# ON THE STRUCTURE OF CANCELLATIVE CONJUGATION SEMIGROUPS

MÁRIA B. SZENDREI

Dedicated to Richárd Wiegandt on his 90th birthday

Abstract. As an abstraction of the conjugation on the multiplicative semigroup of quaternions, Garrão, Martins-Ferreira, Raposo, and Sobral [2] introduced the notion of a conjugation semigroup, and studied the category of cancellative conjugation semigroups. In this note the conjugations of a group are shown to be in one-to-one correspondence with the endomorphisms of the group whose ranges are in the center. Moreover, cancellative conjugation semigroups are proved to be, up to isomorphism, the conjugation subsemigroups of conjugation groups.

Keywords. Cancellative conjugation semigroup, Cancellative conjugation monoid, Conjugation group

Mathematics Subject Classification. 20M10 and 20M50

## 1. Introduction

The notion of a conjugation semigroup has been introduced by Garrão, Martins-Ferreira, Raposo, and Sobral [2] in order to present and investigate new weakly Mal'tsev categories that fail to be Mal'tsev. A *conjugation* on a semigroup $S = (S; \cdot)$ is a unary operation $^-$ on $S$ such that the following equalities hold for every $x, y \in S$:

$$\overline{x}x = x\overline{x}, \tag{1}$$

$$x\overline{y}y = y\overline{y}x, \tag{2}$$

$$\overline{xy} = \overline{y}\,\overline{x}. \tag{3}$$

By a *conjugation semigroup* we mean a unary semigroup $S^- = (S; \cdot, ^-)$ where $^-$ is a conjugation on the semigroup $S$. If $S$ is a monoid, that is, it has an identity element 1, then $^-$ need not satisfy the equality $\overline{1} = 1$. In the context of monoids, we usually require that conjugations have this property. When this is not the case, for example, when applying results obtained for semigroups in the context of monoids, we distinguish the conjugations on monoids where the equality $\overline{1} = 1$ is not required to be satisfied from those where it is required by calling the former *semigroup conjugations* and the latter *monoid conjugations*. A *conjugation monoid* is defined to be a unary monoid $S^-$ where $^-$ is a monoid conjugation on the monoid $S$. If $S$ is a cancellative semigroup (monoid) or, in particular, a group then $S^-$ is a *cancellative conjugation semigroup (monoid)* or, in particular, a *conjugation group*. For example, rules $\overline{x} = x$, $\overline{x} = 1$, and $\overline{x} = x^{-1}$ define conjugations on

The author was partially supported by the National Research, Development and Innovation Office (Hungary), grants K115518 and K128042.

every commutative semigroup, commutative monoid and group, respectively. We denote these conjugations by id, **1** and inv, respectively. Note that the mappings defined by the first two rules on any semigroup and on any monoid, respectively, are endomorphisms, and we denote them also by id and **1**.

Notice that conjugation semigroups (monoids, groups) form a variety of unary semigroups (monoids, groups). Therefore a unary subsemigroup (submonoid, subgroup) of a conjugation semigroup (monoid, group) is a conjugation semigroup (monoid, group), and so we call it a *conjugation subsemigroup (submonoid, subgroup).* Obviously, if the conjugation is id (**1**, inv) in a commutative conjugation semigroup (commutative conjugation monoid, conjugation group) $S^-$ then each subsemigroup (submonoid, subgroup) of $S$ is a conjugation subsemigroup (submonoid, subgroup) of $S^-$. A *homomorphism between conjugation semigroups (monoids, groups)* is meant to be a unary homomorphism (which maps an identity element to an identity element).

It is clear by (3) that each conjugation on a semigroup (monoid, group) is necessarily an anti-endomorphism of the semigroup (monoid, group), that is, a homomorphism from the semigroup (monoid, group) into its left-right dual. In particular, on a commutative semigroup (commutative monoid, Abelian group), the conjugations are just the endomorphisms since (1) and (2) are implied by commutativity. For example, the conjugations id and **1** are of this kind.

