**Meccanica manuscript No.**
(will be inserted by the editor)

# An extension of the Levi-Weckesser method to the stabilization of the inverted pendulum under gravity

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Received: date / Accepted: date

**Abstract** Sufficient conditions are given for the stability of the upper equilibrium of the mathematical pendulum (inverted pendulum) when the suspension point is vibrating vertically with high frequency. The equation of the motion is of the form

$$\ddot{\theta} - \frac{1}{l}(g + a(t))\theta = 0,$$

where $l$,$g$ are constants and $a$ is a periodic step function. M. Levi and W. Weckesser gave a simple geometrical explanation for the stability effect provided that the frequency is so high that the gravity $g$ can be neglected. They also obtained a lower estimate for the stabilizing frequency. This method is improved and extended to the arbitrary inverted pendulum not assuming even symmetricity between the upward and downward phases in the vibration of the suspension point.

*Key words:* second order linear differential equations, step function coefficients, periodic coefficients, hyperbolic and elliptic rotations, impulsive effects

*2000 Mathematics Subject Classification:* 34A26, 34D30, 70J25

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## 1 Introduction

The mathematical pendulum has two equilibria: the lower one is stable, the upper one is unstable. It was a surprising discovery [1, 11] that the unstable upper equilibrium can be stabilized by vibrating of the point of suspension vertically with sufficiently high frequency. (One can directly experience this phenomenon by the simulation on the instructive web site [18].) Many papers (see, e.g., [2–4, 6–8, 12–16, 20, 22, 24] and the references in them) have been devoted to the description of this phenomenon (see also [1, 5, 9, 17]) and related problems in physics [19, 21, 23]. M. Levi and W. Weckesser [15] gave a simple geometrical explanation for the stability effect provided that the frequency is so high that the gravity can be neglected, and the two half-periods of the periodic excitation of the parameter is symmetric. They obtained also a lower estimate for the frequency in this gravity-free case. In its original form, the Levi-Weckesser method does not work in the case when there acts gravitation, so it is a very natural challenge to find an extension of the method to this more natural case. On the other hand, in applications (e.g., in control theory) it is also important to consider cases when the excitation is not symmetric [10, 12, 17].

In this paper we extend the Levi-Weckesser method to the arbitrary inverted pendulum not assuming even symmetricity between the upward and downward phases in the vibration of the suspension point. Meanwhile we can improve the method and give a sharper estimate for the frequency in the gravity-free case, too.

In Section 2 we set up the model and review some definitions and facts from the theory of periodic linear differential equations. In Section 3 we establish our method for analyzing the phase plane of non-autonomous second order differential equation with step function coefficient describing the motion of the inverted pendulum, and estimate the angles of rotations during the different phases of motions. In Section

4 we deduce a sufficient condition for the stabilization of the inverted pendulum and compare the result and its corollaries with the earlier conditions.

## 2 Theoretical background

As is well-known [1, 5, 17] the *mathematical pendulum* is a particle of mass $m$ connected by an absolute rigid and weightless rod to a base by means of a pin joint so that the particle can move in a plane. If the friction at the pin joint and the drag is neglected, and the particle is only subject to gravity, then motions of the mathematical pendulum are described by the second order differential equation

$$\ddot{\psi} + \frac{g}{l}\sin\psi = 0, \tag{1}$$

where the state variable $\psi$ denotes the angle between the rod of the pendulum and the direction downward measured counter-clockwise; $g$ and $l$ are the gravity acceleration and the length of the rod, respectively. The lower equilibrium position $\psi = 0$ is stable, and the upper one $\psi = \pi$ is unstable. We want to stabilize the upper equilibrium position, so we use the new angle variable $\theta = \psi - \pi$. Rewriting the equation of motion (1) with this state variable, and setting $\theta$ instead of $\sin\theta$, we obtain the linear second order differential equation

$$\ddot{\theta} - \frac{g}{l}\theta = 0,$$

which describes the small oscillations of the pendulum around the upper equilibrium position $\theta = 0$.

Suppose now that the suspension point is vibrating vertically with the $T$-periodic acceleration

$$a(t) := \begin{cases} A_h & \text{if } kT \le t < kT + T_h, \\ -A_e & \text{if } kT + T_h \le t < (kT + T_h) + T_e, \\ (k = 0, 1, \dots)\ ; \end{cases} \tag{2}$$

$A_h, A_e, T_h, T_e$ are positive constants ($A_e > g,\ T_h + T_e = T$) so that the motion of the suspension point is $T$-periodic. (Here, and in what follows, indices $_h$ and $_e$ point to the hyperbolic and the elliptic phase of the motion, respectively; for the attributives see (10) and (13) later.) If $Q$ and $P$ denote the amplitude and the velocity in the vibration of the suspension point respectively, and $Q(0) = 0$, $P(0) < 0$, then it can be seen that the motion of the point is represented by the function

$$Q(t) := \begin{cases} \frac{1}{2}A_h(t - kT)(t - kT - T_h) \\ \qquad\qquad \text{if } kT \le t < kT + T_h, \\ -\frac{1}{2}A_e(t - kT - T_h)^2 + \frac{1}{2}A_eT_e(t - kT - T_h) \\ \qquad\qquad \text{if } kT + T_h \le t < (k+1)T, \\ \qquad (k = 0, 1, \dots). \end{cases}$$

