On the variety generated by planar modular lattices

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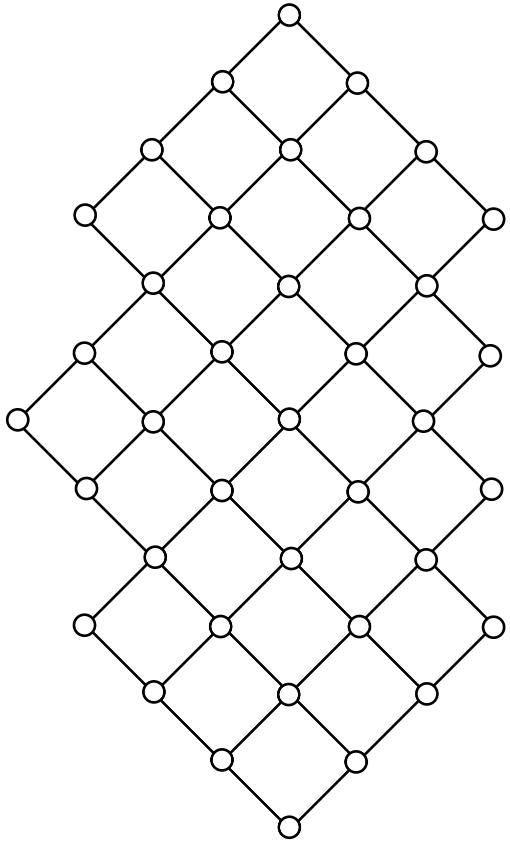
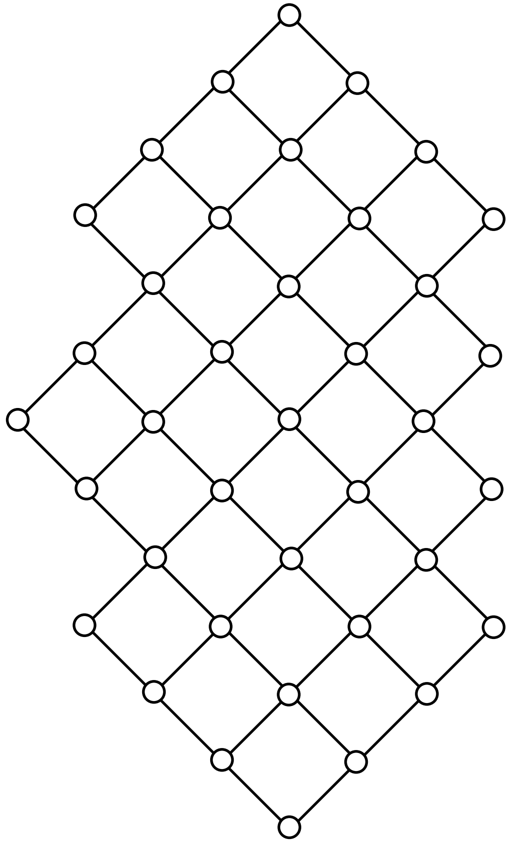
*http://www.math.u-szeged.hu/∼czedli/

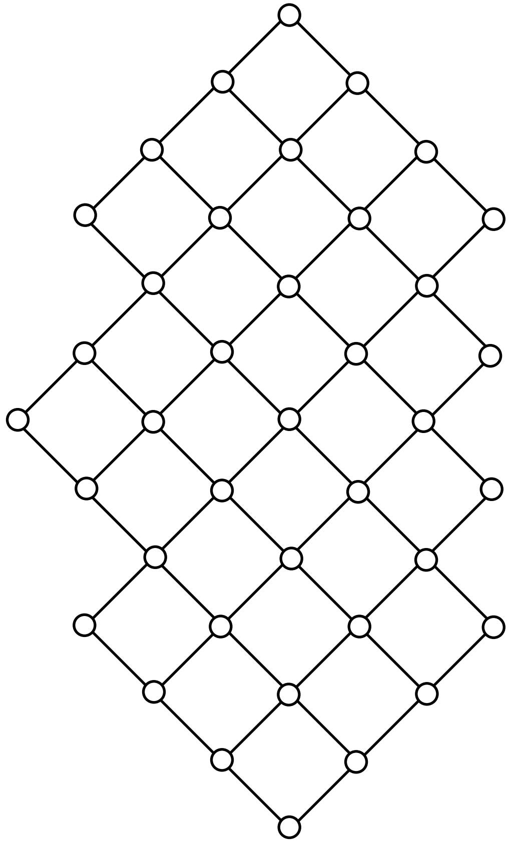
G. Grätzer and R.W. Quackenbush (2008) tells us how to obtain all (finite) planar modular lattices in two steps.

First we take a planar distributive lattice:

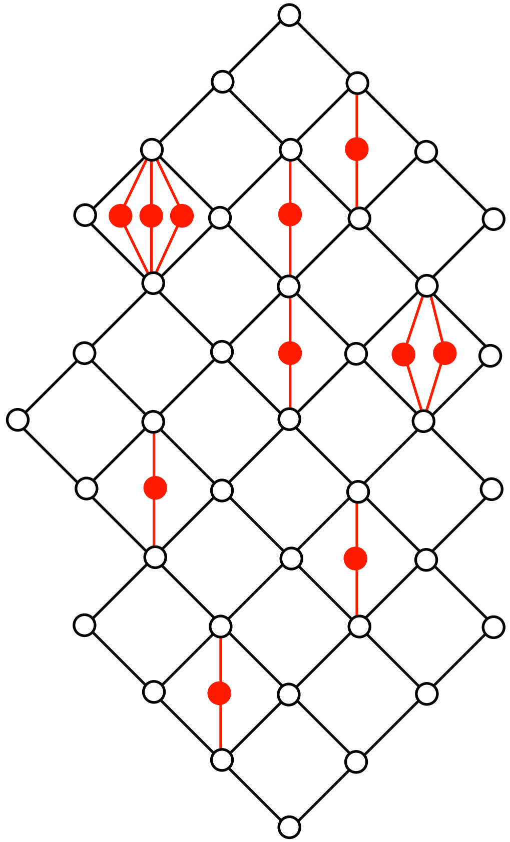


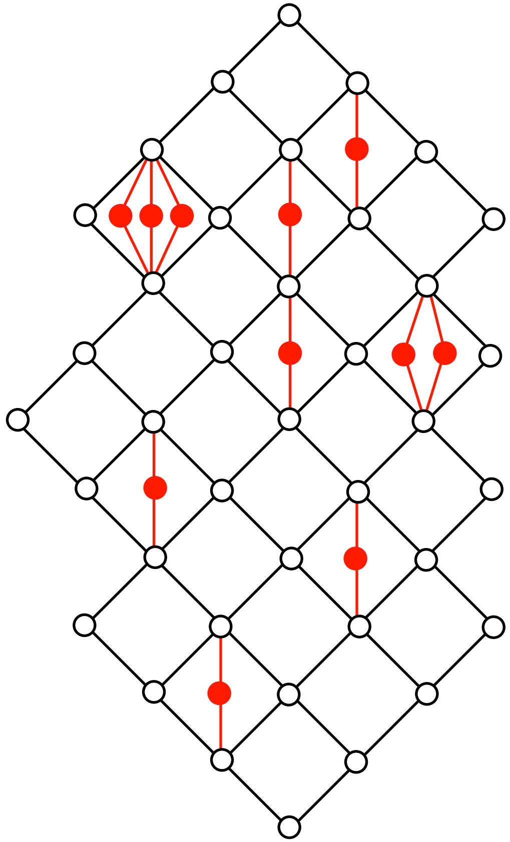


Then we add doubly irreducible elements in some *cells*



Then we add doubly irreducible elements in some *cells* of the grid. (This way some cells change into M_3, M_4, etc.)





Let $\mathcal{V}$ denote the variety generated by planar modular (i.e, by these) lattices.

1. Theorem. *(Ralph Freese 1972. Kirby Baker a bit sooner.)*
$\mathcal{V}$ *has continuum many subvarieties.*

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Our contribution: a new, very elementary proof.