The examples in [2] motivating the notion of a conjugation semigroup are the multiplicative semigroups with underlying sets

$$T_{\mathbb{K}} = \{u \in \mathbb{K} : 0 < |u| < 1\} \quad (\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\})$$

together with the conjugation id if $\mathbb{K} = \mathbb{R}$ and with the usual conjugation otherwise. The same conjugation on the multiplicative group $\mathbb{K}^*$ of the field $\mathbb{K}$ defines a conjugation group $(\mathbb{K}^*)^-$, therefore $(T_{\mathbb{K}})^-$ is a cancellative conjugation subsemigroup in $(\mathbb{K}^*)^-$. Note also that $(T_{\mathbb{K}})^- = (T_{\mathbb{H}})^- \cap (\mathbb{K}^*)^-$, and $(\mathbb{R}^*)^-$ is a conjugation subgroup in $(\mathbb{C}^*)^-$ and $(\mathbb{C}^*)^-$ in $(\mathbb{H}^*)^-$. The conjugation defined on $\mathbb{H}^*$ has the property that $\overline{u} = u^{-1}|u|^2$ for every $u \in \mathbb{H}^*$ where the mapping $u \mapsto |u|^2 (= u\overline{u})$ is an endomorphism of the group $\mathbb{H}^*$, and its range is $\mathbb{R}^+ = \{u \in \mathbb{R} : u > 0\}$ which is contained in the center of $\mathbb{H}^*$. Obviously, this property is inherited by the conjugation subgroups $(\mathbb{C}^*)^-$ and $(\mathbb{R}^*)^-$ since they contain $\mathbb{R}^+$. The aim of Section 2 is to describe any conjugation on a group in a similar way.

**Theorem 1.** *For every group $G$, there is a one-to-one correspondence between the conjugations of $G$ and the endomorphisms of $G$ whose ranges are in the center of $G$. For any such endomorphism $\zeta$ of $G$, the respective conjugation is defined by the rule $\overline{g} = g^{-1} \cdot g\zeta$ ($g \in G$).*

For example, the conjugations id and **1** on Abelian groups correspond to the endomorphisms $x \mapsto x^2$ and id, respectively, and the conjugation inv corresponds to the endomorphism **1**.

It is an important observation in [2] that a cancellative semigroup possessing a conjugation is necessarily embeddable in a group. The question naturally arises whether every cancellative conjugation semigroup is embeddable in a conjugation group (as a unary subsemigroup). In Section 3 we give an affirmative answer to this question.

**Theorem 2.** *The cancellative conjugation semigroups (monoids) are, up to isomorphism, just the conjugation subsemigroups (submonoids) of conjugation groups.*

As it has been mentioned, the motivation for introducing and studying conjugation semigroups and monoids in [2] has come from the topic of weakly Mal'tsev categories. Therefore we formulate our results in Section 4 in terms of category theory.

## 2. Conjugations on groups

This section is mainly devoted to proving Theorem 1. Before focusing on conjugations on groups, we introduce a weaker version of a conjugation on any semigroup which is defined by the properties crucial from the point of view of Theorem 1, and show that on cancellative semigroups, it coincides with the original notion.

Let $S$ be an arbitrary semigroup. The *center* of $S$, denoted by $Z(S)$, consists of all elements of $S$ which commute with each element of $S$. If $Z(S)$ is non-empty then it forms a subsemigroup in $S$. Note that if $S$ is a monoid then $1 \in Z(S)$. Consequently, the center of a monoid (group) always forms a submonoid (subgroup).

If $^-$ is a unary operation on a semigroup $S$ then we call $^-$ a *weak conjugation* if the equalities

$$xy\overline{y} = y\overline{y}x, \tag{4}$$

$$\overline{xy} = \overline{y}\,\overline{x}. \tag{5}$$

hold for every $x, y \in S$. For weak conjugations on monoids we use the terms *weak semigroup conjugation* and *weak monoid conjugation* in a similar way as for conjugations.

Clearly, (1) and (2) imply (4), and (3) coincides with (5), so that a conjugation is necessarily a weak conjugation. Conversely, (4) implies the equality $xx\overline{x} = x\overline{x}x$ ($\overline{y}y\overline{y} = y\overline{y}\,\overline{y}$) for every $x \in S$ ($y \in S$), whence we obtain $x\overline{x} = \overline{x}x$ ($\overline{y}y = y\overline{y}$), if $S$ is left (right) cancellative. Let us also mention that if $S$ has an identity element 1 then (5) implies $\overline{1} \cdot 1 = \overline{1} = \overline{1} \cdot \overline{1}$ ($1 \cdot \overline{1} = \overline{1} \cdot \overline{1}$) whence $1 = \overline{1}$ follows if $S$ is left (right) cancellative. Thus we verified the following.

