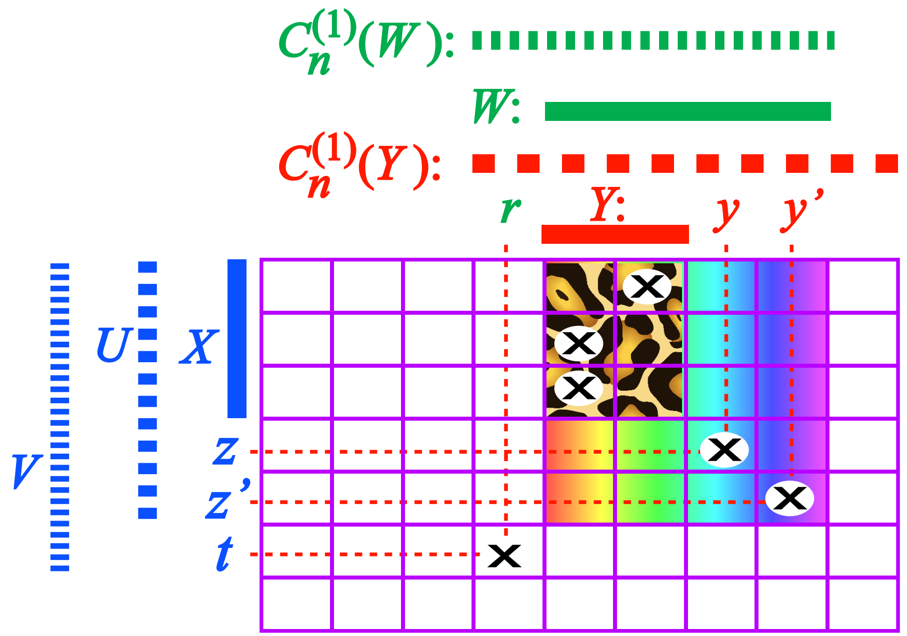
$$W := Y \cup \{y, y', \ldots\} \in \psi(U). \; W \subseteq \mathcal{C}_n^{(1)}(Y) \Longrightarrow \mathcal{C}_n^{(1)}(W) = \mathcal{C}_n^{(1)}(Y).$$

$$V = \mathcal{C}^{(0)}_{n+1}(U) := \mathcal{C}^{(0)}_{n}(U) \cap \bigcap_{W \in \psi(U)} \bigcup_{r \in \mathcal{C}^{(1)}_n(W)} \{r\}\rho_1$$



$W \in \psi(U)$ and the formula above for $W = \mathcal{C}^{(0)}_{n+1}(U)$ yield: $\exists$

$$\mathcal{C}^{(0)}_{n+1}(X) := \mathcal{C}^{(0)}_{n}(X) \cap \bigcap_{Y \in \psi(X)} \bigcup_{y \in \mathcal{C}^{(1)}_{n}(Y)} \{y\}\rho_1$$



$$r \in \mathcal{C}^{(1)}_{n}(W) = \mathcal{C}^{(1)}_{n}(Y).$$

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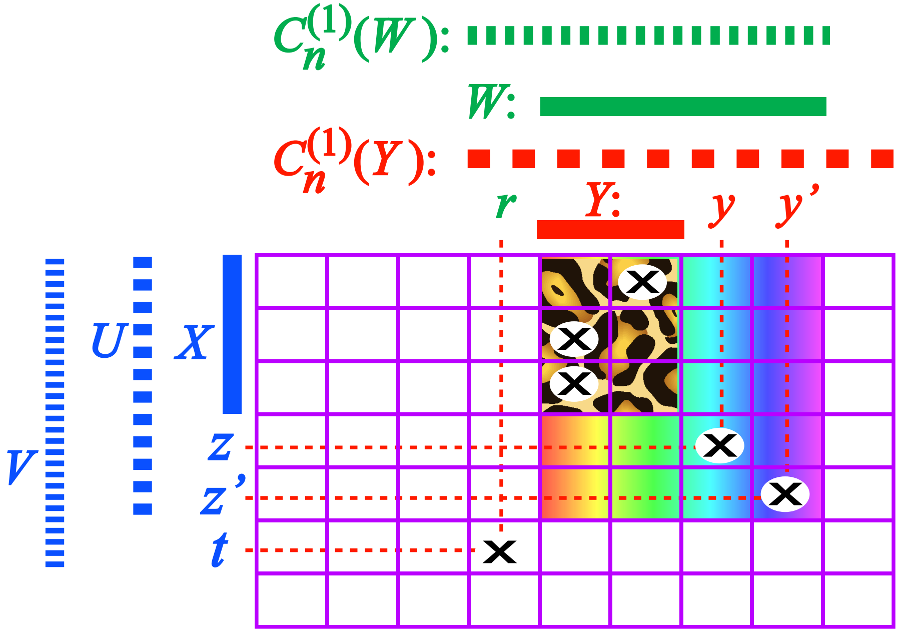