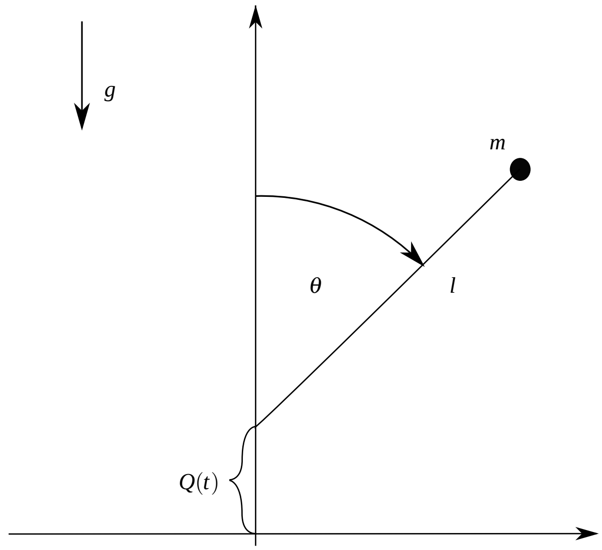


**Fig. 1** Vertically excited inverted pendulum

The maximum amplitudes of the vibration in the first and second phase within one period $T_h + T_e = T$ are expressed by the formulae

$$D_h = \frac{1}{8}A_h{T_h}^2, \qquad D_e = \frac{1}{8}A_e{T_e}^2,$$

and, presuming the natural condition that the velocity of the point of suspension is continuous, the six parameters of the vibration satisfy the following two assumptions:

$$\frac{A_h}{A_e} = \frac{T_e}{T_h}, \qquad \frac{D_h}{D_e} = \frac{T_h}{T_e}. \tag{3}$$

Since the suspending rod is rigid, the acceleration of the vibration is continuously added to the gravity, and the equation of motion of the pendulum is

$$\ddot{\theta} - \frac{1}{l}(g + a(t))\theta = 0 \tag{4}$$

(see Figure 1). For this linear equation we use the stability notions accepted in [1, 15]. Equation (4) is called *stable* if $(x(t), \dot{x}(t))$ is bounded on $(-\infty, \infty)$ for every solution $x$. (4) is called *strongly stable* if it is stable together with all of its sufficiently small perturbation, i.e., there exists an $\varepsilon > 0$ such that $\ddot{\theta} - ((g + \tilde{a}(t))/l)\,\theta = 0$ is stable if $(\tilde{A}_h - A_h)^2 + (\tilde{A}_e - A_e)^2 + (\tilde{T}_h - T_h)^2 < \varepsilon^2$, where the step function $\tilde{a}$ belongs to $\tilde{A}_h, \tilde{A}_e, \tilde{T}_h$ in the sense of the definition (2), provided that $\tilde{T}_e = T - \tilde{T}_h$, and the first equality in (3) is satisfied for the parameters with $\tilde{}$.

Let $\theta_1$, $\theta_2$ denote the solutions of (4) satisfying the initial conditions

$$\theta_1(0) = 1, \quad \dot{\theta}_1(0) = 0; \qquad \theta_2(0) = 0, \quad \dot{\theta}_2(0) = 1.$$

The matrix

$$M := ((\theta_1(T), \dot{\theta}_1(T))^T, (\theta_2(T), \dot{\theta}_2(T))^T), \tag{5}$$

where $(.,.)^T$ denotes the column vector in $\mathbb{R}^2$ transposed to the row vector $(.,.)$, is called the *monodromy matrix* of equation (4). By the Liouville Theorem [1,5,9], $\det(M) = 1$, i.e., the linear transformation $x \mapsto Mx$ preserves the area on the plane $\mathbb{R}^2$. Such a *matrix* $M$ is called *stable* if $\left(M^k\right)_{k\in\mathbb{Z}}$ have a uniform bound in norm. Such a *matrix* is called *strongly stable* if all area preserving matrices near $M$ are stable. It follows from the Floquet Theory [1,5,9] that (4) is strongly stable if and only if its monodromy matrix $M$ is strongly stable. It can be seen that $M$ is strongly stable if $M$ has no real eigenvalues.

Levi's and Weckesser's method is based upon the geometrical observation that the linear transformation $M$ has no real eigenvalues if it turns every non-zero vector of $\mathbb{R}^2$ with a non-zero angle $\pmod{\pi}$. In the next section we estimate this angle for an arbitrary vector of $\mathbb{R}^2$.

## 3 The method

Every motion of (4) has two phases during every period, a hyperbolic and an elliptic one, that are described by the equations

$$\ddot{\theta} - \omega_h^2\theta = 0 \qquad (kT \le t < kT + T_h) \tag{6}$$

and

$$\ddot{\theta} + \omega_e^2\theta = 0 \qquad (kT + T_h \le t < (kT + T_h) + T_e), \tag{7}$$

where

$$\omega_h := \sqrt{\frac{A_h + g}{l}}, \qquad \omega_e := \sqrt{\frac{A_e - g}{l}}$$

denotes the hyperbolic and the elliptic frequency of the pendulum, respectively.

Now we introduce two different phase planes for the two different phases of the motions. Starting with the hyperbolic case, we introduce the new phase variables

$$x_h = \theta, \quad y_h = \frac{\dot{\theta}}{\omega_h}, \tag{8}$$

in which (6) has the following symmetric form:

$$\dot{x}_h = \omega_h y_h, \qquad \dot{y}_h = \omega_h x_h. \tag{9}$$

Using polar coordinates $r_h, \varphi_h$ and the transformation rules

$$x_h = r_h\cos\varphi_h, \quad y_h = r_h\sin\varphi_h \qquad (r_h > 0,\ -\infty < \varphi_h < \infty),$$

the second order differential equation (6) can be rewritten into the system

$$\dot{r}_h = r_h\omega_h\sin 2\varphi_h, \qquad \dot{\varphi}_h = \omega_h\cos 2\varphi_h. \tag{10}$$

The derivative of $H_h(x,y) := x_h^2 - y_h^2$ with respect to system (9) equals identically zero, i.e., $H_h$ is a first integral of (9), so the trajectories of the system are hyperbolae; (10) describes "hyperbolic rotations" (see Figure 2).

Let us repeat the same procedure for the second phase of the period with the new phase variables

$$x_e = \theta, \quad y_e = \frac{\dot{\theta}}{\omega_e}. \tag{11}$$

Then we get systems

$$\dot{x}_e = \omega_e y_e, \qquad \dot{y}_e = -\omega_e x_e, \tag{12}$$

$$\dot{r}_e = 0, \qquad \dot{\varphi}_e = -\omega_e. \tag{13}$$

Now $H_e(x,y) := x_e^2 + y_e^2$ is a first integral, and the trajectories of (12) are circles around the origin; (13) describes uniform "elliptic (ordinary) rotations".