**Proposition 1.** (i) *Every conjugation on a semigroup (monoid) is a weak conjugation.*

(ii) *If a semigroup (monoid) is left or right cancellative then a unary operation on it is a conjugation if and only if it is a weak conjugation.*

(iii) *If a monoid is left or right cancellative then a (weak) semigroup conjugation on it is a (weak) monoid conjugation.*

The starting point towards Theorem 1 is the following general observation on (weak) conjugations.

**Proposition 2.** *For every semigroup $S$ and every weak conjugation $^-$ on $S$, the mapping*

$$\zeta\colon S \to S,\ a\zeta = a\overline{a} \quad (a \in S) \tag{6}$$

*is an endomorphism of the semigroup $S$ such that $S\zeta \subseteq Z(S)$.*

*Proof.* Property (4) implies that $S\zeta \subseteq Z(S)$. By applying (5) and (4), we obtain for every $a, b \in S$ that

$$(ab)\zeta = ab\overline{ab} = ab\overline{b}\overline{a} = a\overline{a}b\overline{b} = a\zeta \cdot b\zeta.$$

Thus $\zeta$ is, indeed, an endomorphism of the semigroup $S$. □

Now we turn our attention to conjugations on groups, and prove Theorem 1. Let $G$ be an arbitrary group. To any conjugation $^-$ on $G$, Proposition 2 assigns an endomorphism $\zeta$ of $G$ whose range is contained in $Z(G)$. It is straightforward by the definition of $\zeta$ that $^-$ can be expressed by means of $\zeta$ as follows:

$$\overline{g} = g^{-1} \cdot g\zeta \quad \text{for every } g \in G. \tag{7}$$

Consequently, the assignment $^- \mapsto \zeta$ given by (6) defines an injective mapping from the set of all conjugations of $G$ to the set of all endomorphisms of $G$ whose ranges are contained in $Z(G)$. In order to complete the proof of Theorem 1, it suffices to show that, for any endomorphisms $\zeta$ of $G$ with $G\zeta \subseteq Z(G)$, the unary operation $^-$ defined by the rule in (7) is a conjugation on $G$. By Proposition 1(ii), this follows if we check that $^-$ is a weak conjugation. Let $\zeta$ be an endomorphism of $G$ such that $G\zeta \subseteq Z(G)$. Then equality (4) is implied since $g\overline{g} = g\zeta \in Z(G)$ by assumption, and property (5) can be checked as follows where the same assumption is applied in the last but one step:

$$\overline{gh} = (gh)^{-1} \cdot (gh)\zeta = h^{-1}g^{-1} \cdot g\zeta \cdot h\zeta = h^{-1} \cdot h\zeta \cdot g^{-1} \cdot g\zeta = \overline{h}\overline{g}.$$

This completes the proof of Theorem 1.

## 3. Cancellative conjugation semigroups

The aim of this section is to prove Theorem 2.

Throughout the section, let $S^- = (S; \cdot, ^-)$ be a cancellative conjugation semigroup, and put $S = (S; \cdot)$.

It is observed in [2] that, due to property (2), conjugation semigroups satisfy the condition that $aS \cap bS \neq \emptyset$ for every $a, b \in S$. This implies by Ore's theorem [1, Theorem 1.23] that the semigroup $S$ is embeddable in a group. Since (2) is left-right symmetric, the dual condition $Sa \cap Sb \neq \emptyset$ $(a, b \in S)$ also holds in $S$. Thus we see by [1, Theorems 1.24 and 1.25] that there exists a group $G$ containing $S$ as a subsemigroup such that

$$\text{every } g \in G \text{ is of the form } g = ab^{-1} = c^{-1}d \text{ for some } a, b, c, d \in S, \tag{8}$$

and $G$ is uniquely determined up to isomorphism. Such a group $G$ is called the *group of quotients of* the semigroup $S$.

The main step of the proof of Theorem 2 is that we extend the conjugation $^-$ on $S$ to the group of quotients $G$ of $S$ such that we obtain a conjugation $^\sim$ on $G$. Before introducing $^\sim$, we need two lemmas.

**Lemma 1.** *For every $a, b, c, d \in S$, the following implications hold in $G$:*

(i) *if $ab^{-1} = c^{-1}d$ then $\overline{b}^{-1}\overline{a} = \overline{d}\,\overline{c}^{-1}$;*

(ii) *if $ab^{-1} = cd^{-1}$ then $\overline{b}^{-1}\overline{a} = \overline{d}^{-1}\overline{c}$.*

*Proof.* (i) If $ab^{-1} = c^{-1}d$ in $G$ then $ca = db$ in $S$ whence we see by (3) that $\overline{a}\,\overline{c} = \overline{ca} = \overline{db} = \overline{b}\,\overline{d}$ in $S$. This implies the equality $\overline{b}^{-1}\overline{a} = \overline{d}\,\overline{c}^{-1}$ in $G$.