$$r \in \mathcal{C}^{(1)}_n(W) = \mathcal{C}^{(1)}_n(Y).\ \forall Y$$

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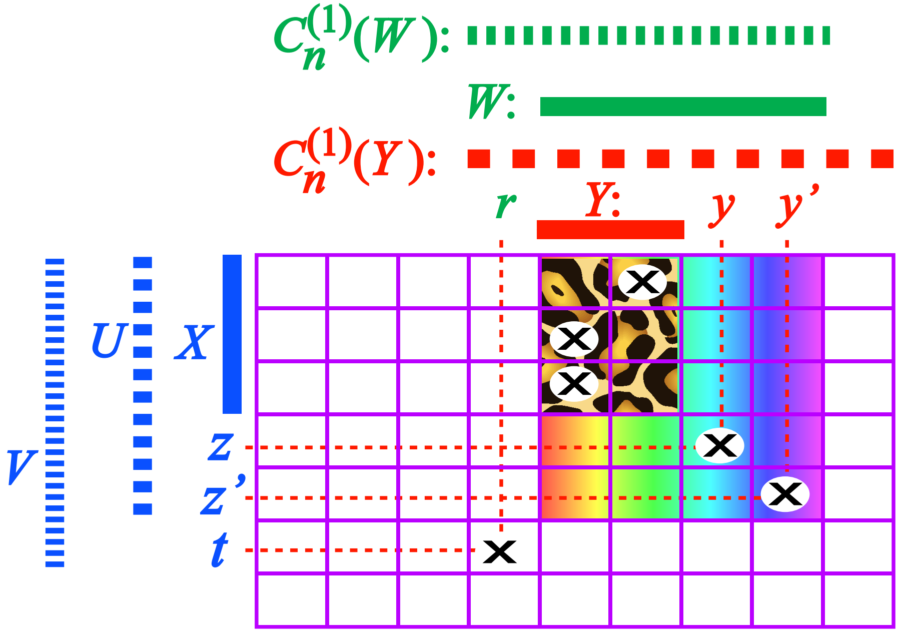

$r \in \mathcal{C}^{(1)}_n(W) = \mathcal{C}^{(1)}_n(Y)$. $\forall Y \Longrightarrow t \in \mathcal{C}^{(0)}_{n+1}(X) = U$. Contradiction.

The omitted details are similar (but easier). **Q.e.d.**

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Now let us see an

Example:

X_0

	B	V	R	H
a	x			x
s	x		x	
r	x			
f	x	x		
m		x	x	x

X_0

	B	V	R	H
a	x			x
s	x		x	
r	x			
f	x	x		
m		x	x	x

Objects: **a**pple-eating, **s**nake, **r**obot, **f**ountain, „**m**ixed”

Attibutes: **b**alls, **v**ertical, **r**esting time, **h**ead touch.

X_0

	B	V	R	H
a	x			x
s	x		x	
r	x			
f	x	x		
m		x	x	x

Question: what can we associate with $X = X_0 = \{s, f\}$? Or: if we can perform **s**nake and **f**ountain, what should we learn next?

