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A form of the Zermelo—von Neumann theorem under minimal assumptions*

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To the memory of L. Kalmár (1905-1976)

Abstract

A simple and general version of the classical result in the title is formulated and proved in the form of a proposition concerning formal languages.

The fundamental game-theoretical theorem of von Neumann asserts that in a two-player zero-sum game both players have optimal (possibly mixed) strategies. A stronger statement is true for *chesslike games*, i.e. discrete finite games in which there are no chance moves, and there is complete information for both players. In such games, either one of the two players has a pure winning strategy or both players have pure safe strategies. We call this fact the Zermelo—von Neumann theorem, as Zermelo was the first to state an equivalent claim in [15], although he did not use the notion of a *strategy*, which was introduced and developed later in works of such pioneers as Borel [4], Steinhaus [14], von Neumann [10], and Kalmár [7]. In this note we prove a simple and general version of the Zermelo—von Neumann theorem. Here it takes the form of an assertion on formal languages, with no assumption on chance moves or complete information. The proof utilizes an idea of Fremlin [5] which dates back to the solution of the game Nim by Bouton [2]. The title alludes to the title of an article of Kalmár ([8]) in which a simple and general form of Gödel's incompleteness theorem is proposed and proved. Note that a fine analysis of interconnections between [15], [7], and a closely related article [9] by D. König may be found in a recent survey paper of Schwalbe and Walker ([13]), in which the definitive formulation of the Zermelo—von Neumann theorem is convincingly credited to Kalmár.

As usual, generators, elements and subsets of free monoids $F(P)$ will be called *letters*, *words*, and *languages*, respectively. A word w_1 is a *prefix* of the word w if $w = w_1w_2$ for some word w_2; we write $w_1 \leq w$ in this case. The *prefix closure* of a language L is the set of all prefixes of all words in L. We say that L is *complete*, if

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every infinite chain $w_1 \leq w_2 \leq \ldots$ of elements of the prefix closure of L stabilizes (i.e., there exists an i such that $w_i = w_{i+1} = \ldots$). A prefix w_1 of w is *proper* if $w_1 \neq w$. A language L is *prefix-free* if no proper prefix of $w \in L$ is in L (cf., e.g., [6], [12], where such languages are called prefix codes). Here by a *game* we shall mean a nonempty complete prefix-free language $L \subseteq F(P)$ trisected into pairwise disjoint sets L_N, L_M, and L_T. For games $L \subseteq F(P)$ we adopt the following terminology: the letters (i.e., the elements of P) are the *positions*, all nonempty prefixes of the words in L are the *states*, and all pairs (s_1, s_2) of states such that there exist a position p with $s_2 = s_1 p$ are the *moves* of L. The words in L are the *terminal states*, the one-element prefixes of them are the *initial states* of the game L; finally, the words in L_N, L_M, and L_T are the *normal*, *misère*, and *tie* terminal states (cf. [1]), respectively. We also refer to terminal states of L as *L-games*.

The rationale of this terminology is that whenever two persons (say, White and Black) play a common finite discrete game G (as Nim, Chess, Go, card games, etc.), the whole process of playing—i.e., the G-game—is fully determined by the sequence $g = p_1 \ldots p_n$ of subsequent positions, and every move of G consists of choosing a further position p_{i+1} to continue a prefix $p_1 \ldots p_i$ $(1 \leq i < n)$ of g, according, of course, to the rules of the considered game. For several simple games (e.g., for Nim) the set of options depends only upon p_i. However, it may depend upon the parity of i, and, more generally, upon each position in $p_1 \ldots p_i$. This is the case, e.g., in Chess, in virtue of some special rules such as castling, en passant capturing, and the threefold repetition rule that prevents infinite games of Chess. Thus, we can consider every game $L \subseteq F(P)$ as an abstract form of a concrete two-player discrete game $\mathcal{L}$ with possible positions $p \in P$. The rule of moves of $\mathcal{L}$ is implicit in the set of all pairs of states of form $(p_1 \ldots p_i, p_1 \ldots p_i p_{i+1})$. As L is complete, this rule excludes the possibility of an infinite sequence of moves in $\mathcal{L}$; i.e., $\mathcal{L}$ is a finite game. The idea of considering states rather than positions goes back to the article [7], in which Kalmár introduced the script form of a game (Schriftspiel). The result of $\mathcal{L}$ is encoded into the components L_N, L_M, L_T of L: for $g \in L$, $g \in L_N$ means that the player unable to move *loses* (as in Nim), $g \in L_M$ indicates that he/she *wins* (e.g., L_M is empty if L stands for Chess), and $g \in L_T$ means that the L-game g ends in a *tie*.