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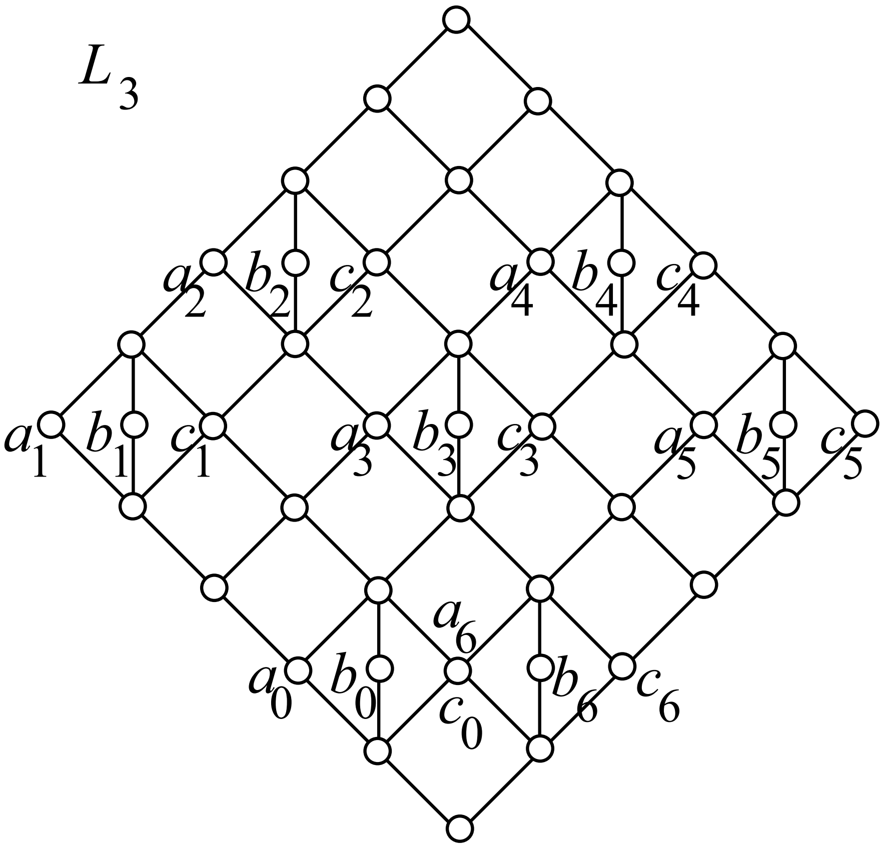
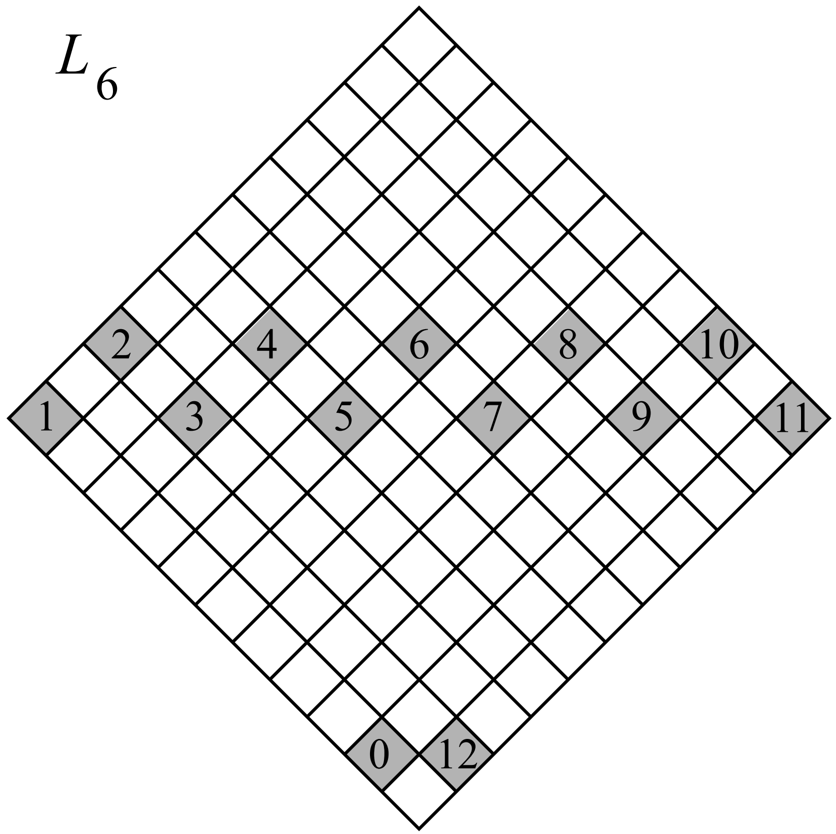
Our lattices, the L_n, are here:





There are $2n$ copies of the diamond, M_3.

For $\vec{u} = (x, y, z) \in L^3$, define the **diamond terms**

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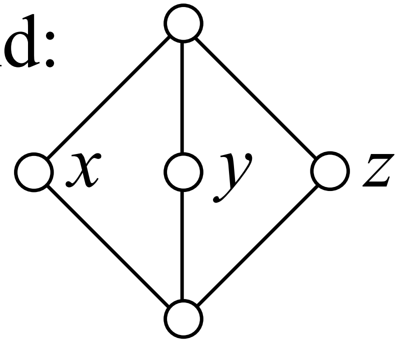
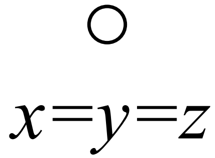
diamond:



singleton:



$$\vec{\mathbf{d}}(\vec{u}) = ((x + yz)(y + z)\,,$$

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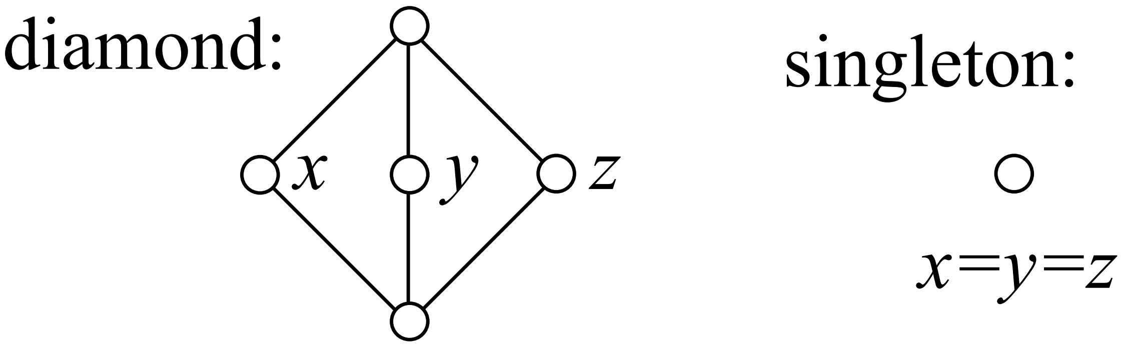

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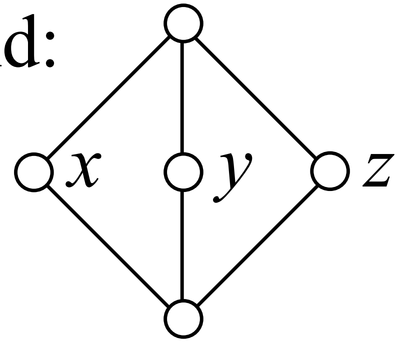

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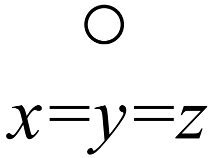

$$\vec{\mathbf{d}}(\vec{u}) = ((x + yz)(y + z)\,,(y + xz)(x + z)\,,(z + xy)(x + y)). \quad (1)$$

Purpose: $\vec{\mathbf{d}}(\ \ )$ should

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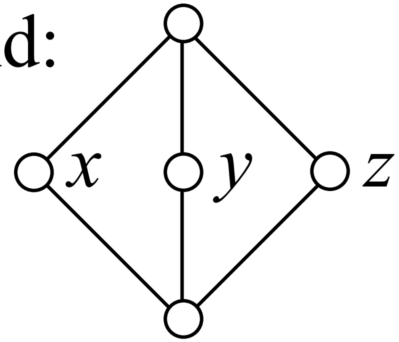
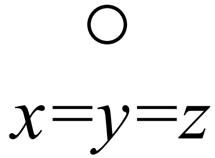


singleton:



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Purpose: $\vec{\mathbf{d}}(\ \ )$ should create either a diamond or a singleton

For $\vec{u} = (x, y, z) \in L^3$, define the **diamond terms**

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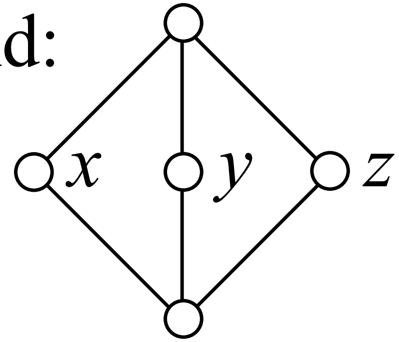
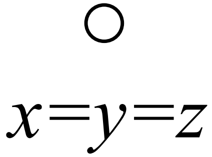


singleton:



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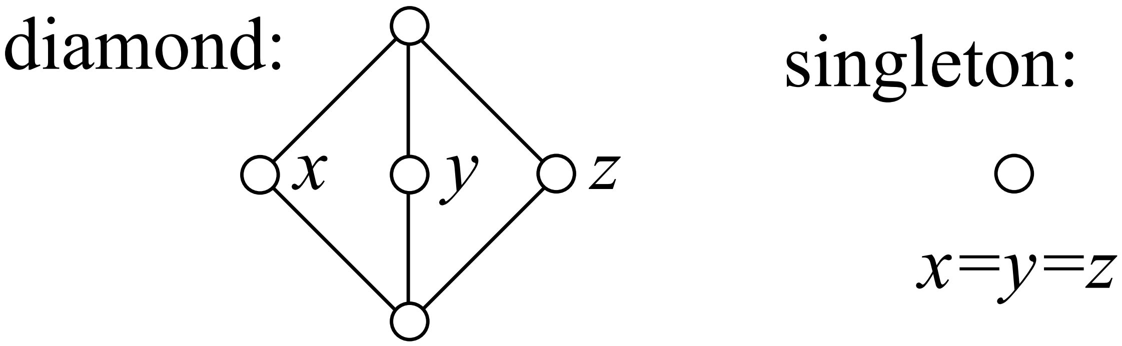


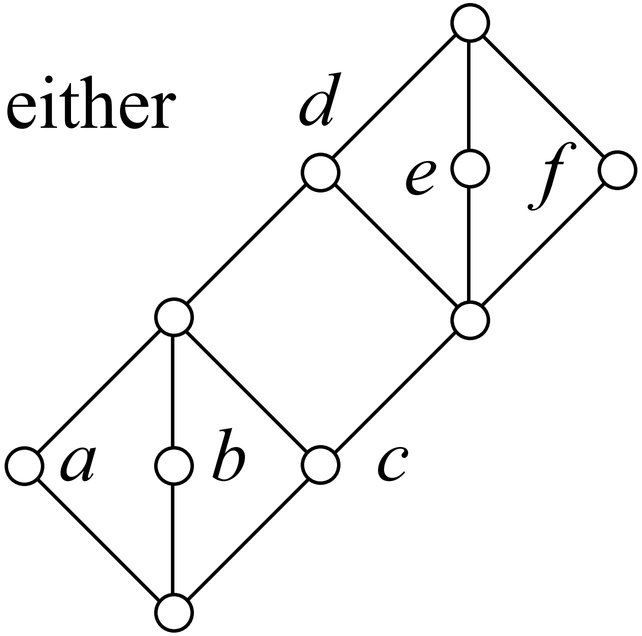
$$\vec{\mathbf{d}}(\vec{u}) = ((x + yz)(y + z)\,, (y + xz)(x + z)\,, (z + xy)(x + y)). \quad (1)$$

Purpose: $\vec{\mathbf{d}}(\ \ )$ should create either a diamond or a singleton and it should keep diamonds unchanged.

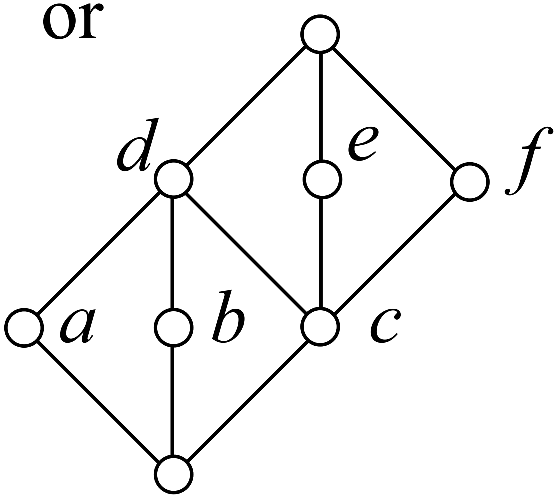
It does the job in modular lattices. (Well-known.)

Analogously, we can define the $\vec{\mathbf{m}}(,)$-terms

either



or



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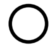

$a=b=c=$
$=d=e=f$

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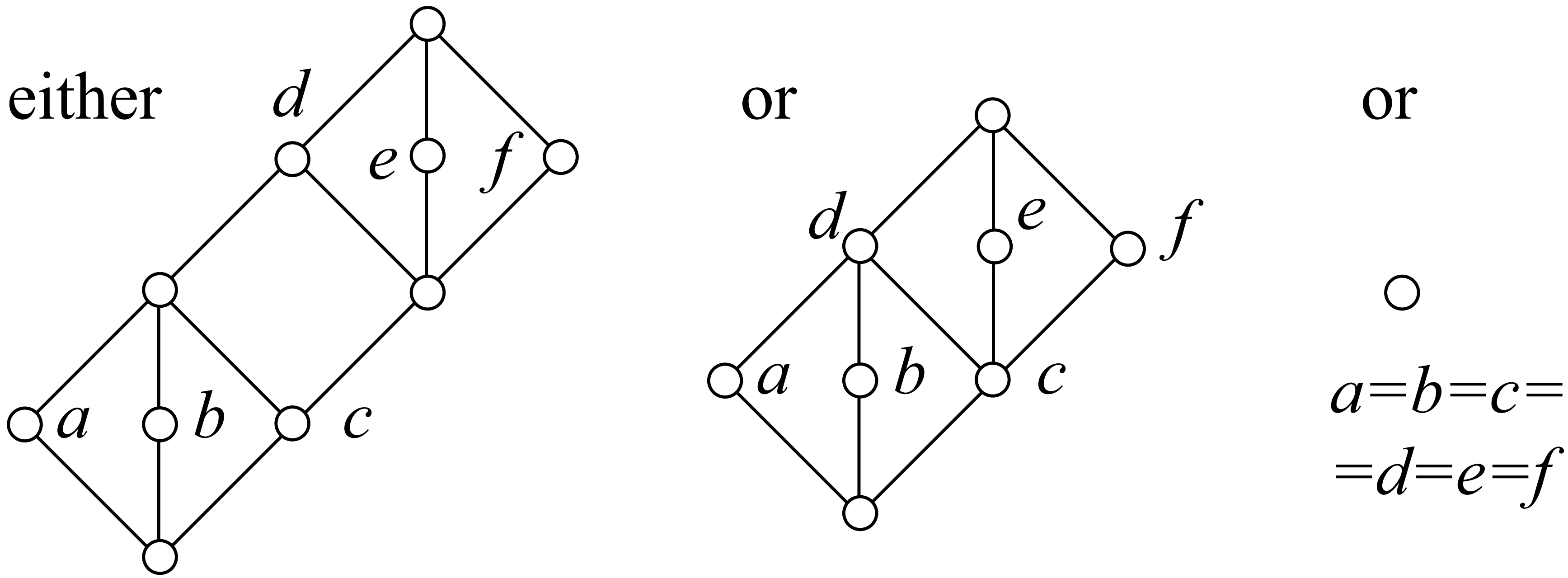

Their job is to produce at least one of the above configurations from $\vec{u} = (a, b, c) \in L^3$ and $\vec{v} = (d, e, f) \in L^3$, and