X_0

	B	V	R	H
a	×			×
s	×		×	
r	×			
f	×	×		
m		×	×	×

Compute the G–W closure:

X_0

	B	V	R	H
a	X			X
s	X		X	
r	X			
f	X	X		
m		X	X	X

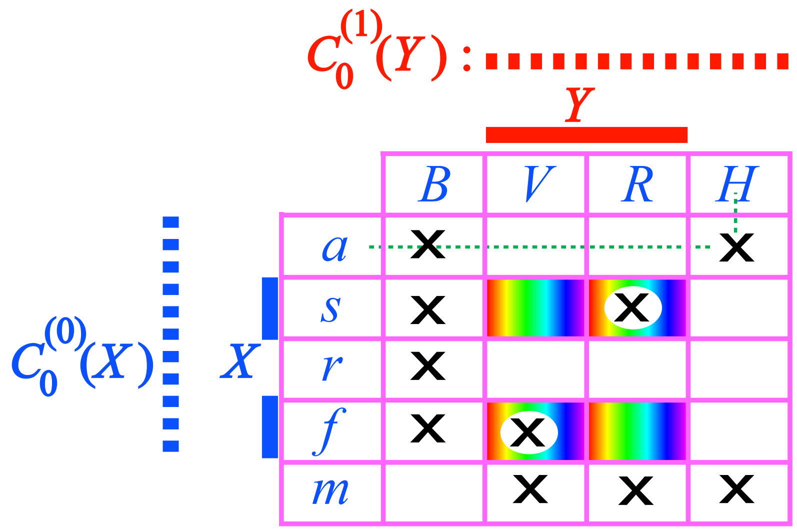
Compute the G–W closure: $\mathcal{C}_0^{(0)}(\{s, f\}) = \{s, f, a, r\}$. This says **a**pple-eating or **r**obot. And we are still hesitating.

X_0

	B	V	R	H
a	x			x
s	x		x	
r	x			
f	x	x		
m		x	x	x

Compute the G–W closure: $\mathcal{C}_0^{(0)}(\{s,f\}) = \{s,f,a,r\}$. This says **a**pple-eating or **r**obot. And we are still hesitating. Calculate $\mathcal{C}_1^{(0)}(\{s,f\})$!

$$\mathcal{C}^{(i)}_{n+1}(X) := \mathcal{C}^{(i)}_{n}(X) \cap \bigcap_{Y \in X\psi_i} \; \bigcup_{y \in \mathcal{C}^{(1-i)}_{n}(Y)} \{y\}\rho_{1-i}$$



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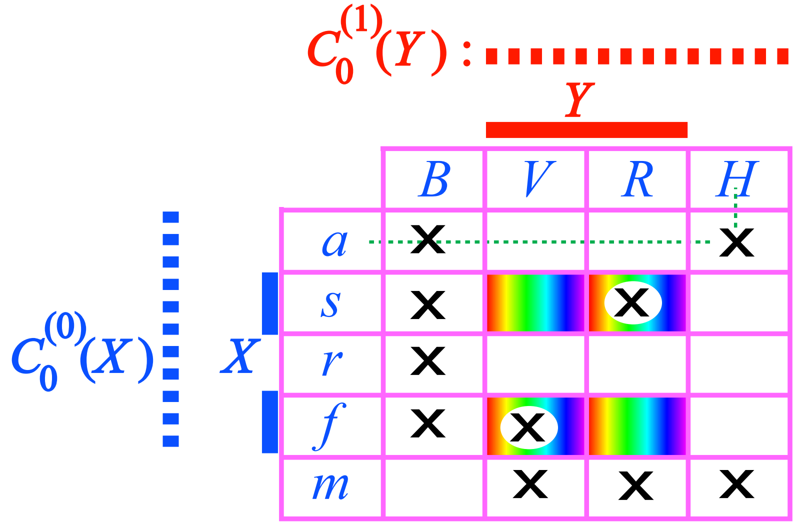