The second differential equation in (10) is separable, consequently, it is integrable. If $\cos 2\varphi_h(0) = 0$, then

$$\cos 2\varphi_h(t) \equiv 0 \ (t \in [0, T_h));$$

the phase point $(r_h(0), \varphi_h(0))$ does not turn. If $\cos 2\varphi_h(0) \neq 0$, then

$$\int_0^{T_h} \frac{\dot{\varphi}_h(t)}{\cos 2\varphi_h(t)}dt = \int_{\varphi_h(0)}^{\varphi_h(T_h - 0)} \frac{d\tau}{\cos 2\tau} = \omega_h T_h.$$

To estimate $|\varphi_h(T_h - 0) - \varphi_h(0)|$ we can assume without loss of generality that $|\varphi_h(0)| < \pi/4$ (see Figure 2, (a)). If we introduce the notation $f(\xi - 0)$ for the left-hand side limit $\lim_{u\to\xi-0} f(u)$, then

$$\varphi_h(T_h - 0) = G^{-1}[\omega_h T_h + G(\varphi_h(0))],$$

where $G^{-1} : \mathbb{R} \to (-\frac{\pi}{4}, \frac{\pi}{4})$ denotes the inverse function of

$$G(\varphi) := \int_0^{\varphi} \frac{d\tau}{\cos 2\tau} = \ln\sqrt{\frac{1+\tan\varphi}{1-\tan\varphi}} \quad \left(-\frac{\pi}{4} < \varphi < \frac{\pi}{4}\right),$$

so we get

$$\varphi_h(T_h - 0) = \arctan\frac{e^{2\omega_h T_h}\dfrac{1+\tan\varphi_h(0)}{1-\tan\varphi_h(0)} - 1}{e^{2\omega_h T_h}\dfrac{1+\tan\varphi_h(0)}{1-\tan\varphi_h(0)} + 1}.$$

$G^{-1}$ is an odd function, which is concave in $[0,\infty]$; therefore,

$$\max_{-\frac{\pi}{4}\le\varphi_h(0)\le\frac{\pi}{4}} |\varphi_h(T_h - 0) - \varphi_h(0)| = 2\arctan\frac{e^{\omega_h T_h} - 1}{e^{\omega_h T_h} + 1}, \tag{14}$$

and we have obtained the desired upper estimate for the hyperbolic turn. By the second equation in (13), for the elliptic turn we have

$$\varphi_e(T_h + T_e - 0) - \varphi_e(T_h) = -\omega_e T_e. \tag{15}$$

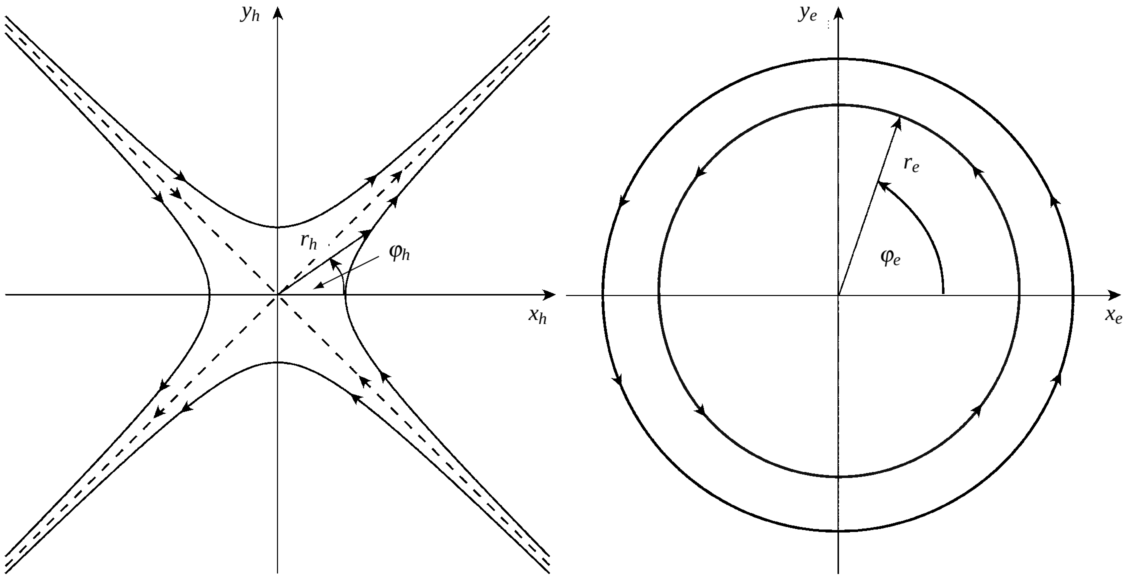


**Fig. 2** (a) Hyperbolic rotation; (b) Elliptic rotation

Besides the hyperbolic and elliptic phases, two impulsive effects, so called “jumps” happen to the phase point during the interval $[0,T]$ at $t = T_h$ and $t = T$. Now we estimate the turns

$$\varphi_e(T_h) - \varphi_h(T_h - 0), \qquad \varphi_h(T_h + T_e) - \varphi_e(T_h + T_e - 0)$$

during these jumps.

Equation (4) has a piecewise continuous coefficient, so we have to modify the standard definition of a solution of a continuous second order differential equation. A function $\theta : \mathbb{R} \to \mathbb{R}$ is a solution of (4) if it is continuously differentiable on $\mathbb{R}$, it is twice differentiable on the set

$$S := \mathbb{R} \setminus (\{kT\}_{k\in\mathbb{Z}} \cup \{kT - T_e\}_{k\in\mathbb{Z}}),$$

and it satisfies equation (4) on the set $S$. Any solution $\theta$ consists of solutions $x_h : [kT, kT + T_h) \to \mathbb{R}$ and $x_e : [kT + T_h, (k+1)T) \to \mathbb{R}$ of (9) and (12) respectively ($k \in \mathbb{Z}$). To guarantee the continuity of $\dot{\theta}$ on $\mathbb{R}$ we have to require the “connecting conditions”

$$\begin{aligned} x_e(kT + T_h) &= \lim_{t\to kT+T_h-0} x_h(t), \\ x_h((k+1)T) &= \lim_{t\to(k+1)T-0} x_e(t); \\ \omega_e y_e(kT + T_h) &= \lim_{t\to kT+T_h-0} \omega_h y_h(t), \\ \omega_h y_h((k+1)T) &= \lim_{t\to(k+1)T-0} \omega_e y_e(t). \end{aligned} \tag{16}$$