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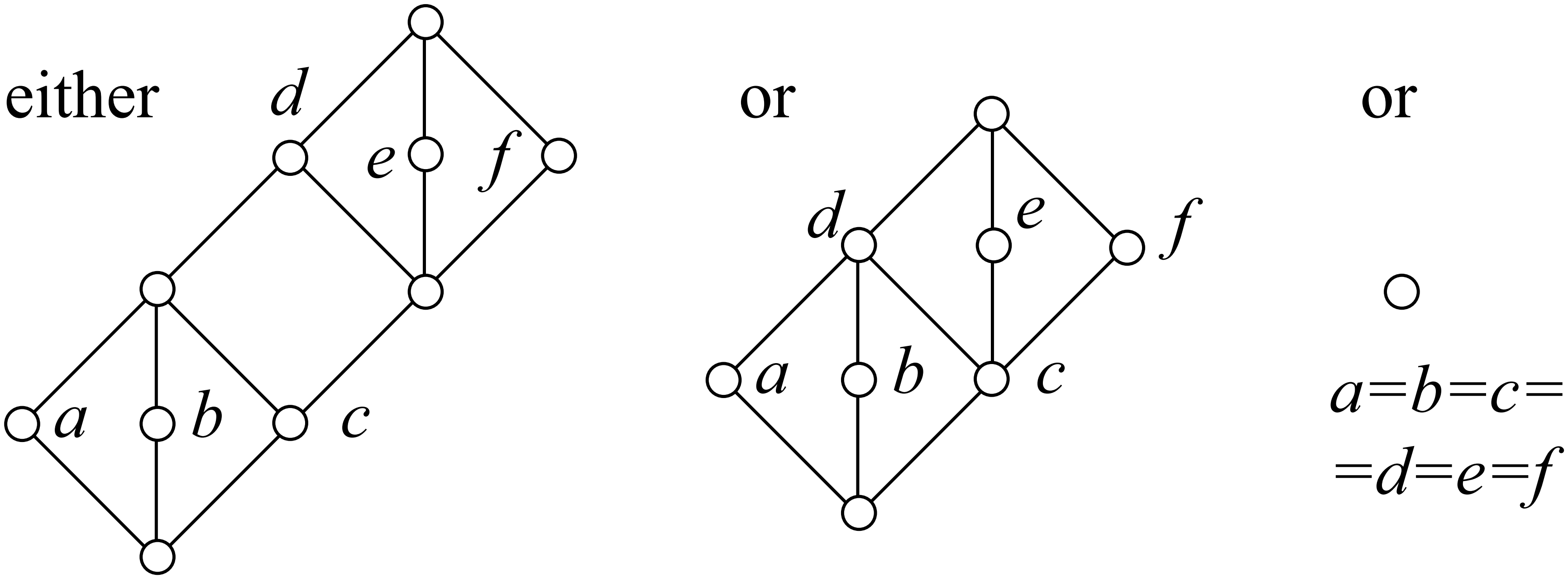
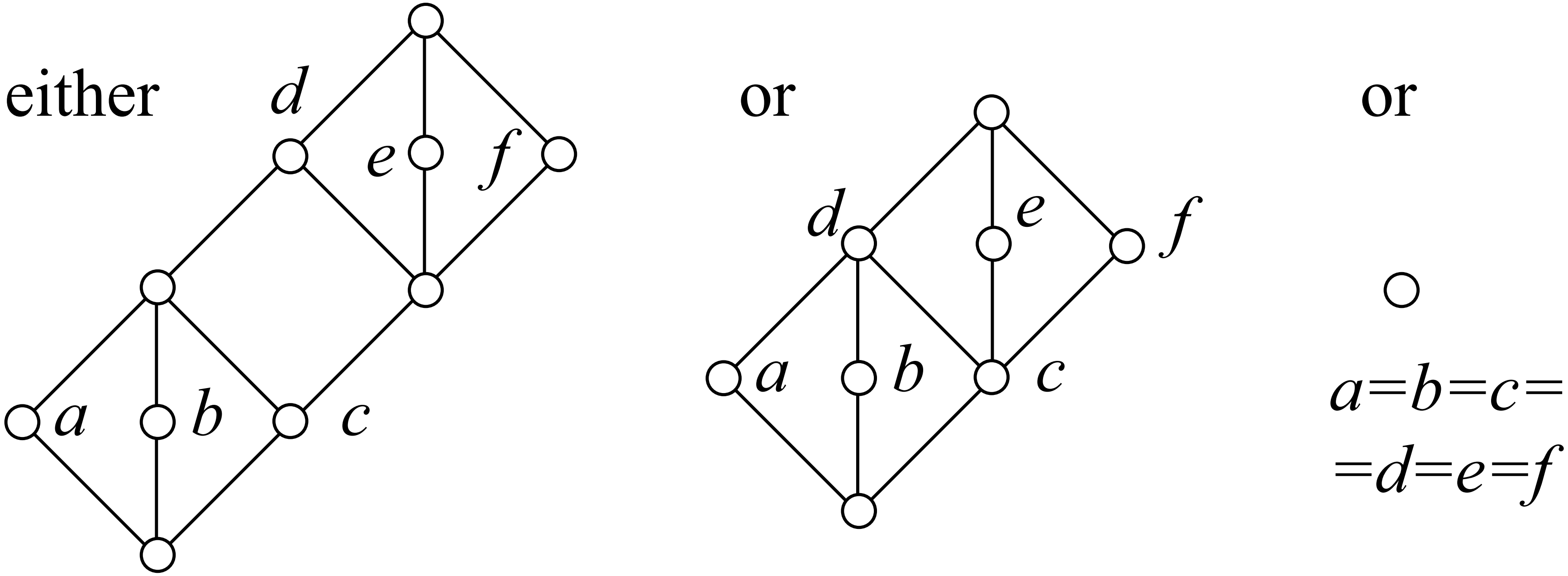


Their job is to produce at least one of the above configurations from $\vec{u} = (a, b, c) \in L^3$ and $\vec{v} = (d, e, f) \in L^3$, and leave any of these configurations unchanged.

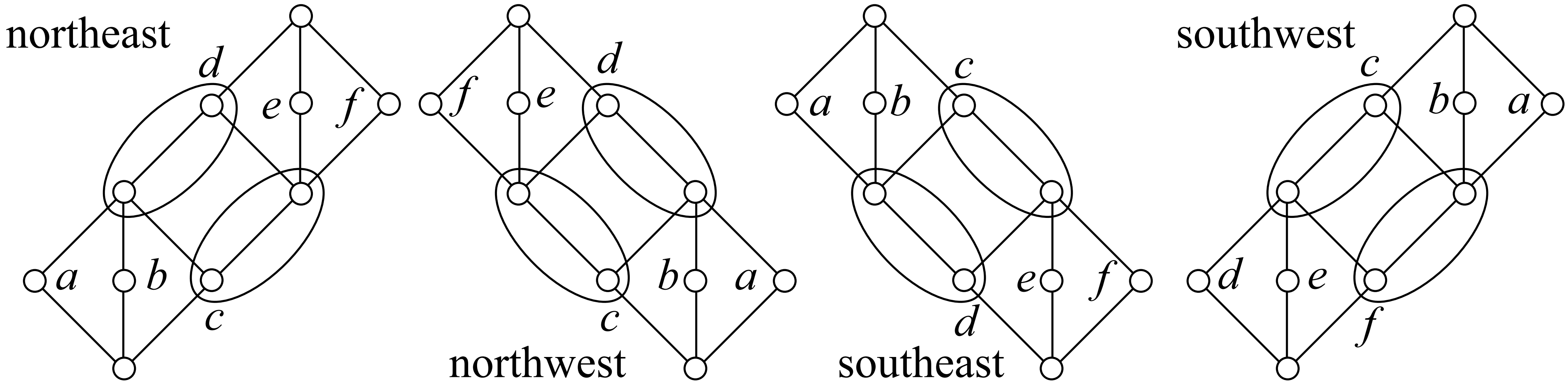
The formal definition is this:

$$\begin{array}{rcccc}
\vec{v}' & = & (d', e', f') & := & (c+d, c+e, c+f), \\
\vec{v}'' & = & (d'', e'', f'') & := & \vec{\mathbf{d}}(\vec{v}'), \\
\vec{v}''' & = & (d''', e''', f''') & := & (a+b+e''f'', e'', f''), \\
\vec{\mathbf{m}}(\vec{u}, \vec{v}) = \vec{v}^* & = & (d^*, e^*, f^*) & := & \vec{\mathbf{d}}(\vec{v}''').
\end{array}$$

It is easy to show that this does the job (it suffices to prove that they work in our particular lattices L_n).