(ii) If $ab^{-1} = cd^{-1}$ in $G$ then property (8) ensures that $ab^{-1} = x^{-1}y = cd^{-1}$ for some $x, y \in S$. Thus applying (i) for both equalities, we obtain the equality to be verified. □

**Lemma 2.** *For every $a \in S$ and $g \in G$, we have $a\overline{a}g = ga\overline{a} = g\overline{a}a = \overline{a}ag$.*

*Proof.* We show the first equality, the rest follows since $^-$ satisfies (1). By (8) assume that $g = bc^{-1} = d^{-1}e$ for some $b, c, d, e \in S$. Then $db = ec$ and $db\overline{a}a = ec\overline{a}a$ in $S$. By (2), we obtain that $da\overline{a}b = ea\overline{a}c$ in $S$. Hence we deduce that $a\overline{a}g = a\overline{a}bc^{-1} = d^{-1}ea\overline{a} = ga\overline{a}$ in $G$. □

Now we are ready to define the unary operation $^\sim$ on $G$. For any $g \in G$, if $g = ab^{-1}$ for some $a, b \in S$ then let $\widetilde{g} = \overline{b}^{-1}\overline{a}$. Lemma 1 shows that the operation $^\sim$ is well defined.

**Proposition 3.** *The unary operation $^\sim$ on $G$ is a conjugation, and it extends the conjugation $^-$ on $S$.*

*Proof.* First we check that $^\sim$ extends $^-$. Assume that $c = ab^{-1}$ for some $a, b, c \in S$. Then $a = cb$ in $S$, and we have $\overline{a} = \overline{cb} = \overline{b}\,\overline{c}$ by (3) which implies that $\overline{c} = \overline{b}^{-1}\overline{a} = \widetilde{c}$.

By Proposition 1(ii), it suffices to prove that the unary operation $^\sim$ is a weak conjugation, that is, properties (4) and (5) hold. Consider arbitrary elements $g = ab^{-1}$ and $h = cd^{-1}$ in $G$ where $a, b, c, d \in S$.

To verify the equality $hg\widetilde{g} = g\widetilde{g}h$, first observe that, by the definition of $^\sim$ and by Lemma 2, we have

$$g\widetilde{g} = ab^{-1}\widetilde{ab^{-1}} = ab^{-1}\overline{b}^{-1}\overline{a} = a(\overline{a}^{-1}\overline{b}b)^{-1} = a(b\overline{b}\overline{a}^{-1})^{-1} = a\overline{a}(b\overline{b})^{-1}.$$

Thus we have to check that $ha\overline{a}(b\overline{b})^{-1} = a\overline{a}(b\overline{b})^{-1}h$. Applying Lemma 2, we obtain that, indeed,

$$ha\overline{a}(b\overline{b})^{-1} = a\overline{a}h(b\overline{b})^{-1} = a\overline{a}(b\overline{b}h^{-1})^{-1} = a\overline{a}(h^{-1}b\overline{b})^{-1} = a\overline{a}(b\overline{b})^{-1}h.$$

To show the equality $\widetilde{gh} = \widetilde{h}\widetilde{g}$, notice that $\widetilde{h}\widetilde{g} = \widetilde{cd^{-1}}\widetilde{ab^{-1}} = \overline{d}^{-1}\overline{c}\overline{b}^{-1}\overline{a}$ by definition. On the other hand, we have $b^{-1}c = xy^{-1}$ for some $x, y \in S$ by (8). This implies that $gh = ab^{-1}cd^{-1} = axy^{-1}d^{-1} = ax(dy)^{-1}$, and by Lemma 1(i) that $\overline{y}^{-1}\overline{x} = \overline{c}\overline{b}^{-1}$. By applying these equalities and (3) for $^-$, we deduce that

$$\widetilde{gh} = \overline{dy}^{-1}\overline{ax} = (\overline{y}\overline{d})^{-1}\overline{x}\,\overline{a} = \overline{d}^{-1}\overline{y}^{-1}\overline{x}\,\overline{a} = \overline{d}^{-1}\overline{c}\overline{b}^{-1}\overline{a}.$$

Thus the equality $\widetilde{gh} = \widetilde{h}\widetilde{g}$ follows. □

Proposition 3 shows that the conjugation semigroup $S^-$ is, indeed, a conjugation subsemigroup in the conjugation group $(G; \cdot, ^\sim)$, and so Theorem 2 is proved for cancellative conjugation semigroups. By Proposition 1(iii), the statement for cancellative conjugation monoids is a straightforward consequence.