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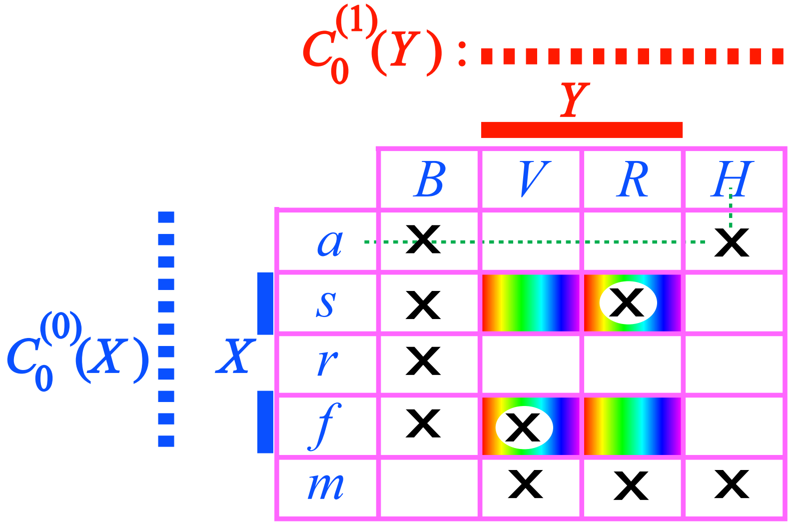

Observe: $a \in \mathcal{C}_1^{(0)}(X)$ but $r \notin \mathcal{C}_1^{(0)}(X)$.

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Thus, in the stronger sense, the **a**pple-eating can be associated with $X = \{$**s**nake, **f**ountain$\}$ but the **r**obot cannot.

So we should choose the **a**pple-eating.

More serious applications

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For lattices we have:

1. Proposition. *If L is a finite* **modular** *lattice and we consider the context $(J, M, \leq)$ then $\mathcal{C}_0$ (the pair of G–W closures) equals $\mathcal{C}$.*

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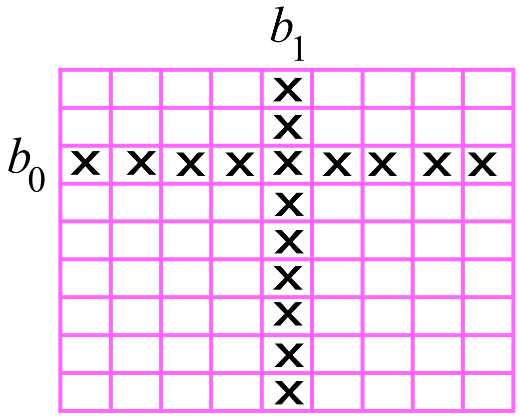
The converse is false. $\Longrightarrow$ **Problem:** Find a better statement.

According to Wille, $(J, M, \leq)$ is interesting.

Since lattices have majority term, the rest of this talk also gives another connection between lattices and $\mathcal{C}$.

In many cases, $\mathcal{C}$ coincides with $\mathcal{C}^{(0)}$. It is trivial that:



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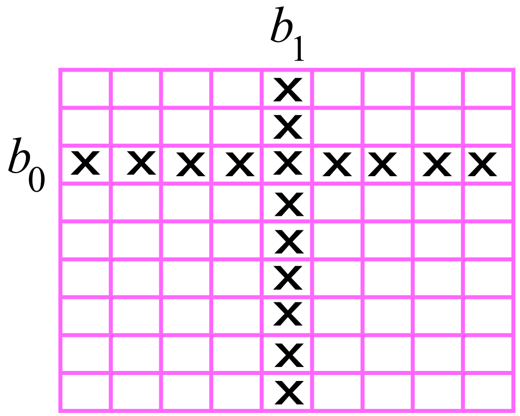

In this case $\mathcal{C}_0 = \mathcal{C}_n = \mathcal{C}$.

Majority term versus $\mathcal{C}$

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A ternary term m of an algebra A is called a **majority term** if

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x$$

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Example: lattices, where $m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$.