Geometrically this means that at the ends of the hyperbolic and elliptic phases there acts on the phase point $(x,y)$ a linear transformation (a contraction or a dilatation)

$$(x,y) \mapsto (x, qy) =: (x, \hat{y}) \qquad (0 < q = \text{const.},\ q \neq 1)$$

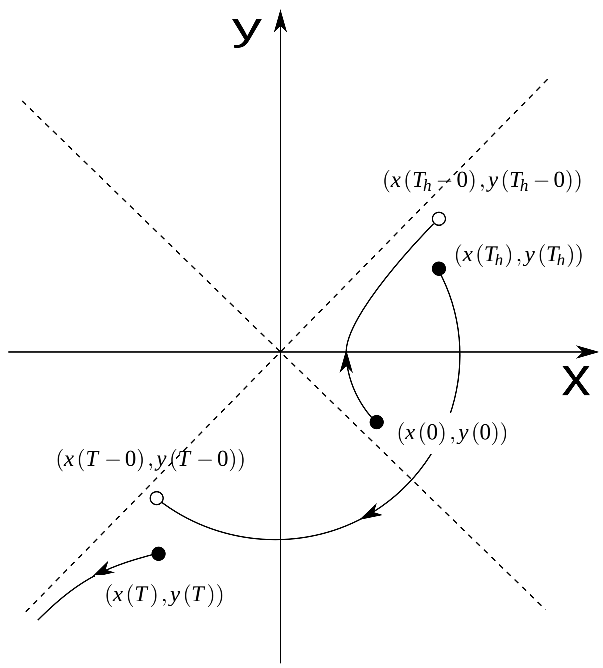


**Fig. 3** A piece of a trajectory during a period

in the direction of $y$-axis (see Figure 3). Now we estimate the turn of the phase point during this “jump”.
If $\varphi \in (-\pi/2, \pi/2)$, then $\hat{\varphi} \in (-\pi/2, \pi/2)$, where $\hat{\varphi}$ denotes the polar angle of the point $(x, \hat{y})$, and

$$\begin{aligned} f_q(\varphi) := \Delta\varphi = \hat{\varphi} - \varphi &= \arctan(q\frac{y}{x}) - \arctan\frac{y}{x} \\ &= \arctan(q\tan\varphi) - \varphi\ ; \\ f_q'(\varphi) &= q\frac{1+\tan^2\varphi}{1+q^2\tan^2\varphi} - 1. \end{aligned}$$

Therefore,

$$\max_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} |f_q(\varphi)| = |\arctan\sqrt{q} - \arctan\frac{1}{\sqrt{q}}|.$$

Obviously, the same estimate is true also for $\varphi \in (\pi/2, 3\pi/2)$, and consequently

$$\max_{0\le\varphi\le 2\pi} |\Delta\varphi| \le |\arctan\sqrt{q} - \arctan\frac{1}{\sqrt{q}}|.$$

By the use of the trigonometric additional theorems, there can be proved the identity

$$\arctan x - \arctan\frac{1}{x} = 2\arctan x - \frac{\pi}{2} \qquad (x>0);$$

therefore,

$$\max_{0\le\varphi\le 2\pi} |\Delta\varphi| \le 2|\arctan\sqrt{q} - \frac{\pi}{4}|. \tag{17}$$

**Remark** Jumps are mathematical tools needed by the difference between the two transformations (8) and (11), i.e., between the two corresponding phase planes $x_h, y_h$ and $x_e, y_e$. The size of jumps can be expressed by

$$q-1 = \frac{\omega_h}{\omega_e} - 1 = \frac{(A_h - A_e) + 2g}{\sqrt{(A_e-g)(A_h+g)} + (A_e - g)},$$

so we can say that it measures the deviation from the case when the vibration is symmetric and there acts no gravitation (this case was considered by Levi and Weckesser). If the vibration is symmetric ($A_h = A_e = A$), then

$$q-1 = \frac{g}{A} + o(\frac{g}{A}) \quad (A\to\infty)$$

asymptotically equals the proportion of the accelerations $g$ and $A$.

## 4 The results

The main result is concerned with the general case of the excited inverted pendulum when the particle is objected to gravity and the vibration of the suspension point is not supposed to be symmetric.

**Theorem 1** *Let* $\mathrm{Rem}(\varphi;\pi)$ *denote the reminder of the real number* $\varphi\in\mathbb{R}$ *modulo* $\pi$ $(0\le \mathrm{Rem}(\varphi;\pi)<\pi)$.

*If*

$$\begin{aligned} &2\arctan\frac{e^{\omega_h T_h}-1}{e^{\omega_h T_h}+1} + 4\left|\arctan\sqrt{\frac{\omega_h}{\omega_e}} - \frac{\pi}{4}\right| \\ &< \min\{\mathrm{Rem}(\omega_e T_e;\pi);\ \pi - \mathrm{Rem}(\omega_e T_e;\pi)\}, \end{aligned} \tag{18}$$

*then equation* (4) *is strongly stable.*

*Proof* We formalize the geometrical thoughts of the previous section. Let $R_h(\omega_h, T_h)$, and $R_e(\omega_e, T_e)$, denote the matrix of the rotation

$$(x_h(0), y_h(0)) \mapsto (x_h(T_h-0), y_h(T_h-0)),$$

and

$$(x_e(T_h), y_e(T_h)) \mapsto (x_e(T_h+T_e-0), y_e(T_h+T_e-0)),$$

defined by (9), and (12), respectively, and introduce the notation

$$C(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \qquad (\lambda>0,\ \lambda\ne 1).$$

