Sums and tolerances of lattices*

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http://server.math.umanitoba.ca/homepages/gratzer/

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Traditional look: by a pair of adjoint functors, see, e.g.,

Bandelt, H.-J.; *Płonka sums of complete lattices*, Simon Stevin 55 (1981), no. 3, 169-171.

Graczyńska, G.: *On the sum of double systems of lattices*, Proc. Colloq. Szeged (1975), North Holland 1977.

Graczyńska, G.; Grätzer, G.: *On double systems of lattices*, Demonstratio Math. 13 (1980), 743-747.

Grätzer, G.; Kelly, D.: *Products of lattice varieties*, AU 21 (1985), 33-45.

Romanowska, A.: *Building bisemilattices from lattices and semi-lattices*, Contributions to General Algebra, 2 (Klagenfurt, 1982), 343-358.

Particular case: P. Jedlička: Semidirect product of lattices, AU 57 (2007), 259-272.

A new look at sums [Czédli, *Order* **26** (2009), 309-318.] **:**

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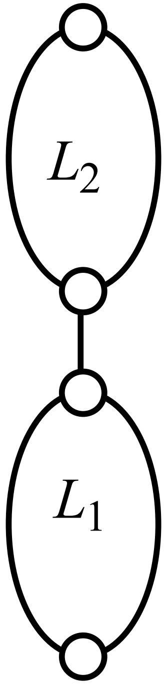
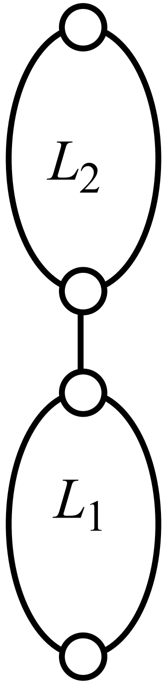
Generally, $\varrho \subseteq L_1 \times L_2$ will be called an **atop relation**, if $((L_1 \times \{1\}) \cup (L_2 \times \{2\}); \leq_1' \cup \leq_2' \cup \varrho')$ is a lattice.

Four examples:

Example 1: $\varrho = L_1 \times L_2$:

Example 1: $\varrho = L_1 \times L_2$: 

(ordinal sum)



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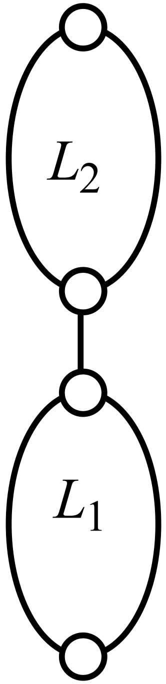
Ex.2: $L_1 = L_2 = L$, $\varrho =$ „$\leq$":



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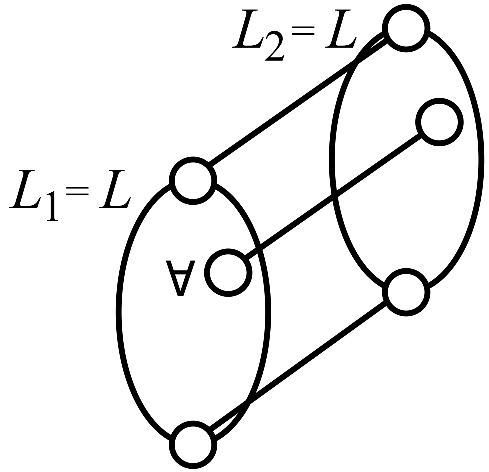
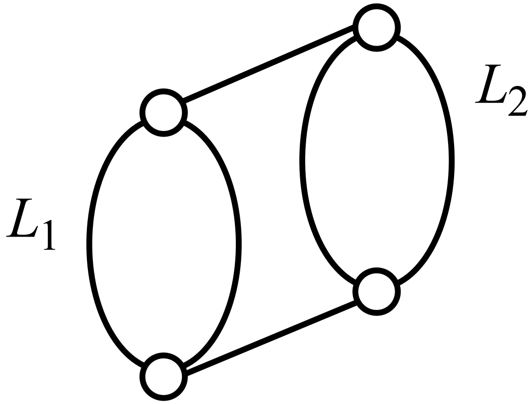


