Graph Theory (2025 fall semester)
Practice 2

1. Is there a graph with the following degree sequences?
a) $11, 3, 3, 3, 3$;
b) $11, 3, 3, 3$.

2. Is there a graph without loop edges with the following degree sequences?
a) $12, 3, 3, 2, 2$;
b) $6, 6, 6, 6, 3, 3, 2, 2$.

3. Is there a simple graph with the following degree sequences?
a) $3, 3, 3, 2, 2, 2, 1, 1, 1$;
b) $6, 6, 5, 4, 4, 3, 2, 2, 1$;
c) $6, 6, 6, 6, 3, 3, 2, 2$.

4. Is there a tree on 8 vertices with the following degree sequences: **a)** 1, 1, 2, 2, 2, 2, 2, 2; **b)** 1, 2, 2, 2, 2, 2, 2, 3; **c)** 1, 1, 1, 2, 2, 2, 3, 4; **d)** 1, 1, 1, 2, 2, 2, 2, 3?

5. Let G be a simple graph with the degree sequence $1, 2, 3, 4, 5, 6, x, y$. What is the maximum number of edges of G? (Additional condition: the Havel-Hakimi algorithm cannot be used to prove that a certain degree sequence cannot be realized as a simple graph.)

6. Can we put 15 knights on a 20×20 chessboard such that all of them
a) attacks exactly 3 other knights?
b) attacks exactly 2 other knights?

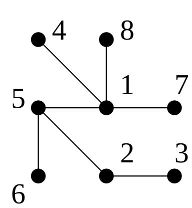
7. Construct a finite, connected graph with the following properties:
a) has an edge e such that $G - e$ is still connected;
b) has a vertex v such that $G - v$ is still connected;
c) for any edge e: $G - e$ is connected;
d) for any vertex v: $G - v$ is connected;
e) has an edge e such that $G - e$ is not connected;
f) has a vertex v such that $G - v$ is not connected;
g) for any edge e: $G - e$ is not connected;
h) for any vertex v: $G - v$ is not connected.

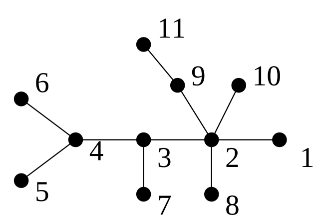
8. Let $G(V, E)$ be a simple, connected graph on $n \geq 2$ vertices, and $v \in V$. Prove that if $G - v$ is still connected then G has a spanning tree T such that v is a leaf of T.

9. Construct a tree such that there are only two distinct degrees: one of them appears 9 times, and the other one 92 times. What could be those two degrees?

10. How should we choose the value of x such that the following Prüfer code $2, 3, x, 3, 2, 4, 4, 3$ encodes a tree with only odd degrees?

11. Determine the Prüfer code of these trees:





12. Draw the trees with Prüfer codes: **a)** $1, 2, 3, 4, 5, 4, 3$; **b)** $2, 5, 10, 1, 9, 3, 8$.

13. Show that for positive integers $d_1 \leq d_2 \leq \ldots \leq d_n$ the following holds:
$\sum_{i=1}^{n} d_i = 2(n-1) \iff$ there exists a tree on n vertices such that $d(i) = d_i$.

14. Consider a Prüfer code $1, 3, x, 3, 2, 2, x, 4, 5$ of a tree T. What can be the smallest value of x such that $\Delta(T) \leq 3$? (also draw down T)

15. Consider n coins with pairwise different weights, and a two-arm scale. Prove that the heaviest can be identified with $n-1$ weighings but cannot be identified with less weighings! (The two-arm scale can compare the weights of two coins, it is not allowed to put more coins on it.)