Then we can represent the monodromy matrix $M$ (see (5)) in the form of the product

$$\begin{aligned} M &= C^{-1}\left(\frac{1}{\omega_e}\right) R_e(\omega_e,T_e) C\left(\frac{1}{\omega_e}\right) C^{-1}\left(\frac{1}{\omega_h}\right) \\ &\quad\times R_h(\omega_h,T_h) C\left(\frac{1}{\omega_h}\right) \\ &= C^{-1}\left(\frac{1}{\omega_h}\right) C\left(\frac{\omega_e}{\omega_h}\right) R_e(\omega_e,T_e) C\left(\frac{\omega_h}{\omega_e}\right) \\ &\quad\times R_h(\omega_h,T_h) C\left(\frac{1}{\omega_h}\right) = C^{-1}\left(\frac{1}{\omega_h}\right)\tilde{M} C\left(\frac{1}{\omega_h}\right). \end{aligned}$$

Since $\left(M^k\right)_{k\in\mathbb{Z}}$ is bounded if and only if $\left(\tilde{M}^k\right)_{k\in\mathbb{Z}}$ is bounded, it is enough to prove that $\tilde{M}$ has no real eigenvalues, i.e., $\tilde{M}$ turns every non-zero vector in $\mathbb{R}^2$ with a nonzero angle $\pmod{\pi}$. But $R_e(\omega_e, T_e)$ turns every vector exactly with $-\omega_e T_e$ (see (13)), and the turns of $R_h(\omega_h, T_h)$, $C(\frac{\omega_h}{\omega_e})$, and $C(\frac{\omega_e}{\omega_h})$ are estimated by (14) and (17), so condition (18) guarantees that $\tilde{M}$ turns every vector in $\mathbb{R}^2$ trough an angle different from $0 \pmod{\pi}$. □

Now let us compare Theorem 1 with earlier results. Levi and Weckesser established their method for the very special case $\omega_h = \omega_e$ in (6)-(7), i.e., when $g=0$ and $A_h = A_e$ in (4), and proved the following theorem.

**Theorem A (M.Levi and W.Weckesser [15])** *Consider the inverted pendulum* (4) *in the gravitation-free case* $(g=0)$ *provided that the suspension point is vibrated symmetrically* $(A_h = A_e = A >> 1$*; consequently,* $T_h = T_e = T/2)$*. If*

$$\omega T < \pi \left(\omega := \sqrt{\frac{A}{l}}\right), \tag{19}$$

*then* (4) *is strongly stable.*

Applying Theorem 1 to this case, we get the following extension and approvement of Levi's and Weckesser's result:

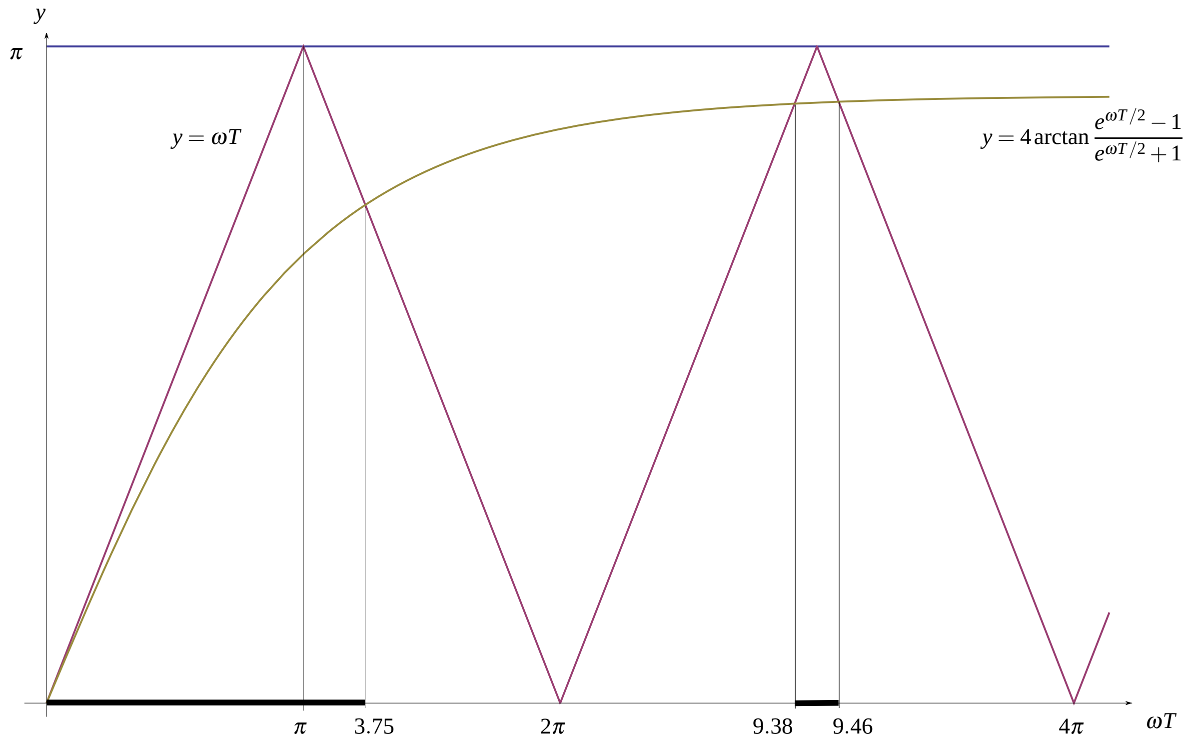


**Fig. 4** Stability intervals for $\omega T$

**Corollary 1** *Suppose that* $g = 0$ *and* $A_h = A_e = A$ *in* (4). *If*

$$4\arctan\frac{e^{\omega T/2}-1}{e^{\omega T/2}+1} < \\ < \min\{\mathrm{Rem}(\omega T;2\pi);\ 2\pi - \mathrm{Rem}(\omega T;2\pi)\}, \tag{20}$$

*then* (4) *is strongly stable.*

We have to admit that condition (20) in our corollary is essentially more complicated than condition (19) in Levi's and Weckesser's theorem. The reason is that Corollary 1 essentially improves Theorem A. In fact,

$$4\arctan\frac{e^{\omega T/2}-1}{e^{\omega T/2}+1} < \omega T \qquad (0 < \omega T < \pi),$$

so the first stability interval on the $\omega T$-axis satisfying (20) is $(0, 3.75\ldots)$ (see Figure 4) instead of $(0,\pi)$ yielded by (19); this is an improvement of 19%. Besides, Corollary 1 also extends Theorem A finding stability intervals on $\omega T$-axis after $2\pi$ (see the thickened intervals on Figure 4). This can be interpreted mechanically that stabilization is possible with arbitrarily large $\omega T = T\sqrt{A/l}$, which cannot be deduced from (19).