Ex.2: $L_1 = L_2 = L$, $\varrho =$ „$\leq$": $(L \times \mathbf{2})$

Ex.3: $\varrho = (\{0\} \times L_2) \cup \{(1,1)\}$:



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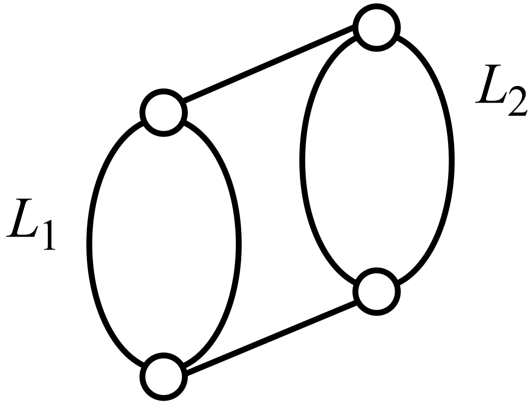

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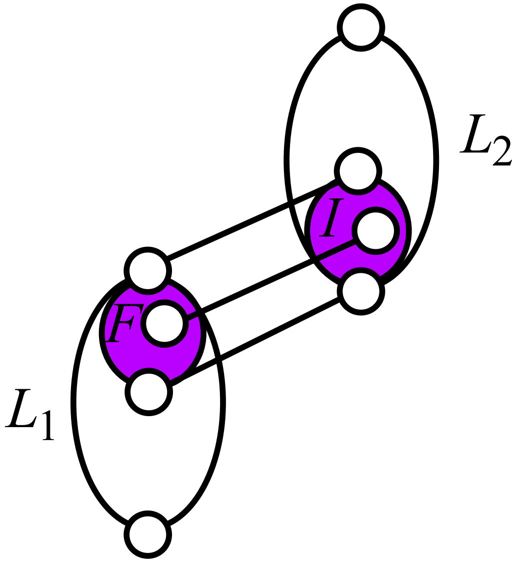

Example 4: $\varphi\colon F \to I$, $(x,y) \in \varrho$ iff $\exists z \in I$ with $x \leq z$ and $\varphi(z) \leq y$, „the Hall-Dilworth gluing without gluing".