2. Proposition. *Let* $A^{(0)}$ *and* $A^{(1)}$ *be algebras with a majority term* m*, and let* $\rho \leq_{sd} A^{(0)} \times A^{(1)}$ *be a subdirect product of them. Then for any* $i \in \{0, 1\}$ *and any* $a, b, c \in A^{(i)}$ *we have*

$$m(a, b, c) \in \mathcal{C}^{(i)}(\{a, b\}) \cap \mathcal{C}^{(i)}(\{a, c\}) \cap \mathcal{C}^{(i)}(\{b, c\}),$$

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1. Corollary. *If* $\rho \leq_{sd} A^{(0)} \times A^{(1)}$ *has a majority term then* **all** *the triangles of* $(A^{(0)}, A^{(1)}, \rho)$ *are degenerate.*

Proof of Proposition 2: Let $a_0, b_0, c_0 \in A^{(0)}$, $u = m(a_0, b_0, c_0)$ and $X := \{a_0, b_0\}$. By symmetry, it suffices to show that

$$u \in \mathcal{C}_n^{(0)}(X), \qquad \text{for all } n.$$

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$$u \in \mathcal{C}_0^{(0)}(X) = (X\rho_0)\rho_1 = \bigcap_{y \in X\rho_0} \{y\}\rho_1,$$