Investigating the symmetric case $A_h = A_e = A$, $T_h = T_e = T/2$, V. Arnold [1] introduced the parameters

$$\varepsilon := \sqrt{\frac{D}{l}}, \qquad \mu := \sqrt{\frac{g}{A}},$$

and supposed that these parameters were small ($\varepsilon << 1$, $\mu << 1$). Using series expansion for the trace of the monodromy matrix, one can prove that $\mu < \varepsilon/3$ is sufficient for the strong stability [1]. Let us apply Theorem 1 to get a *global* stability map on the $\varepsilon - \mu$ plane.

**Corollary 2** *Suppose* $A_h = A_e = A$, $T_h = T_e = T/2$. *If*

$$2\arctan\frac{e^{2\sqrt{2}\varepsilon\sqrt{1+\mu^2}}-1}{e^{2\sqrt{2}\varepsilon\sqrt{1+\mu^2}}+1} + 4\left|\arctan\sqrt[4]{\frac{1+\mu^2}{1-\mu^2}} - \frac{\pi}{4}\right| < \\ < \min\{\mathrm{Rem}(2\sqrt{2}\varepsilon\sqrt{1-\mu^2};\ \pi); \\ \pi - \mathrm{Rem}(2\sqrt{2}\varepsilon\sqrt{1-\mu^2};\ \pi)\}, \tag{21}$$

*then* (4) *is strongly stable.*

Obviously, the stability region on the $\varepsilon - \mu$ plane has infinitely many components separated by the curves $2\sqrt{2}\varepsilon\sqrt{1-\mu^2} = k\pi \quad (k = 0,1,2,\ldots)$. Since the first term on the left-hand side of (21) tends to $\pi/2$, as $\varepsilon \to \infty$, and the second term tends to $\pi$, as $\mu \to 1-0$, the stability region has no point close to the line $\mu = 1$, and the "heights" and the "width" of the components tend to zero, as $k \to \infty$ (see Figure 5).

The right-hand side of (21) takes its maximal value $\pi/2$ along the curves

$$G_k: \quad 2\sqrt{2}\varepsilon\sqrt{1-\mu^2} = (2k+1)\frac{\pi}{2} \quad (k = 0,1,\ldots), \tag{22}$$

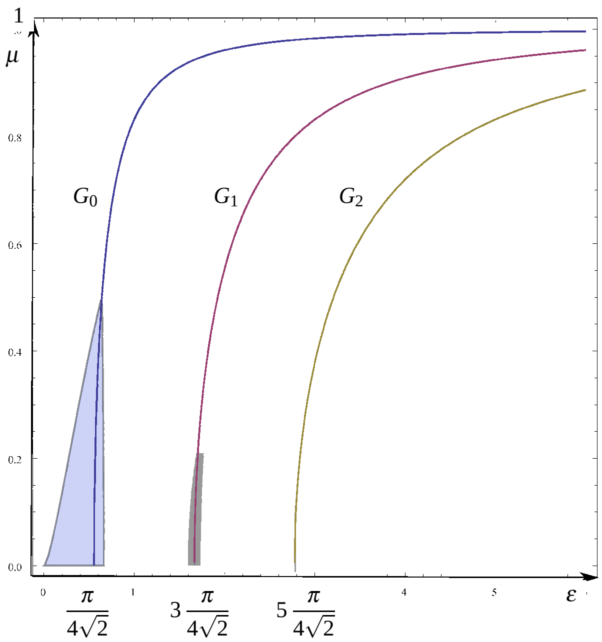


**Fig. 5** Solution set $S$ to inequality (21).

so the $k$th component of the solution set $S \subset \mathbb{R}^2_+$ of the inequality (21) is located along $G_k$. In fact, let us denote by $S_{\mu=0}$ the intersection of $S$ and the $\varepsilon$-axis. Then the points of $S_{\mu=0}$ satisfy the inequality

$$2 \arctan \frac{e^{2\sqrt{2}\varepsilon}-1}{e^{2\sqrt{2}\varepsilon}+1} < \min\{\mathrm{Rem}\,(2\sqrt{2}\varepsilon;\pi);\ \pi - \mathrm{Rem}\,(2\sqrt{2}\varepsilon;\pi)\},$$

which is fulfilled at $\varepsilon = (2k+1)\pi/4\sqrt{2}$ $(k = 0,1,\ldots)$. Since $S$ is open, it has a component along $G_k$ for every $k$ near the $\varepsilon$-axis. On the other hand, since the function $x \mapsto (x-1)/(x+1)$ $(x \geq 0)$ is increasing, every component of $S_{\mu=0}$ has to contain the endpoint of $G_k$ for some $k$. Furthermore, it can be seen that every component of $S$ contains points on axis $\varepsilon$, i.e., in $S_{\mu=0}$. This completes the proof of the fact that $S$ is located along $G_k$. In other words, we can say that curves $G_k$ are the "backbones" of $S$ (see Figure 5).

The larger $\varepsilon$ is the harder to stabilize (4). Since $D = \varepsilon^2 l$, we can practically say that the larger maximum amplitudes of the vibration of the suspension point is the harder to stabilize the inverted pendulum. Nevertheless, there exist critical values of maximum amplitudes

$$D^{(k)} = \frac{(2k+1)^2\pi^2}{32} l \qquad (k = 0,1,\ldots), \tag{23}$$

tending to $\infty$ as $k \to \infty$ such that the pendulum can be stabilized by appropriate accelerations $A^{(k)}$. Of course, $A^{(k)} \to \infty$ as $k \to \infty$; see (21).

