Graph Theory (2025 fall semester)
Practice 1

1. How many simple graphs exist on the vertices A, B and C? What happens if there are 10 labeled vertices?

2. Draw all simple non-isomorphic graphs on at most 4 vertices!

3. Draw the graph where the vertex set is the subsets of the set $A = \{1, 2, 3\}$, and two vertices are adjacent if and only if one of the corresponding subsets contain the other one.
a) What is the length of the shortest walk between the vertices $\{1, 2\}$ and $\{3, 1\}$? (The length of the walk is the number of edges visited by the walk.)
b) What is the length of the longest path in this graph?
c) What is the length of the longest walk in the graph?
d) Are there any trail of length 4 between the vertices $\{1, 2\}$ and $\{2\}$ which is not a path?
e) Are there any cycle of length 4, 5 or 6 in this graph? Or any closed walk of length 6 which is not a cycle?

4. Consider a graph G on the vertex set $V = \{1, 2, \ldots, 100\}$ and the vertices i and j are adjacent if and only if **a)** $1 < |i - j| < 4$; **b)** $i - j$ is even and not 0? What is the number of edges of G?

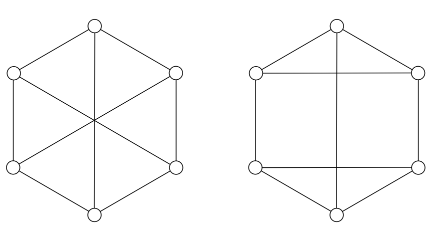
5. 27 players played in a chess tournament where any two players will play each other exactly once. Is it possible that at a certain moment every player played exactly 9 matches so far?

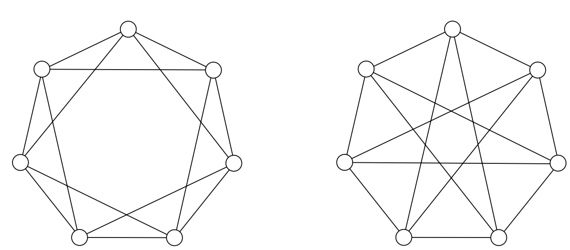
6. Construct simple graphs on 4, 5 and 6 vertices that are isomorphic to their complement! (For a simple graph G, the complement of G, denoted by $\overline{G}$, is a simple graph on the same vertex set such that $ij \in E(G) \iff ij \notin E(\overline{G})$)

7. Prove that in any finite graph G, if the minimum degree, denoted by $\delta(G)$, is at least 2 then G must contain a cycle! Is it true that any vertex of G must belong to a cycle?

8. Prove that in any simple graph G there exist two vertices with the same degree! Is it true even for non-simple graphs?

9. Decide whether the two graphs are isomorphic or not!

a)


b)


10. Construct all non-isomorphic trees on at most 6 vertices!

11. Let G be a graph on vertex set $V = \{1, 2, ..., 100\}$. Is G connected if the edges of G are defined in the following way: i and j are adjacent if and only if
a) $i - j$ is odd;
b) $i - j$ is divisible by 3, and $i \neq j$;
c) $|i - j| = 3$ or $|i - j| = 8$?

12. Consider a simple graph G on n vertices such that $\delta(G) \geq \frac{n-1}{2}$. Prove that G must be connected!

13. What is the number of edges in a graph G on 100 vertices if G has 5 (connected) components such that all of these components are trees? Or in general, how many edges does an n-vertex graph G have if it has k components that are trees?

14. Prove if an n-vertex graph G **a)** does not contain any cycles; **b)** connected and has $n-1$ edges, then G is a tree!

15. Let G be a simple graph on n vertices such that the sum of the degrees of any two non-adjacent vertices is at least $n-1$. Show that G must be connected! Is it still true if G is not simple?

16. Let G be a connected graph on $n \geq 2$ vertices. Prove that there exists a vertex of G that can be deleted from G such that the remaining graph is still connected!