Eighty-three sublattices and planarity

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Paper: dedicated to George Grätzer on his 83rd birthday.

Theorem (G. Czédli, 2019; arXiv:1901.00572, http://www.math.u-szeged.hu/~czedli/ 100 pages, sharp!)

Let L be a finite lattice, and let $n := |L|$ denote the number of its elements. If L has at least $83 \cdot 2^{n-8}$ sublattices, then it is planar.

Planar lattice $\iff$ $\exists$ Hasse diagram without crossing edges.

Motivating earlier results (Note: 83 is "many")

- Claudia Mureşan (2017) many ideals (more than filters), https://arxiv.org/abs/1710.10183
- Czédli, Horváth, Ahmed: the five largest $|\mathrm{Sub}(L)|$ for $|L| = n$, arXiv:1812.11512 , Discuss. Math. Gen. Alg. Appl., to appear
- Czédli (AU 80:16, 2019): $|\mathrm{Con}(L)| > 2^{n-5} \Rightarrow L$ is planar.

$$\boldsymbol{\sigma}(\mathcal{A}) := |\mathrm{Sub}(\mathcal{A})| \cdot 2^{8-n}, \quad \text{for a partial algebra.}$$

Theorem (The main theorem reformulated)

If L is a finite lattice such that $\boldsymbol{\sigma}(L) > 83$, then L is planar. Furthermore, for every natural number $n \geq 9$, there exists an n-element lattice L such that $\boldsymbol{\sigma}(L) = 83$ and L is not planar.

" **$\boldsymbol{\sigma}$-few** " means " $\boldsymbol{\sigma}$ is at most 83 ".
" **$\boldsymbol{\sigma}$-many** " means " $\boldsymbol{\sigma}$ is strictly more than 83 ".

Theorem (Yet another formulation)

Finite lattices with $\boldsymbol{\sigma}$-many sublattices are planar.

Lemma (1st lemma)

If $\mathcal{B} = (B, F_B)$ is a (weak partial) subalgebra of a finite (partial) algebra $\mathcal{A} = (A, F_A)$, then $\boldsymbol{\sigma}(\mathcal{A}) \leq \boldsymbol{\sigma}(\mathcal{B})$.

Proof.

Let $n = |B|$, $k = |A| - |B|$. If $X \in \mathrm{Sub}(\mathcal{A})$, then $X' := X \cap B \in \mathrm{Sub}(\mathcal{B})$. $\#X' \leq |\mathrm{Sub}(\mathcal{B})|$. Each X' extends to a subset of A in 2^k ways $\Rightarrow |\mathrm{Sub}(\mathcal{A})| \leq 2^k \cdot \#X' \leq 2^k \cdot |\mathrm{Sub}(\mathcal{B})|$. Dividing this by 2^{k+n}, we get $\boldsymbol{\sigma}(\mathcal{A}) \leq \boldsymbol{\sigma}(\mathcal{B})$. □

If $\mathcal{B} = (B, F_B)$ is a *weak partial subalgebra* of $\mathcal{A} = (A, F_A)$ if $B \subseteq A$, $\mathrm{Dom}(f_B) \subseteq B^2 \cap \mathrm{Dom}(f_A)$ for every $f \in F$ and $f_B(x, y) = f_A(x, y)$ for all $(x, y) \in \mathrm{Dom}(f_B)$.

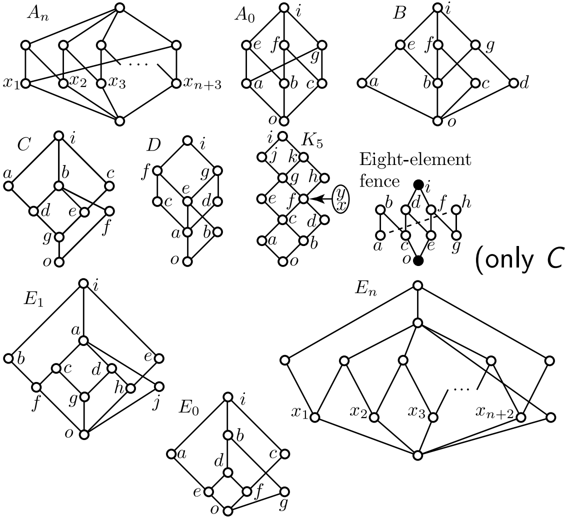
By D. Kelly and I. Rival (1974), a finite lattice L is NON-planar iff one of the following lattices

$$\mathcal{L}_{\mathrm{KR}} := \{A_n, E_n, , F_n, G_n, H_n : n \geq 0\} \cup \{B, C, D\}$$