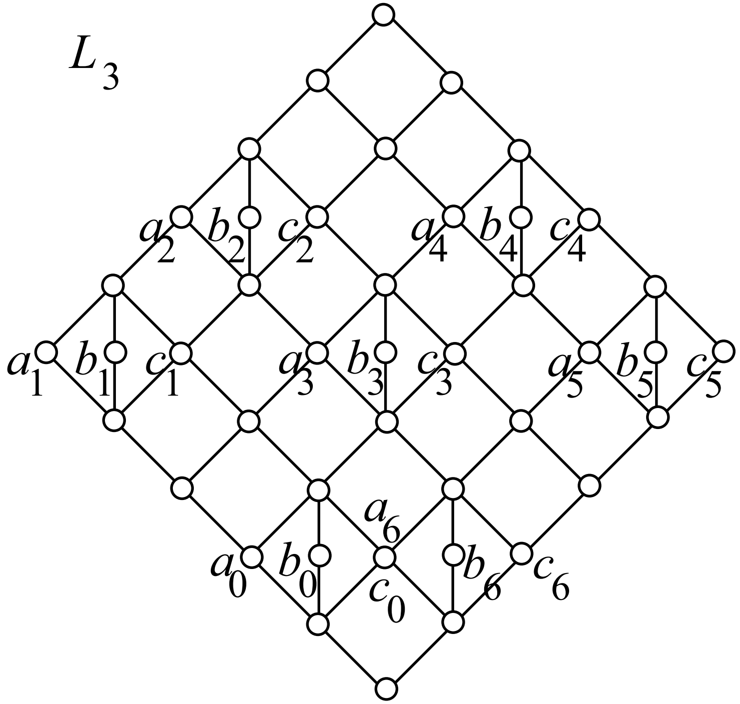


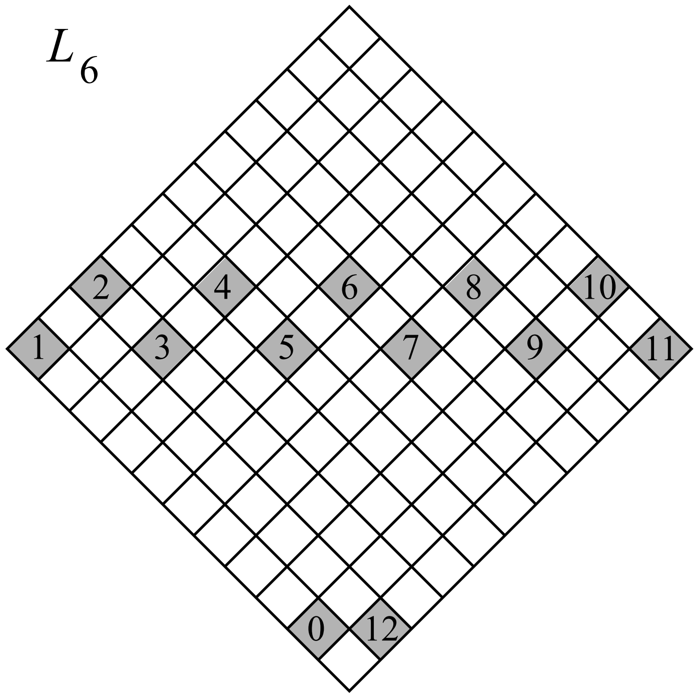
Via dualizing and changing the order of variables we obtain new terms from the $\vec{\mathbf{m}}(\, , \,)$-terms:

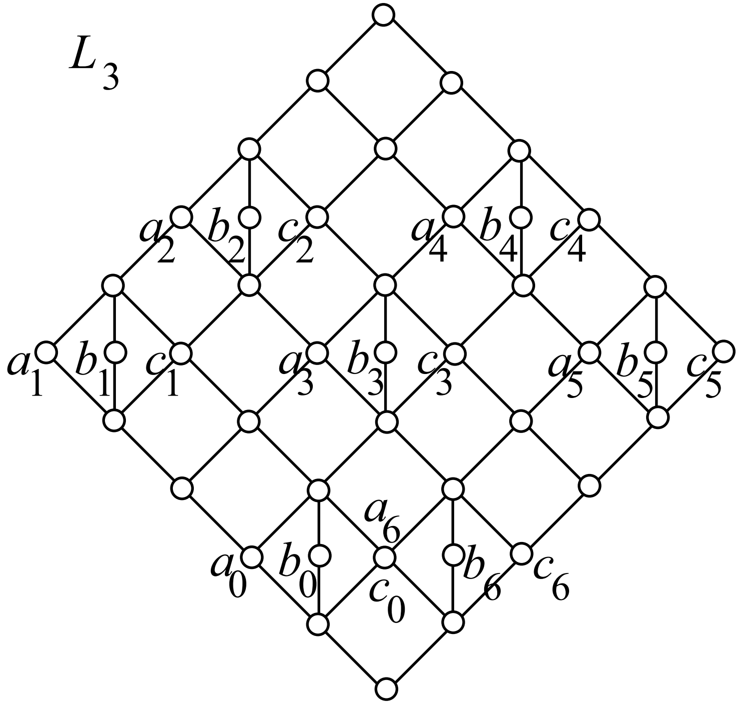


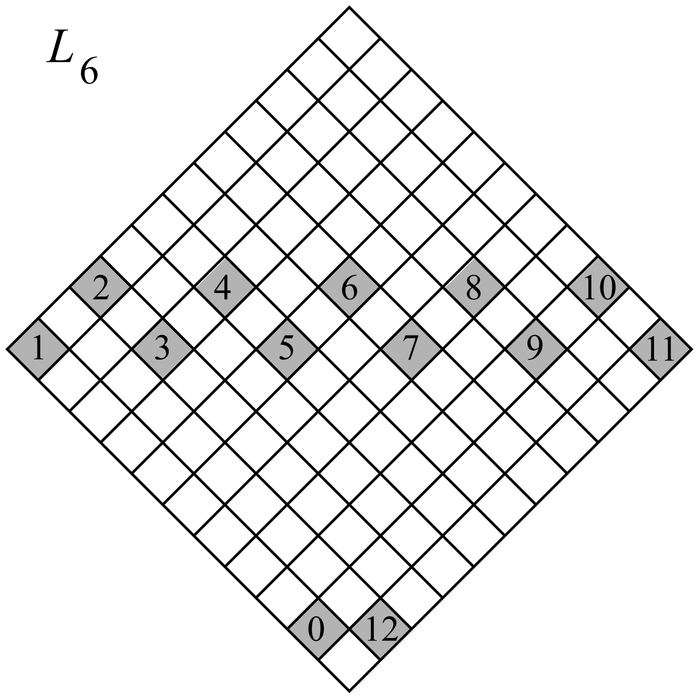
$$\vec{\mathbf{ne}}((a,b,c),(d,e,f)) := \vec{\mathbf{m}}((a,b,c),(d,e,f))$$
$$\vec{\mathbf{nw}}((a,b,c),(d,e,f)) := \vec{\mathbf{m}}((c,b,a),(f,e,d))$$
$$\vec{\mathbf{se}}((a,b,c),(d,e,f)) := \vec{\mathbf{m}}^{\bullet}((a,b,c),(d,e,f))$$
$$\vec{\mathbf{sw}}((a,b,c),(d,e,f)) := \vec{\mathbf{m}}^{\bullet}((c,b,a),(f,e,d))$$

Now consider the variables $\vec{w}_i = (a_i, b_i, c_i)$, $0 \le i \le 2n$, and define terms as follows.