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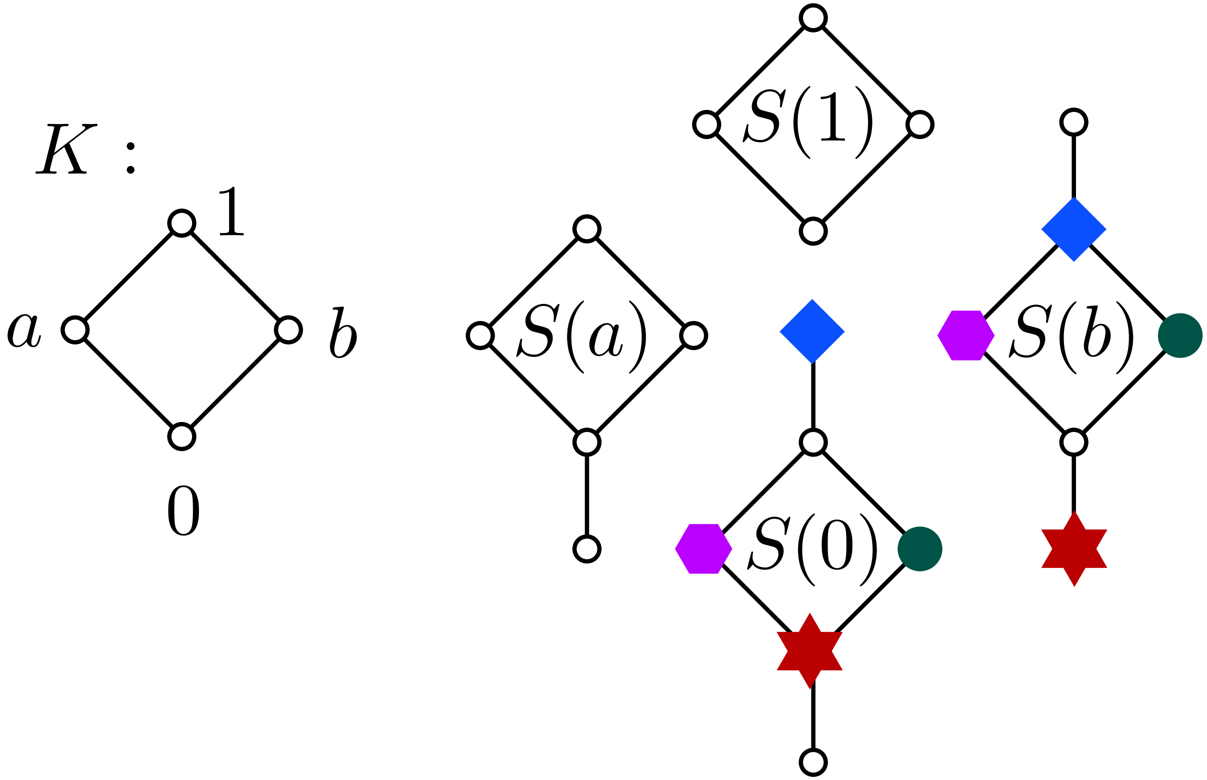
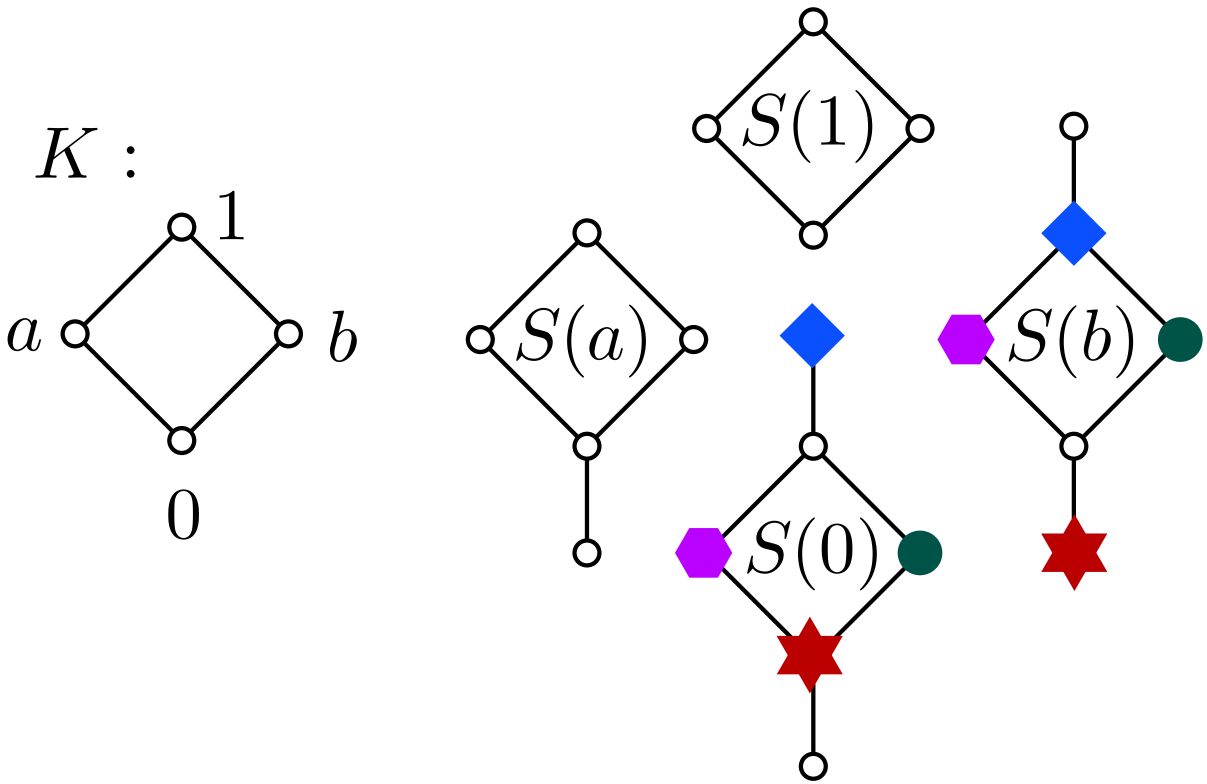
Finite atop systems are exactly the systems that allow forming sums (in the finite case).