## 4. Categories of conjugation groups, and of cancellative conjugation semigroups and monoids

In conclusion, we formulate our results in terms of category theory.

Denote the category of conjugation groups by $\mathcal{G}$, and consider the category $\widehat{\mathcal{G}}$ of all unary groups $(G;\cdot,\zeta)$ where the unary operation $\zeta$ is an endomorphism of the group $(G;\cdot)$ such that $G\zeta \subseteq Z(G)$. We have seen at the end of Section 2 that the assignments $(G;\cdot,^{-}) \mapsto (G;\cdot,\zeta)$ and $(G;\cdot,\zeta) \mapsto (G;\cdot,^{-})$ between the objects of $\mathcal{G}$ and $\widehat{\mathcal{G}}$, where $\zeta$ is defined in the first assignment by (6) and $^{-}$ is defined in the second one by (7), are inverses of each other. Now let $(G;\cdot,^{-})$ and $(G;\cdot,\zeta)$, and similarly, $(H;\cdot,^{-})$ and $(H;\cdot,\zeta)$ be pairs of unary groups assigned to each other by the previous assignments, and let $\phi$ be a homomorphism from the group $(G;\cdot)$ to the group $(H;\cdot)$. Then $\phi$ is easily seen to be a homomorphism from $(G;\cdot,^{-})$ to $(H;\cdot,^{-})$ if and only if it is a homomorphism from $(G;\cdot,\zeta)$ to $(H;\cdot,\zeta)$. For, if $g \in G$ then $\overline{g}\phi = \overline{g\phi}$ if and only if $g\phi \cdot \overline{g}\phi = g\phi \cdot \overline{g\phi}$, where the left hand side is equal to $(g\overline{g})\phi = (g\zeta)\phi$, and the right hand side to $(g\phi)\zeta$. Hence the assignments between the hom-sets of $\mathcal{G}$ and the hom-sets of $\widehat{\mathcal{G}}$ which send each morphism to itself extend the former assignments between the objects of $\mathcal{G}$ and $\widehat{\mathcal{G}}$ to mutual inverse functors between the categories $\mathcal{G}$ and $\widehat{\mathcal{G}}$. This provides the following form of Theorem 1.

**Corollary 1.** *The categories $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are isomorphic to each other.*

As in [2], denote the category of cancellative conjugation semigroups by $\mathcal{S}$, and the category of cancellative conjugation monoids by $\mathcal{M}$. Notice that $\mathcal{G}$ is a full subcategory of $\mathcal{M}$, and $\mathcal{M}$ is a full subcategory of $\mathcal{S}$ (contrary to a remark in [2, p. 814]; recall Proposition 1(iii) and the fact that any homomorphism between left (right) cancellative semigroups with identity elements maps an identity element to an identity element). Consider the category $\operatorname{Sgr}\mathcal{G}$ of all conjugation subsemigroups of the conjugation groups, and similarly, the category $\operatorname{Mon}\mathcal{G}$ of all conjugation submonoids of the conjugation groups. Since each conjugation subsemigroup (submonoid) of a conjugation group is obviously cancellative, $\operatorname{Sgr}\mathcal{G}$ $(\operatorname{Mon}\mathcal{G})$ is a full subcategory of $\mathcal{S}$ $(\mathcal{M})$. Thus Theorem 2 is equivalent to the following statement.

**Corollary 2.** *The category $\operatorname{Sgr}\mathcal{G}$ $(\operatorname{Mon}\mathcal{G})$ coincides with $\mathcal{S}$ $(\mathcal{M})$.*

## References


[1] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I, Mathematical Surveys No. 7, American Mathematical Society, Providence, R.I., 1961.

[2] A. P. Garrão, N. Martins-Ferreira, M. Raposo, M. Sobral, Cancellative conjugation semigroups and monoids, *Semigroup Forum* **100** (2020), 806–836.



Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

*Email address*: m.szendrei@math.u-szeged.hu