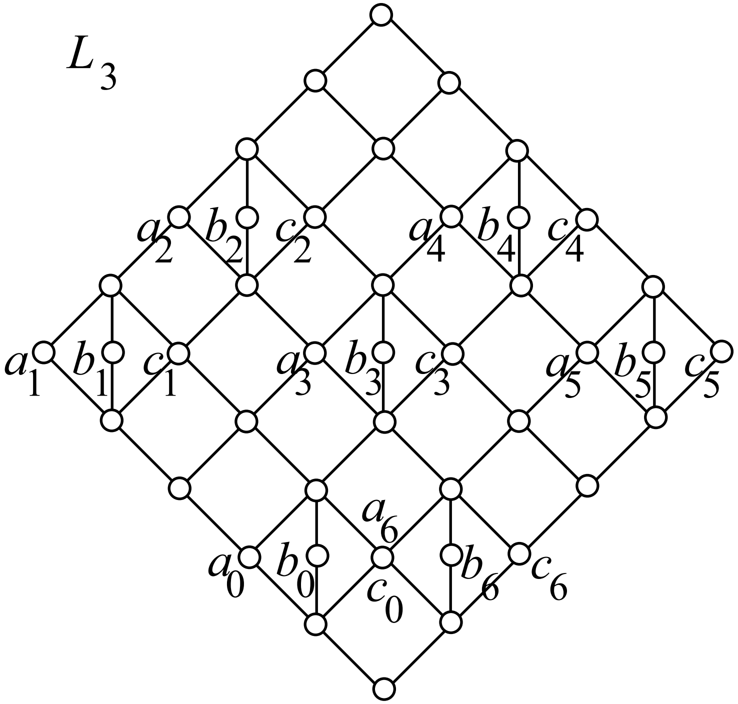


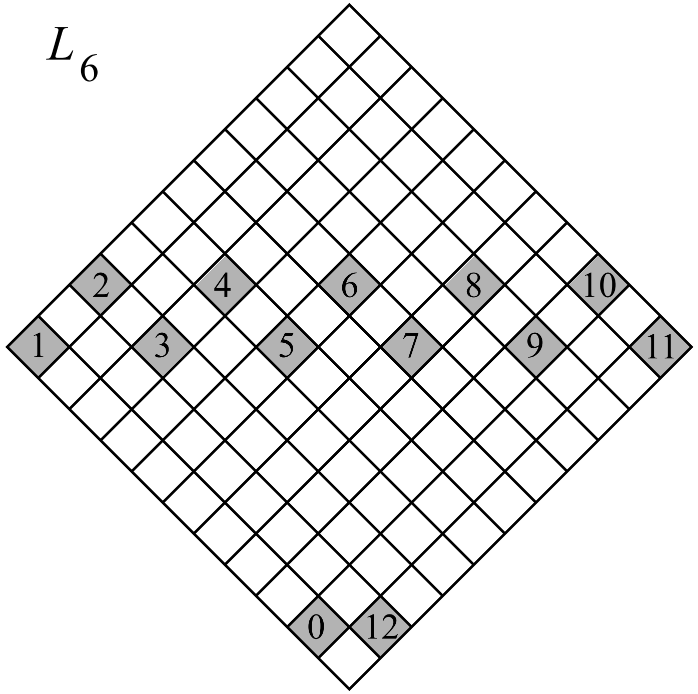




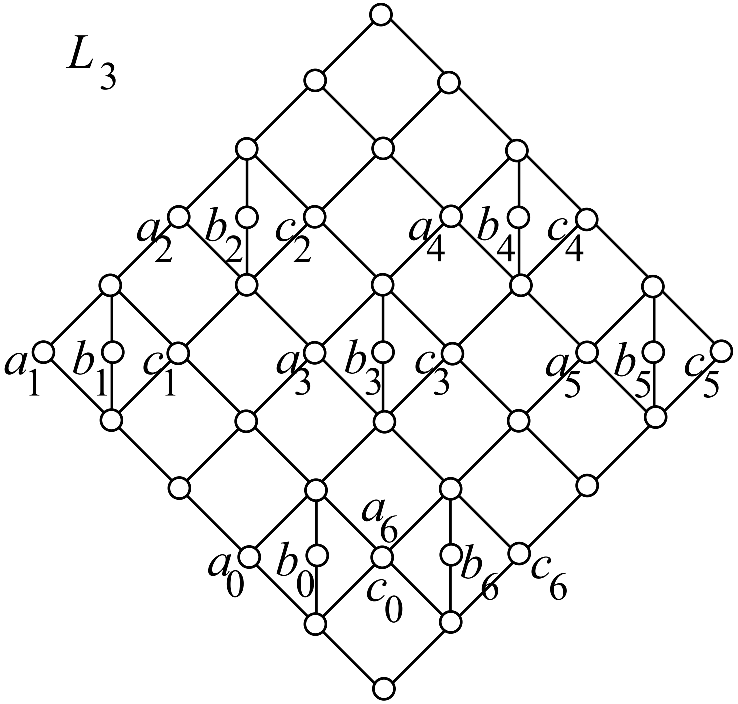


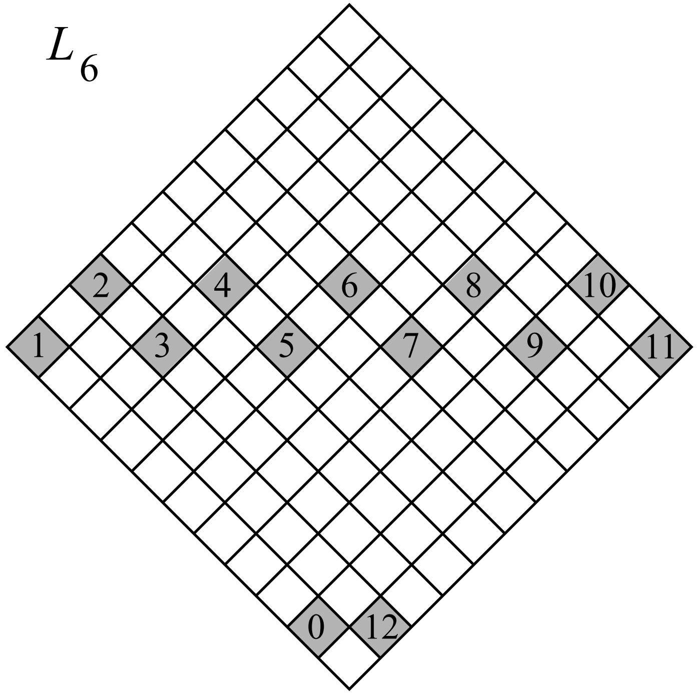
Let $\vec{g_0} = \vec{\mathbf{d}}(\vec{w}_0)$. I.e., we create an initial diamond. (Think of the cell labeled by 0.)



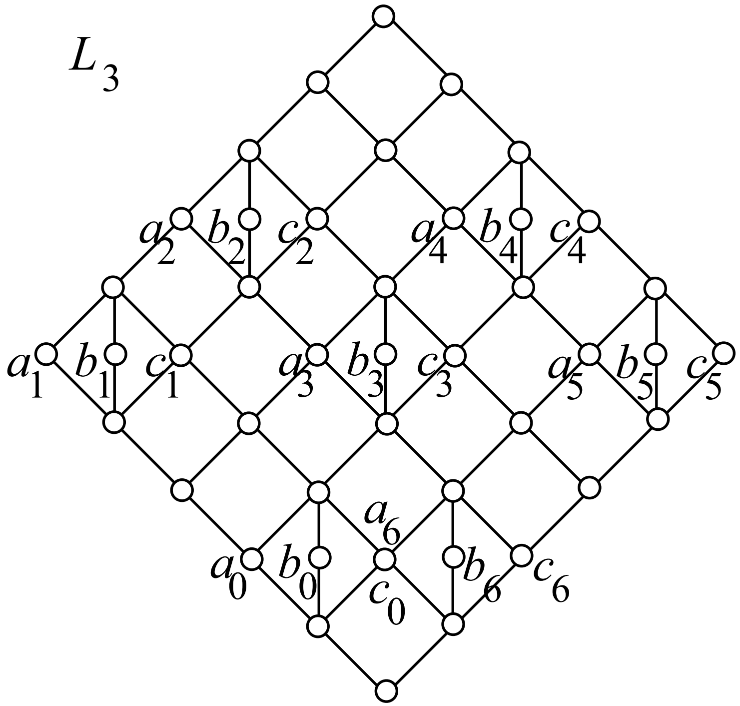


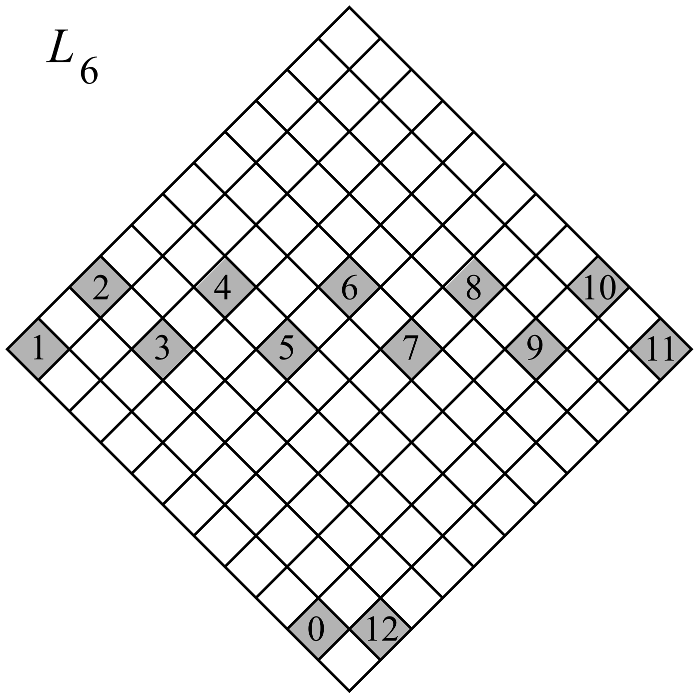
Let $\vec{g_1} = \vec{\mathbf{nw}}(\vec{g}_0, \vec{w_1})$. (I.e., we create the next diamond **northwest** of the previous one. Think of the cell labeled by 1.)