Now let us turn to the general (asymmetric) case choosing Arnold's parameters:

$$\varepsilon_h := \sqrt{\frac{D_h}{l}}, \quad \mu_h := \sqrt{\frac{g}{A_h}}; \qquad \varepsilon_e := \sqrt{\frac{D_e}{l}}, \quad \mu_e := \sqrt{\frac{g}{A_e}}.$$

They are not independent (see (3)). Introducing the new parameter

$$d := \frac{\varepsilon_h}{\varepsilon_e} = \frac{\mu_h}{\mu_e} = \sqrt{\frac{A_e}{A_h}} = \sqrt{\frac{T_h}{T_e}} = \sqrt{\frac{D_h}{D_e}},$$

which measures the "ratio" of the hyperbolic phase to the elliptic one in the vibration of the suspension point, we eliminate $\varepsilon_h, \mu_h$ and use the independent parameters $\varepsilon_e, \mu_e, d$ (the symmetric case is characterized by $d = 1$). Theorem 1 has the following form:

**Corollary 3** *If*

$$2 \arctan \frac{\exp\left[2\sqrt{2}d\varepsilon_e\sqrt{1+d^2\mu_e^2}\right]-1}{\exp\left[2\sqrt{2}d\varepsilon_e\sqrt{1+d^2\mu_e^2}\right]+1} + 4\left|\arctan \frac{\sqrt{\dfrac{1+d^2\mu_e^2}{1-\mu_e^2}}}{d} - \frac{\pi}{4}\right| < \tag{24}$$

$$< \min\{\mathrm{Rem}\,(2\sqrt{2}\varepsilon_e\sqrt{1-\mu_e^2};\pi);\ \pi - \mathrm{Rem}\,(2\sqrt{2}\varepsilon_e\sqrt{1-\mu_e^2};\pi)\},$$

*then equation* (4) *is strongly stable.*

A part of the stability region yielded by this corollary can be seen on Figure 6. The section $d = 1$ of the body on Figure 6 corresponds to the first component of the stability region on Figure 5.

Condition (24) offers the stabilization an essentially greater chance than (21). It is a good situation from the point of stability when the second member of the left-hand side in (24) equals zero and the right-hand side takes its maximal value $\pi/2$, i.e. if

$$1 + d^2\mu_e^2 = d^2(1-\mu_e^2),$$
$$2\sqrt{2}\varepsilon_e\sqrt{1-\mu_e^2} = (2k+1)\frac{\pi}{2} \qquad (k = 0,1,2,\ldots).$$

These define the $\varepsilon_e - \mu_e - d$-space curves

$$C_k : d \mapsto \left(\frac{(2k+1)\pi}{4}\frac{d}{\sqrt{d^2+1}}, \frac{\sqrt{d^2-1}}{\sqrt{2}d}, d\right) \ (d \geq 1) \quad (k = 0,1,2,\ldots). \tag{25}$$

Equation (4) is strongly stable along these curves because the first member of the left-hand side in (24) is always less than $\pi/2$. The components of the stability region in the

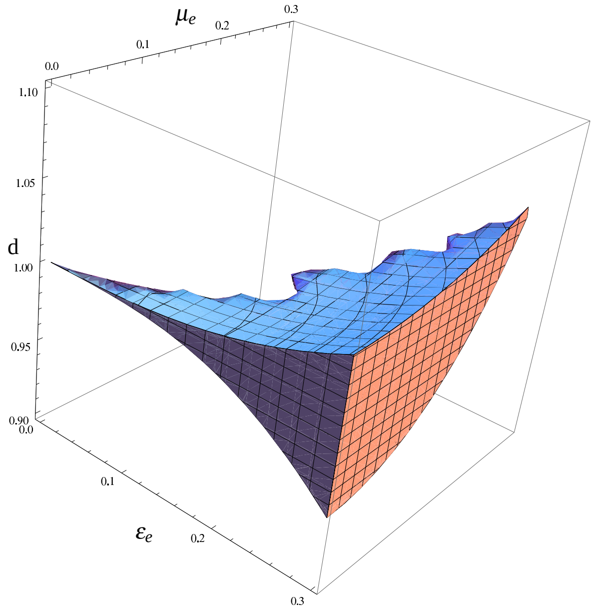


**Fig. 6** A part of stability region.

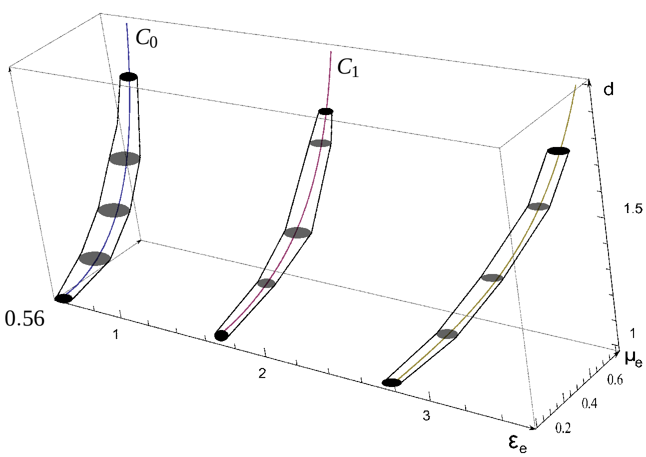


**Fig. 7** Stability region in the $\varepsilon_e$-$\mu_e$-$d$-space.

$\varepsilon_e - \mu_e - d$-space are located "along" these curves (see Figure 7).