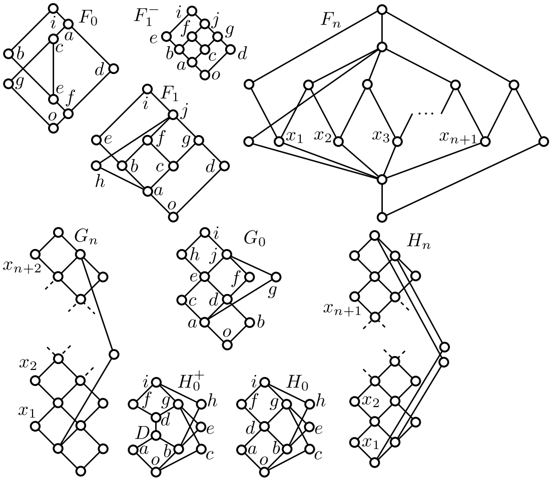
is sitting in L or in the dual of L. We need not care with "dual" since $\mathrm{Sub}(L) = \mathrm{Sub}(\text{dual of } L)$.

The plan is clear:

- First, compute $\boldsymbol{\sigma}(K)$ for each $K \in \mathcal{L}_{\mathrm{KR}}$,
- Second, apply the Kelly–Rival Theorem.



(only C and D)


F_0
F_1^-
F_n
F_1
G_n
G_0
H_n
H_0^+
H_0

Lemma (2nd Lemma; on small Kelly–Rival lattices)

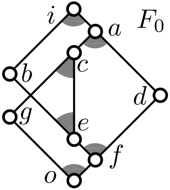
$\boldsymbol{\sigma}(A_0) = 74$, $\boldsymbol{\sigma}(B) = 54$, $\boldsymbol{\sigma}(C) = 68.5$, $\boldsymbol{\sigma}(D) = 76$,
$\boldsymbol{\sigma}(E_0) = 60.5$, $\boldsymbol{\sigma}(F_0) = \underline{\underline{83}}$, $\boldsymbol{\sigma}(G_0) = 54.25$,
$\boldsymbol{\sigma}(H_0) = 49.75$, $\boldsymbol{\sigma}(E_1) = 31.125$, *and* $\boldsymbol{\sigma}(F_1) = 41.125$.

Using fences and K_5, there are arguments for the large Kelly–Rival lattices.

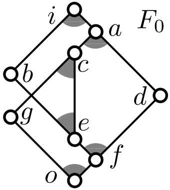
If L is nonplanar, then some $K \in \mathcal{L}_{\mathrm{KR}}$ is sitting in L, whereby our lemmas yield $83 \geq \boldsymbol{\sigma}(K) \geq \boldsymbol{\sigma}(L)$, i.e., L has $\boldsymbol{\sigma}$-few sublattices. Consequently, if L has $\boldsymbol{\sigma}$-many sublattices, then it is planar. ???

WRONG! What does "sitting" mean? In the Kelly–Rival Theorem, "sitting" means "subposet". But the first lemma is about subalgebras; this lemma is not applicable ! Very tedious work is needed for each $K \in \mathcal{L}_{\mathrm{KR}}$. We only outline it for F_0.

Assuming F_0 is a subposet of L; we want to show $\boldsymbol{\sigma}(L) \leq 83$.



$$b \wedge c = e, \; e \vee g = c, \; c \vee d = a, \; d \wedge e = f, \; a \vee b = i, \; f \wedge g = o. \quad (1)$$



(C1): $b \vee c = i$; then $b \vee g = b \vee e \vee g = b \vee c = i$ also holds.
(C1a): $e \wedge g = o$; then $b \wedge g = b \wedge c \wedge g = e \wedge g = o$.
(C1a.1): $d \vee e = a$, then $b \vee d = b \vee e \vee d = b \vee a = i$.
(C1a.1a): $c \wedge d = f$; then $d \wedge g = d \wedge c \wedge g = f \wedge g = o$.
(C1a.1a.1): $a \wedge b = e$; then $b \wedge d = b \wedge a \wedge d = e \wedge d = f$.
(C1a.1a.1b): $f \vee g =: x \neq c$. Then $x < c$ is new and $e \vee x = c$ since $c = e \vee g \leq e \vee x \leq c$. $\boldsymbol{\sigma}$(weak part. subalgebra) $= 74.25$.
Now, $\boldsymbol{\sigma}$(each of the 13 leaves) ≤ 83 ; Lemma 1 $\Rightarrow \boldsymbol{\sigma}(L) \leq 83\surd$.

$K \in \mathcal{L}_{KR}$	B	C	D	E_0	E_1	F_0	F_0	G_0	H_0
$\lvert\{\text{leaves}\}\rvert$	11	12	5	37	5	13	19	24	67

Lattice theoretical considerations and some additional (small) parsing trees are necessary to exclude the "large" members of the Kelly–Rival List.

These slides, the paper, the computer program, and the input and output files are all available from

http://www.math.u-szeged.hu/~czedli/

THANK YOU FOR YOUR ATTENTION!