As White and Black move alternately, White always moves *from* a state of odd length. Hence we call such states *White states*, and states of even length will be called *Black states* (including terminal states in both cases). Let S_W and S_B stands for the set of all White states, resp. Black states. Clearly, in an L-game g White wins iff $g \in L_N \cap S_B$ or $g \in L_M \cap S_W$, and Black wins iff $g \in L_N \cap S_W$ or $g \in L_M \cap S_B$. We define a *strategy of White* as a mapping w of the set of nonterminal White states into the set of Black states; similarly, a *strategy of Black* is a mapping $b : S_B \setminus L \to S_W$. Given a one-element word p_1 which is an initial state of L, any pair (w, b) of strategies determines a sequence

$$g(p_1, w, b) = p_1 \ w(p_1) \ b(w(p_1)) \ w(b(w(p_1))) \ b(w(b(w(p_1)))) \ \ldots$$

which, due to the completeness of L, cannot be infinite. Thus, $g(p_1, w, b)$ is an L-game with initial state p_1. A strategy w_0 of White is called a *winning strategy at p_1* if, for any strategy b of Black, White wins the game $g(p_1, w_0, b)$, and, correspondingly, a strategy b_0 of Black is winning at p_1 if, for any strategy w of White, Black wins the game $g(p_1, w, b_0)$. Finally, we call a strategy w_1 of White a *safe strategy at p_1* if, for any strategy b of Black, either White wins $g(p_1, w_1, b)$ or $g(p_1, w_1, b)$ ends in a tie; a safe strategy of Black is defined similarly. We prove the following:

Given a game L and an initial state p_1 of L, either one of Black and White has a winning strategy at p_1 or both of them have safe strategies at p_1.

Consider a game L and let S be the set of all states of L. Call a triple (R_1, R_2, R_x), consisting of disjoint subsets of S, *regular* if it meets the following requirements:

(1) If $s \in R_1$ and (s, s') is a move, then $s' \in R_2$.
(2) If $s \in R_2$ and s is not a terminal state, then there exists a move (s, s') with $s' \in R_1$.
(3) If $s \in R_x$ and (s, s') is a move, then $s' \in R_2 \cup R_x$.
(4) If $s \in R_x$ and s is not a terminal state, then there exists a move (s, s') with $s' \in R_x$.

E.g., (L_N, L_M, L_T) is regular. The set of all regular triples is partially ordered by the rule

$$(R_1, R_2, R_x) \leq (R_1', R_2', R_x') \quad \text{iff} \quad R_1 \subseteq R_1', \ R_2 \subseteq R_2', \ R_x \subseteq R_x'.$$

Clearly, if $(R_1^\alpha, R_2^\alpha, R_x^\alpha)$ is a chain of regular triples, then $(\bigcup_\alpha R_1^\alpha, \bigcup_\alpha R_2^\alpha, \bigcup_\alpha R_x^\alpha)$ is regular, too. Hence there exists a maximal regular triple $(Q_1, Q_2, Q_x) \geq (N, M, T)$. We show that $Q = Q_1 \cup Q_2 \cup Q_x = S$. Suppose not. No states in $S \setminus Q$ are terminal. Therefore, as L is complete, there is a state $s \in S \setminus Q$ with the property that if (s, s') is a move, then $s' \in Q$. If all such s' are in Q_2, then $(Q_1 \cup \{s\}, Q_2, Q_x)$ is regular; if there is such an s' in Q_1, then $(Q_1, Q_2 \cup \{s\}, Q_x)$ is regular; and $(Q_1, Q_2, Q_x \cup \{s\})$ is regular in the remaining case, contradicting the maximality of (Q_1, Q_2, Q_x).

Define a function u on the set $S \setminus L$ of nonterminal states by choosing for $u(s)$

some $s' \in Q_2$ such that (s, s') is a move, if $s \in Q_1$,
some $s' \in Q_1$ such that (s, s') is a move, if $s \in Q_2$,
some $s' \in Q_x$ such that (s, s') is a move, if $s \in Q_x$.

Such a function exists by the axiom of choice and the definition of regular triples. Suppose that the initial state p_1 is in Q_1. Denote the restriction of u to $S_B \setminus L$ by u_1. Then u_1 is a strategy of Black. Consider an arbitrary strategy w of White. The L-game $g_1 = g(p_1, w, u_1)$ is either a White state in L_N or a Black state in L_M, i.e., Black wins in g_1. Thus, u_1 is a winning strategy at p_1 for Black. If $p_1 \in Q_2$, then similarly we obtain that the restriction of u to $S_W \setminus L$ is a winning strategy at p_1 for White. Finally, if $p_1 \in Q_x$, then these restrictions are safe strategies at p_1 for Black and White, respectively.

Note that the standard proof (see, e.g., [11] and [3]) goes by induction on the maximal length $n(p_1)$ of games with initial state p_1. The finiteness of a game does not imply the existence of such an $n(p_1)$; this latter is a slight additional requirement on the game, whose fulfilment is usually guaranteed by postulating that the number of possible moves is finite in every state. This assumption is superfluous under our treatment. A simple example of a finite discrete game with no upper bound $n(p_1)$ on the number of possible consecutive moves from the initial state is the following. Two players place congruent coins onto a centrally symmetric table alternately; however, at most once during a game, instead of placing a coin, a player may choose to reduce arbitrarily but equally the size of all coins to be placed further on. The last player to move wins.

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