let $y \in X\rho_0$ be arbitrary. I.e., $(a_0, y), (b_0, y) \in \rho_0 = \rho$. Since ρ is a subdirect product, $\exists\ z$ with $(c_0, z) \in \rho$. Then

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shows $u \in \{y\}\rho_1$. This settles the case $n = 0$.

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i.e., $u \in \{y\}\rho_1$. Thus, $u \in \mathcal{C}^{(0)}_{n+1}(X)$. Q.e.d.

Def. $\alpha \in$ Con(A) is called

2-uniform if each α-block is 2-element;

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Grätzer, Quackenbush and Schmidt asked: are finite uniform lattices congruence permutable?

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He also proved that „finiteness" cannot be omitted.

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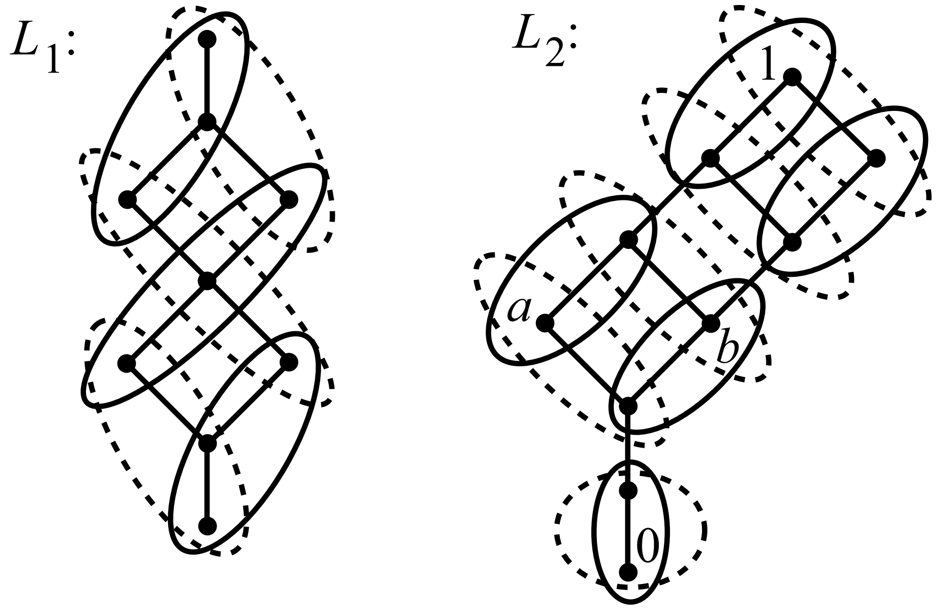
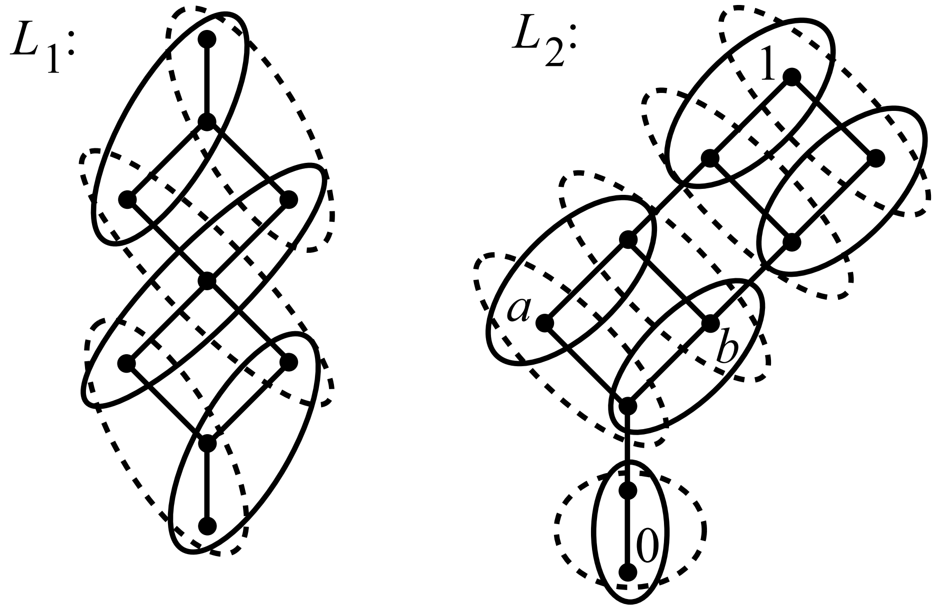
Remarks: (1) But I will prove this one. (2) „Finiteness" is essential. (3) This thm gives something new even for lattices:



L_2 is not a uniform lattice but it does have 2-uniform congruences.



L_2 is not a uniform lattice but it does have 2-uniform congruences. L_1 indicates that we cannot strengthen our result for 3-uniform congruences.

1. Our main theorem. *Let A be a finite algebra and let V denote the variety generated by A. If*

$$\mathbf{1} \notin \mathbf{typ}\, V$$

then any two 2-uniform congruences of A permute.

This is a much stronger statement. The proof uses the previous (weaker) theorem.

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Kaarli/1: for $\alpha, \beta \in \mathrm{Con}(A)$ we may assume that $\alpha \vee \beta = 1$. Otherwise take the $\alpha \vee \beta$ blocks. They are subalgebras by idempotency, and it suffices to show that these subalgebras are CP.

Kaarli/2: We may suppose $\alpha \wedge \beta = 0$. Otherwise we go to $A/(\alpha \wedge \beta)$, and 2-uniformity *on nontrivial blocks* is preserved.

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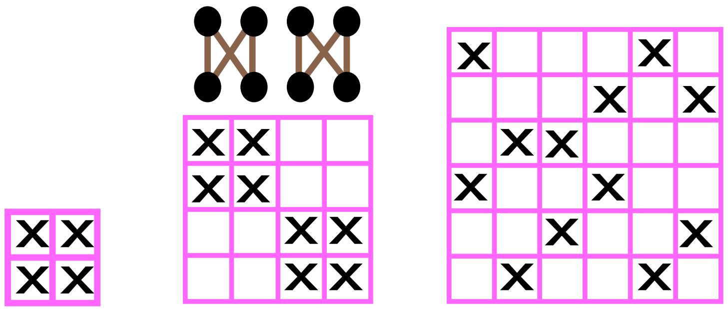
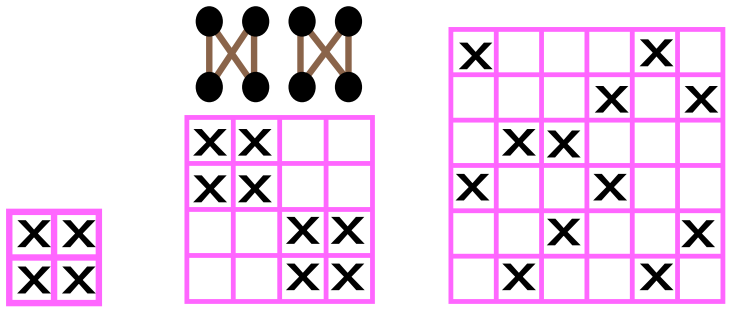
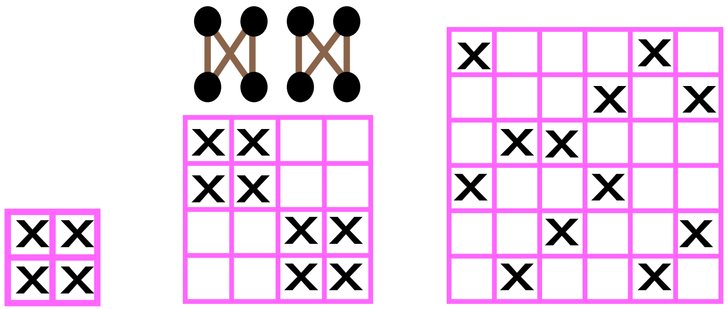
Kaarli/3: $\alpha \wedge \beta = 0$ gives rise to a subdirect product.

These three ideas give easily that only the following proposition needs proving. We call **a context 2-uniform** if each row and each column has exactly two crosses. For example:





Trivial observations: (1) If $|A^{(0)}| = 2$ then CP is evident.



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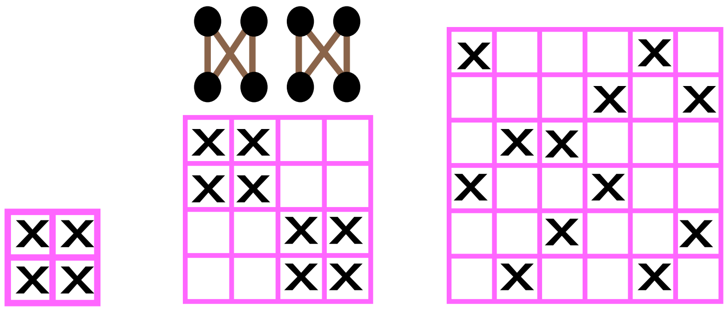
(2) The second one, also given by a bipartite graph, does not correspond to $\alpha \vee \beta = 1$, since the graph **is not connected**. Such contexts are called **decomposable**.



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(3) Only the 3rd one has to be treated. This will be done by

3. Proposition. *Each finite 2-uniform indecomposable context* $(A^{(0)}, A^{(1)}, \rho)$ *with* $|A^{(i)}| \geq 3$ *contains a nondegenerate triangle.*

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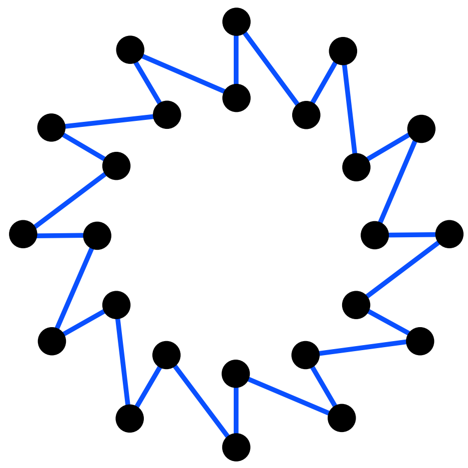
Proof We prove more, namely: $\mathcal{C}$ **is trivial**! Since the context (as a bipartite graph) is 2-regular and connected,it has a Hamiltonian circle. So it can be depicted this way:

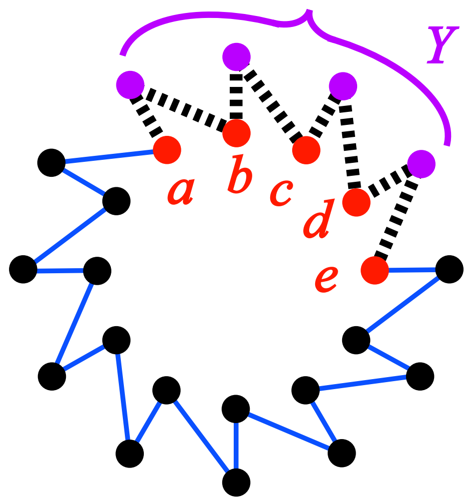


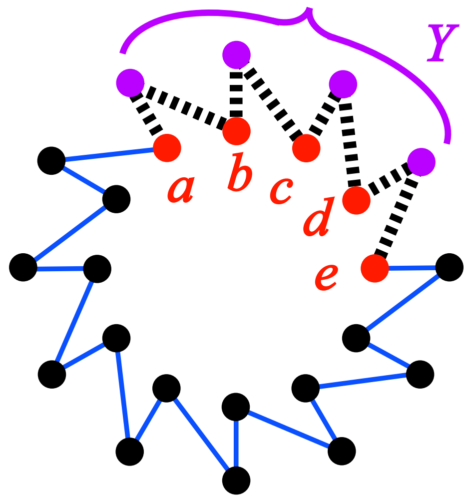
The inner vertices of this „circular saw” constitute $A^{(0)}$ while the outer ones form $A^{(1)}$.

First we show that $\mathcal{C}^{(i)}(X) = X$ for any **arc** in $P(A^{(i)})$. (I.e., $\mathcal{C}$ is trivial on arcs.) Via induction on $|X|$, the length of the arc.

The task is trivial when $|X| < 3$; indeed, then even $\mathcal{C}_0^{(i)}(X) = X$ holds.

Now suppose we have a longer arc, say $X = \{a, b, c, d, e\}$.


Y
a
b
c
d
e



Then $Y \in \psi_{\mathrm{min}}(X)$. Ind.hyp. $\Longrightarrow \mathcal{C}_n^{(1)}(Y) = Y$. Observe that $\{y\}\rho_1 \subseteq X$ for any $y \in Y = \mathcal{C}_n^{(1)}(Y)$. Hence

$$\mathcal{C}_{n+1}^{(0)}(X) \subseteq \bigcap_{Y \in \psi_{\mathrm{min}}(X)} \quad \bigcup_{y \in \mathcal{C}_n^{(1)}(Y)} \{y\}\rho_1 \subseteq X\,,$$

proving $\mathcal{C}^{(0)}(X) = X$. Thus $\mathcal{C}^{(i)}(X) = X$ for any arc $X \in P(A^{(i)})$.

Finally, each $Y \in P(A^{(i)})$ is an intersection of arcs. $\Longrightarrow$

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Using the fact that $\mathcal{C}^{(i)}$ is a closure operator, it is routine to conclude that $\mathcal{C}^{(i)}(Y) = Y$ for all $Y \in P(A^{(i)})$.

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Hence any triangle of three distinct elements is nondegenerate. This proves the proposition and the theorem.

Further open problem: Does any finite k-uniform indecomposable context with $|A^{(i)}| > k$ contain a nondegenerate triangle?

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And another open problem: How far can Kaarli's result be generalized? Majority algebras? CD?

time?

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(In spite of several asymptotic results on the length of the longest circle in the covering graph of the poset

$$(\{X : X \subseteq \{1, 2, \ldots, 2k+1\} \text{ and } |X| \in \{k, k+1\}\}; \subseteq),$$

it is not known if this graph has a Hamiltonian circle.)

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Data miners earn money by investigating the customers' behavior using the context

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given in a huge database. They figure out, e.g., that customers associate sugar with lemon and tee. I.e. when somebody buys lemon and tee then he/she usually buys sugar. I.e., the association rule $\{$lemon,tea$\} \to$ sugar holds. Then the supermarket may reduce the price of lemon and tee while increasing the price of sugar.