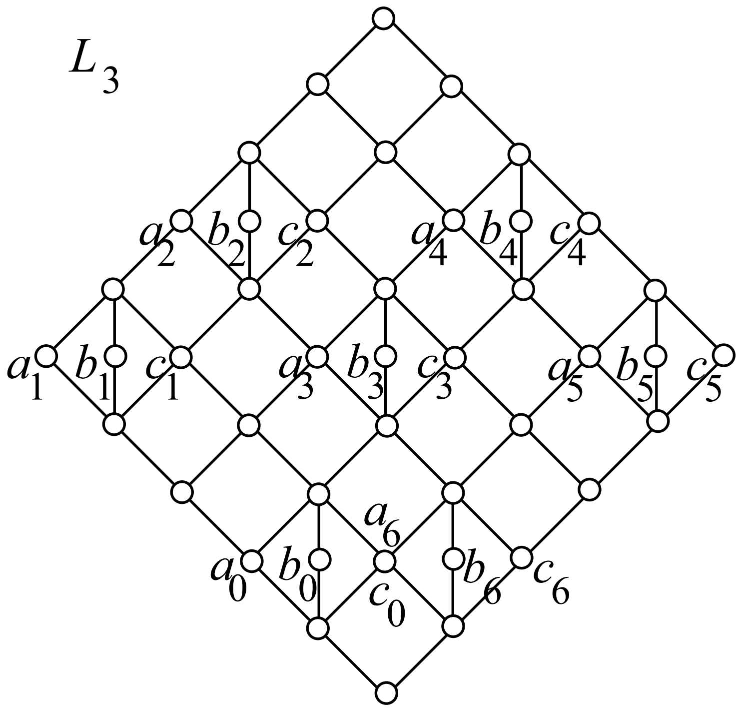


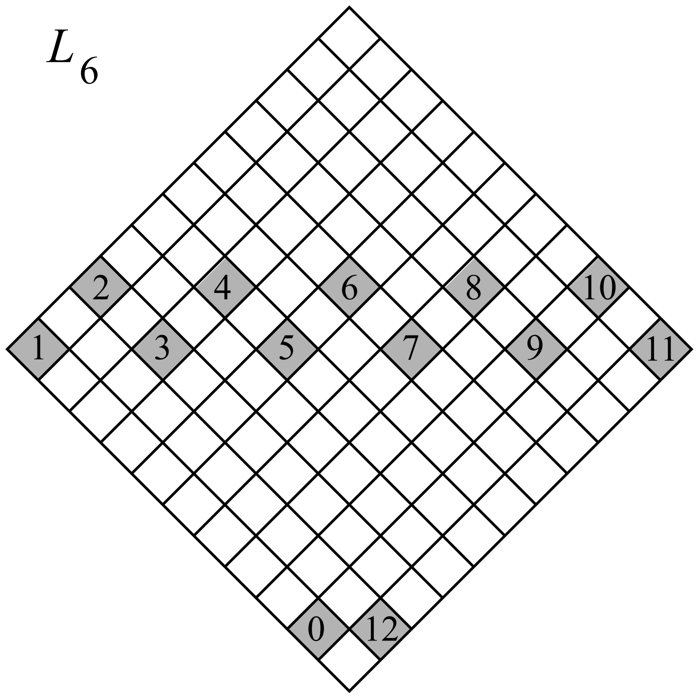
Let $\vec{g_2} = \vec{\mathbf{ne}}(\vec{g_1}, \vec{w_2})$. (I.e., we create the next diamond **northeast** of the previous one. Think of the cell labeled by 2.)



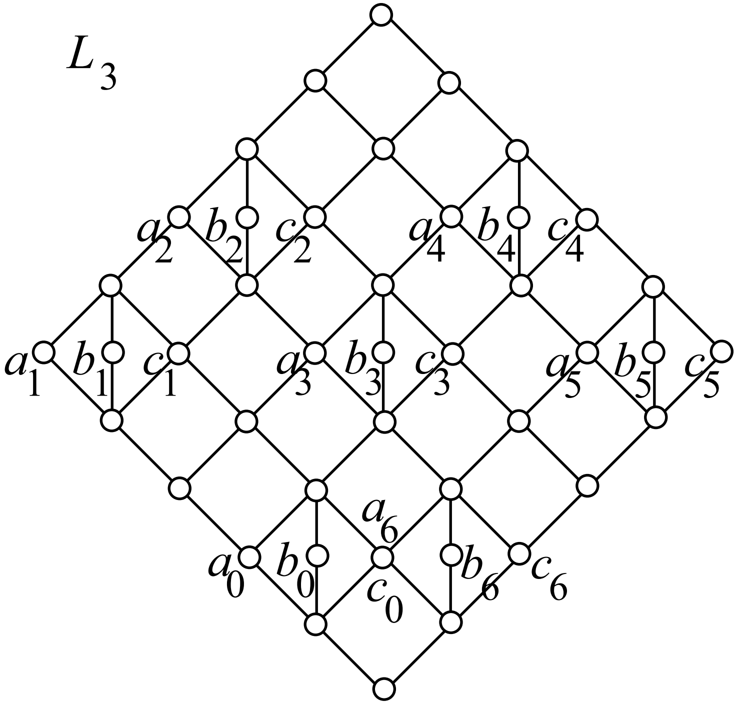


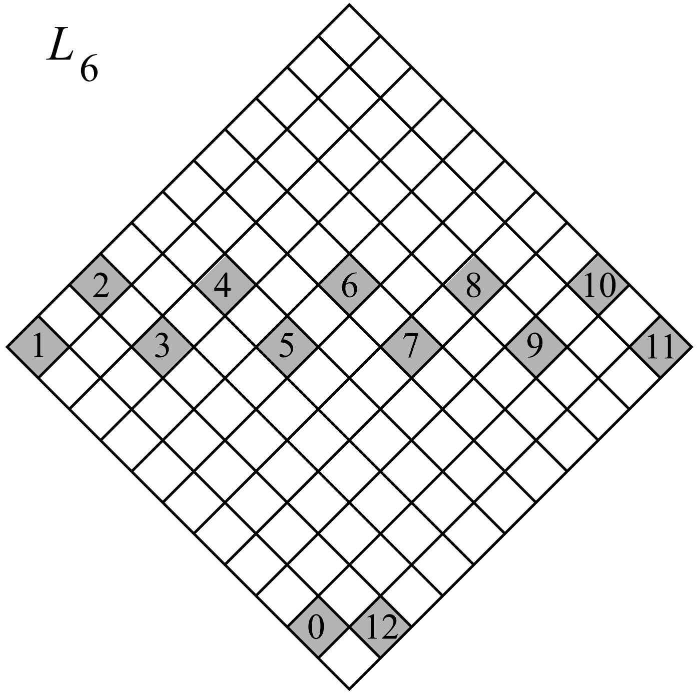
Let $\vec{g_2} = \vec{\mathbf{ne}}(\vec{g_1}, \vec{w_2})$. (I.e., we create the next diamond **northeast** of the previous one. Think of the cell labeled by 2.) . . .