No details, only an example of an atop system S:



For example, the edge $0 \prec b$ is a morphism in K.



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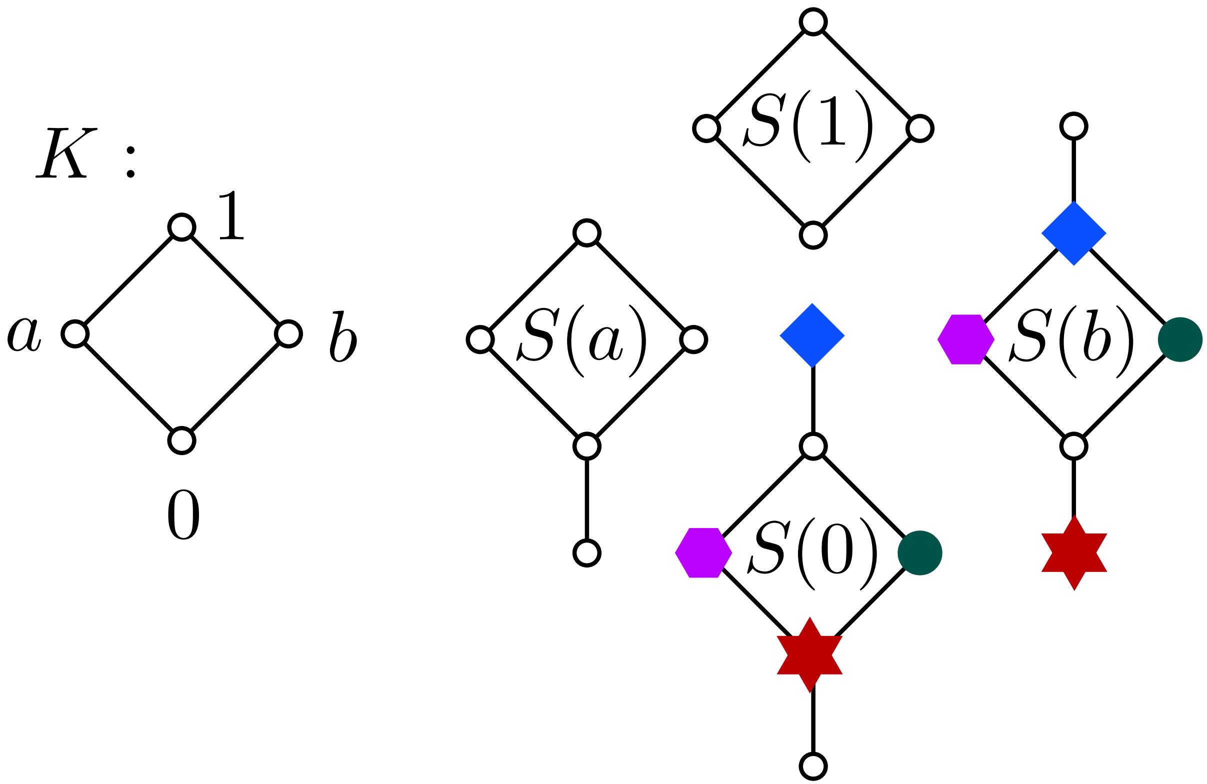
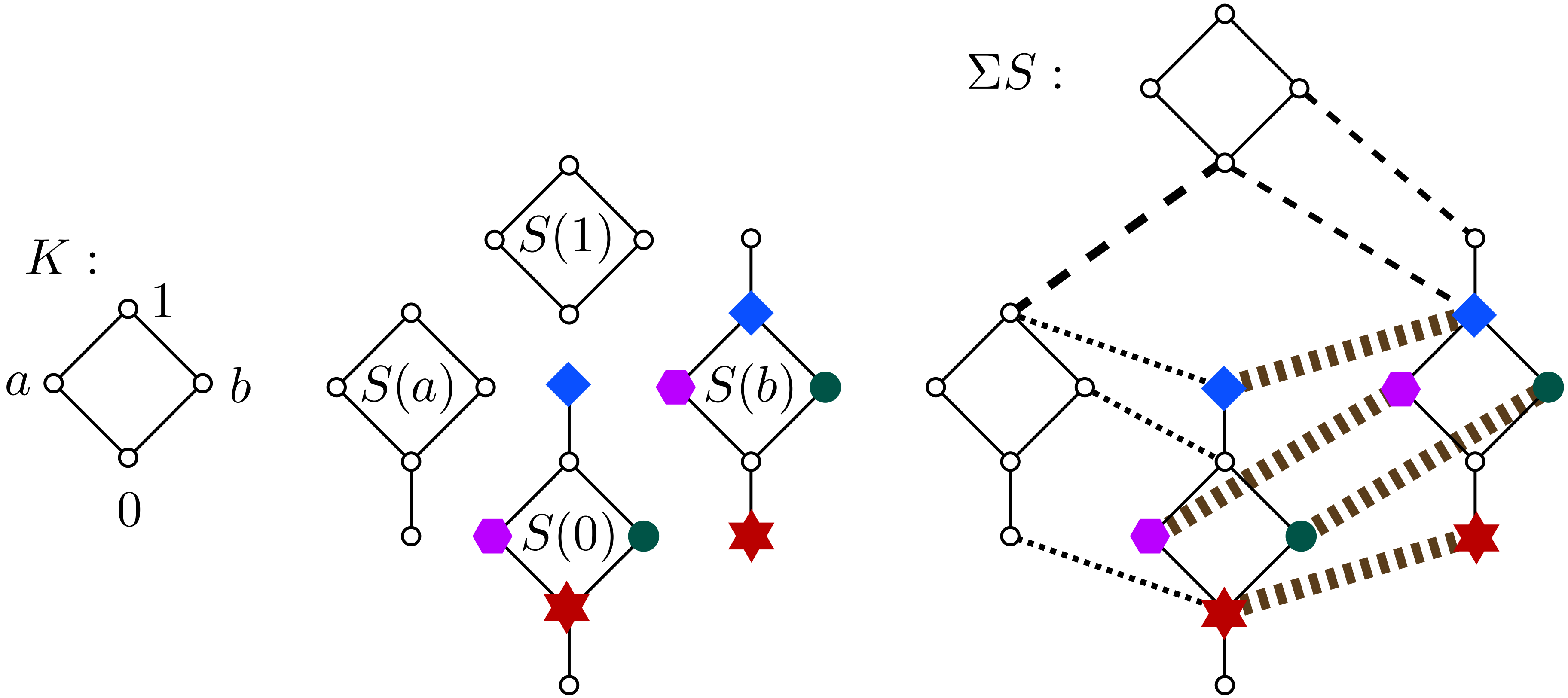


For example, the edge $0 \prec b$ is a morphism in K. S sends it to an atop relation $\varrho \subseteq S(0) \times S(b)$. This ϱ induces a Galois-correspondence, see the color-filled vertices.

By connecting the equally colored vertices, we obtain the sum:



(Make all dotted lines solid.)

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Remark: if Ker(φ) $\subseteq \alpha$, then $\varphi(\alpha)$ is a congruence.

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MAIN THEOREM [Czédli-Grätzer, AU, to appear] For each tolerance ϱ of a lattice L, there is a lattice K, a congruence α of K, and a surjective lattice homomorphism $\varphi\colon K \to L$ such that $\varrho = \varphi(\alpha)$.

Proof:

THM $\forall \varrho$, $\exists$ K, α, and $\varphi : K \to L$ such that $\varrho = \varphi(\alpha)$.

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Connection with sums: K is a sum of the ϱ-blocks.

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Some more references

Herrmann, C.: *S-verklebte Summen von Verbänden*, Math. Z. 130 (1973), 255-274.

Grygiel, J.: *The concept of gluing for lattices*, Czestochowa 2004, 116 pages.

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