As we mentioned in the symmetric case, the inverted pendulum can be stabilized even if the maximum amplitudes of the vibration of the suspension point is arbitrarily large (see the critical values (23)). However, the appropriate values $A^{(k)}$ of the acceleration had to tend to infinity as $k \to \infty$, what is hard to realize. Now the stabilizer has much more chance; namely, the acceleration can be a prescribed fixed value. Mathematically formulating, for every $\overline{\mu}_e \quad (0 \leq \overline{\mu}_e < \sqrt{2}/2)$ there exist $\overline{d} \geq 1$ and $\overline{\varepsilon}_e^{(k)} \quad (\overline{\varepsilon}_e^{(k)} \to \infty$ as $k \to \infty)$ such that equations (4) with parameters $\overline{\varepsilon}_e^{(k)}, \overline{\mu}_e, \overline{d}$ are strongly stable. What is more, the rule of the appropriate device of the parameters is known:

$$\overline{d} = \frac{1}{\sqrt{1-2\overline{\mu}_e^2}}, \qquad \overline{\varepsilon}_e^{(k)} = \frac{(2k+1)\pi}{4\sqrt{2}\sqrt{1-\overline{\mu}_e^2}} \quad (k = 0, 1, \ldots).$$

## 5 Conclusions

The Levi-Weckesser geometric method for the stabilization of the upper equilibrium position of the linearized mathematical pendulum (linearized inverted pendulum) by vertical vibration of the suspension point with high frequency is generalized to the case when the gravity is not neglected. By the use of appropriate coordinates the hyperbolic, and the elliptic pieces of trajectories are transformed into regular hyperbolae, and circles, respectively. The presence of gravity results in impulsive effects ("jumps") in the dynamics on the new phase plane. The method handling these effects is suitable also to handle the asymmetric vibration of the suspension point without any new difficulties, so we considered this more general case. The main result (Theorem 1) gives the new sufficient condition for the stability in the form of an inequality with the parameters. The analysis and application of this result yields, among others, the following conclusions:

– Levi's and Wickesser's condition is improved with 19 %.
– The inverted pendulum can be stabilized by symmetric vibration with arbitrarily large maximum amplitudes choosing sufficiently large acceleration.
– Allowing asymmetric vibration gives the stabilizer much more chance; namely, the inverted pendulum can be stabilized by vibration with arbitrarily large maximum amplitudes and fixed acceleration.

**Acknowledgements** The authors would like to thank the referees for their valuable comments and suggestions.

The authors were supported by the European Union and co-funded by the European Social Fund under the project "Telemedicine-focused research activities on the field of Matematics, Informatics and Medical sciences" of project number "TAMOP-4.2.2.A-11/1/KONV-2012-0073". The second author was supported also by the Hungarian National Foundation for Scientific Research (OTKA K75517) and Analysis and Stochastics Research Group of the Hungarian Academy of Sciences.

## References

1. Arnold VI (2006) Ordinary Differential Equations. Springer-Verlag, Berlin
2. Blackburn JA, Smith HJT and Gronbeck-Jensen N (1992) Stability and Hopf bifurcation in an inverted pendulum. Amer. J. Physics 60: 903–908
3. Butikov E (2011) An improved criterion for Kapitza's pendulum. J. Phys. A: Math. Theor. 44: 1–16
4. Carbo RM, Smith RWM and Poese ME (2010) Stability of the parametrically excited damped inverted pendulum: Theory and experiment. J. Acoust. Soc. Am. 128: 1623–1631

5. Chiccone C (1999) Ordinary Differential Equations with Applications. Springer-Verlag, New York
6. Erdos G and Singh T (1996) Stability of a parametrically excited damped inverted pendulum. J. Sound and Vibration 198: 643–650
7. Formal'skii AM (2006) On the stabilization of an inverted pendulum with a fixed or moving suspension point.(Russian) Dokl. Akad. Nauk. 406 no. 2: 175–179
8. Zhao Haiqing, Wang Guangxue and Zhang Yule (2011) Approximate stability of the parametric inverted pendulum. In: "2011 International Conference on Multimedia Technology (ICMT 2011), July 26-28, Hangzhou, China" 2000–2003
9. Hale J (1969) Ordinary Differential Equations. Wiley-Interscience, New York
10. Hatvani L (2009) On the critical values of parametric resonance in Meissner's equation by the method of difference equations. Electron. J. Qual. Theory Differ. Equ. Special Edition I No. 13: pp. 10
11. Kapitsa PL (1965) Dynamical stability of a pendulum when its point of suspension viberates. In Collected Papers by P. L. Kapitsa vol.II Pergamon Press, Oxford
12. Lavrovskii EK and Formal'skii AM (1993) Optimal control of the rocking and damping of swings. J. Appl. Math. Mech. 57: 311–320
13. Levi M (1988) Stability of the inverted pendulum - a topological explanation. SIAM Rev. 30: 639–644
14. Levi M (1998) Geometry of Kapitsa' potentials. Nonlinearity 11: 1365–1368
15. Levi M and Weckesser W (1995) Stabilization of the inverted, linearized pendulum by high frequency vibrations. SIAM Rev. 37: 219–223
16. Mehidi N (2007) Averaging and periodic solutions in the plane and parametrically excited pendulum. Meccanica 42: 403–407
17. Merkin DR (1997) Introduction to the Theory of Stability. Springer-Verlag, New York
18. Molecular Workbench softwer (2007) An inverted pendulum on an oscillatory base. http://mw.concord.org/modeler1.3/mirror/mechanics/inversependulum.html Copyright 2007 Concord Consortium
19. Yong-Chen Pei and Qing-Chang Tan (2009) Parametric instability of flexible disk rotating at periodically varying angular speed. Meccanica 44: 711–720
20. Seyranian AA and Seyranian AP (2006) The stability of an inverted pendulum with a vibrating suspension point. J. Appl. Math. Mech. 70: 754–761
21. Seiranyan AP and Belyakov AO (2008) The dynamics of a swing. Dokl. Akad. Nauk. 421 no. 1: 54–60
22. Shaikhet L (2005) Stability of difference analogue of linear mathematical inverted pendulum. Discrete Dyn. Nat. Soc. 215–226
23. Sheiklou M, Rezazadeh G and Shabani R (2013) Stability and torsional vibration analysis of a micro-shaft subjected to an electrostatic parametric excitation using variational iteration method. Meccanica 48: 259–274
24. Stephenson A (1908) On a new type of dynamical stability. Manchester Memoirs 52: 1–10