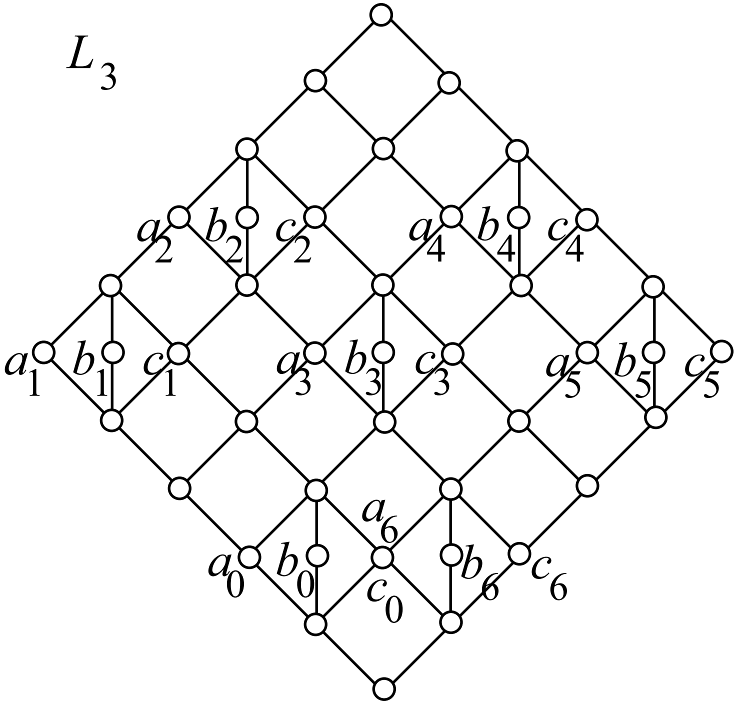


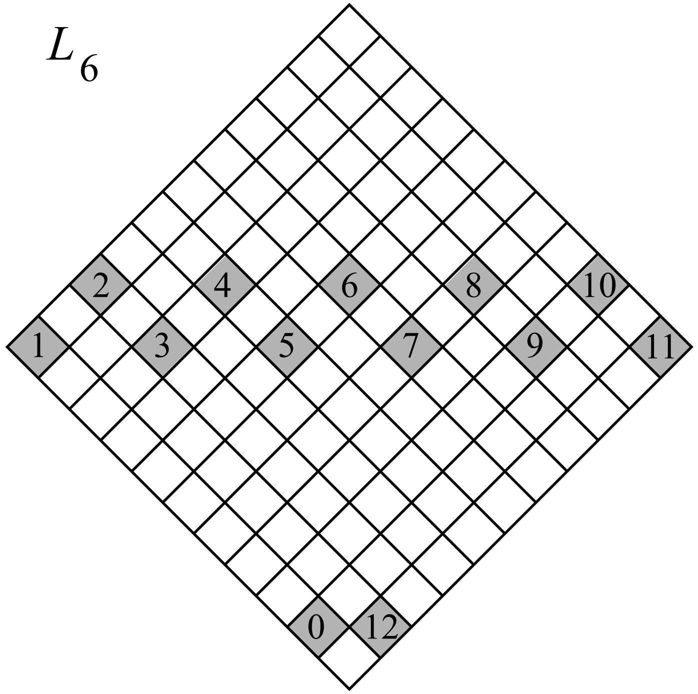
Finally, let $g_{\vec{2n}} =$





Finally, let $g_{\vec{2n}} = (r_{2n}, s_{2n}, s_{2n}) =$



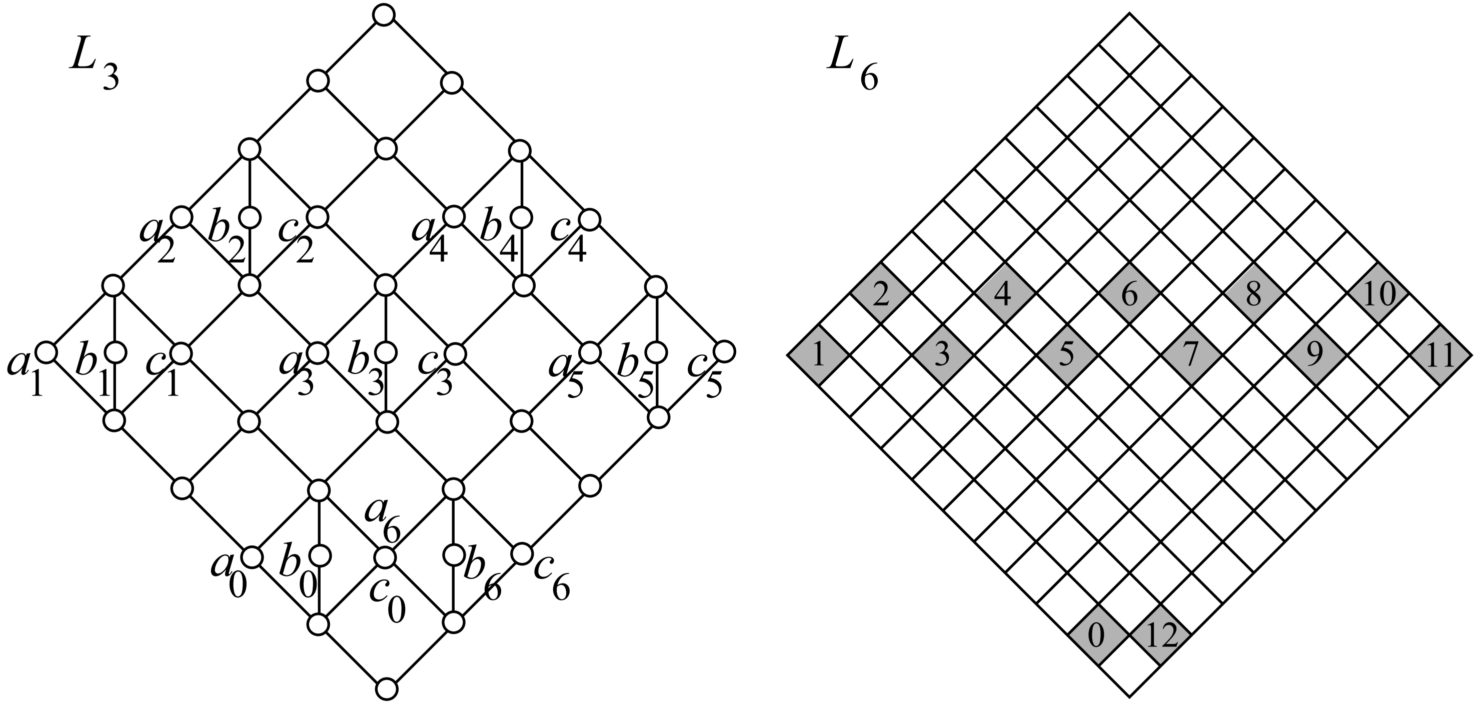


Finally, let $\vec{g_{2n}} = (r_{2n}, s_{2n}, s_{2n}) = \vec{\mathbf{sw}}(\vec{g}_{2n-1}, \vec{w_{2n}})$ (southwest)

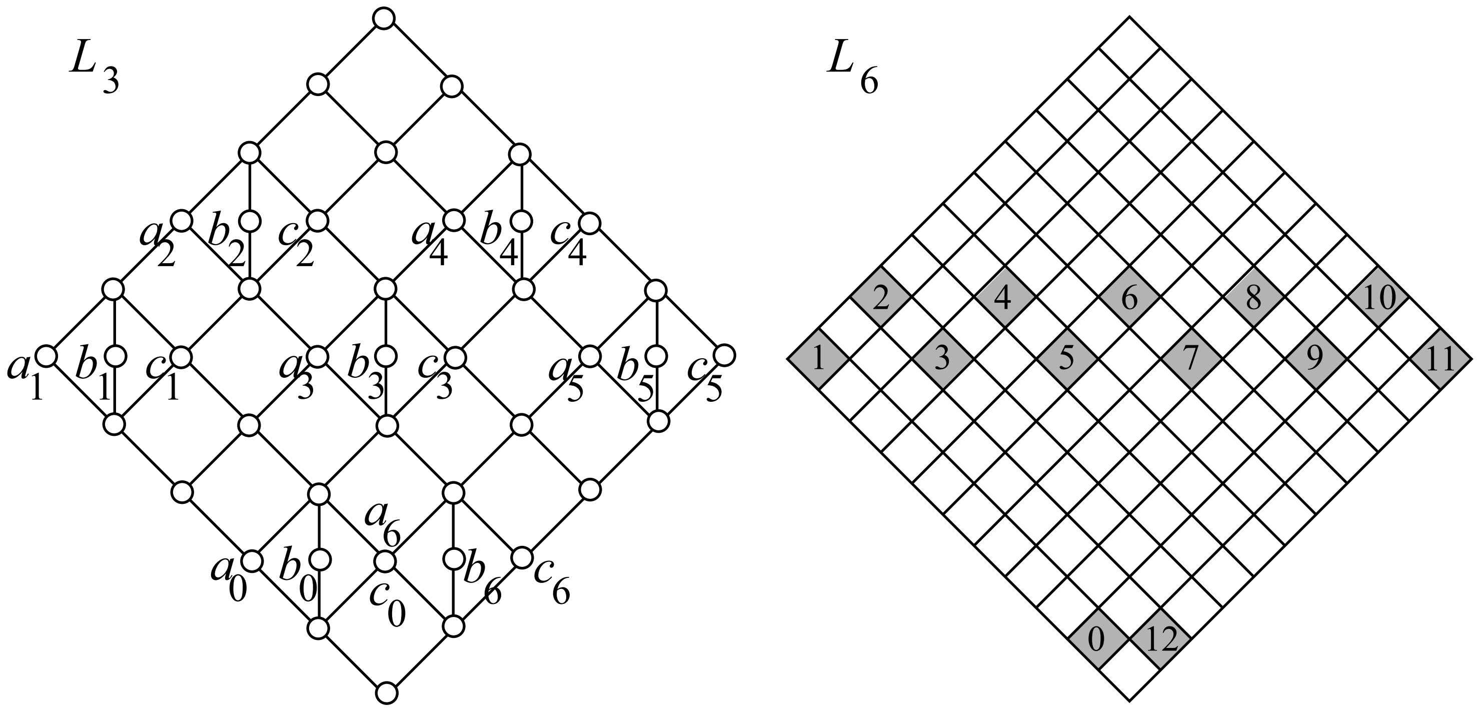
. . .

Now, let λ_n denote the lattice identity $r_{2n} = s_{2n}$.

λ_n tells us how to chase diamonds



$\lambda_n: \; r_{2n} = s_{2n}$ fails in L_n, for the terms we considered keep the diamonds of L_n fixed.



$\lambda_n:\ \ r_{2n} = s_{2n}$ holds in L_k if $k \neq n$ Indeed, we cannot follow the intstruction encoded in λ_n, either too soon or too late we should find a new diamond southwest of the previous one, but there is no diamond there, then the rest of diamonds collapse, whence $r_{2n} = s_{2n}$. Q.e.d.