Practice 9

Integration by partial fraction decomposition

Problem 1.

$$\int \frac{1}{x^2-x-2}\,dx$$

Step 1. Factorize the denumerator.

$$x^2-x-2=0$$

$$x_{1,2}=\frac{1\pm\sqrt{1+8}}{2} \begin{cases} \frac{1+3}{2}=2 \\ \frac{1-3}{2}=-1 \end{cases} \Rightarrow \frac{1}{x^2-x-2}=\frac{1}{(x-2)(x+1)}$$

Step 2. Give the original fraction as a sum of two simplier fractions.

$$\frac{1}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1} \quad \left(\text{Multiply by } (x-2)(x+1)\right)$$

$$\frac{(x-2)(x+1)}{(x-2)(x+1)}=\frac{A(x-2)(x+1)}{x-2}+\frac{B(x-2)(x+1)}{x+1} \quad (\text{Simplify.})$$

$$1=A(x+1)+B(x-2)$$

$$1=Ax+A+Bx-2B$$

$$1=(A+B)x+A-2B$$

$$0x+1=(A+B)x+A-2B$$

$$\begin{rest}\end{rest}$$

$$\ominus\ \left.\begin{array}{l} A+B=0 \\ A-2B=1 \end{array}\right\}$$

$$3B=-1$$

$$B=-\frac{1}{3}$$

$$A-\frac{1}{3}=0 \Rightarrow \underline{A=1/3}$$

Step 3. Integrate the sum separately.

$$\int\frac{1}{x^2-x-2}\,dx=\int\frac{1/3}{x-2}\,dx+\int\frac{-1/3}{x+1}\,dx$$

$$=\frac{1}{3}\int\frac{1}{x-2}\,dx-\frac{1}{3}\int\frac{1}{x+1}\,dx=\underline{\underline{\frac{1}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C}}$$

Problem 2.

$$\int \frac{2}{x^2+8x+15}\,dx = ?$$

① $x^2+8x+15=0$ $\qquad \dfrac{2}{x^2+8x+15} = \dfrac{2}{(x+3)(x+5)}$

$x_1=-3 \quad x_2=-5$

②
$$\frac{2}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5}$$

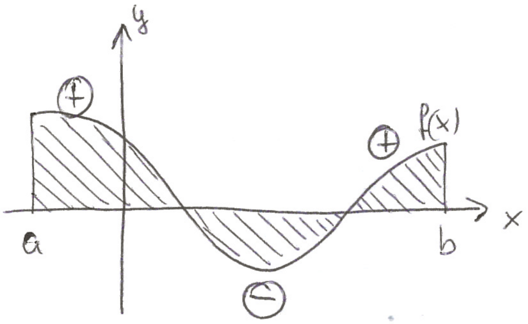
$$2 = A(x+5) + B(x+3)$$

$x=-5 \Rightarrow 2 = B\cdot(-2) \qquad B=-1$

$x=-3 \Rightarrow 2 = 2A \qquad A=1$

③
$$\int \frac{2}{x^2+8x+15}\,dx = \int \frac{1}{x+3}\,dx + \int \frac{-1}{x+5}\,dx = \underline{\underline{\ln|x+3| - \ln|x+5| + C}}$$

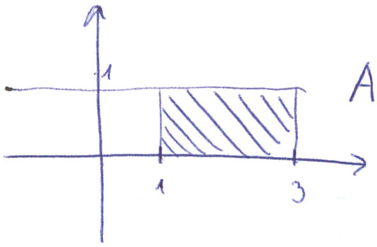
Definite Integral



$$\int_a^b f(x)\,dx = \text{The } \underline{\underline{\text{signed}}} \text{ area of the region between the function } f \text{ and the x-axis from a to b.}$$

Problem 1.

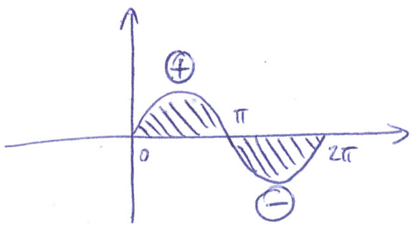
$$\int_1^3 1\,dx = ? \quad 2$$



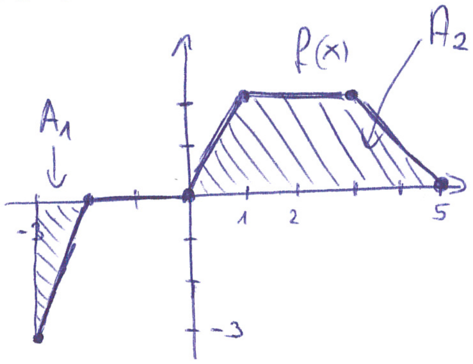
$A = 1\cdot 2 = 2$

Problem 2.

$$\int_0^{2\pi} \sin x\,dx = ? \quad 0$$



Problem 3.

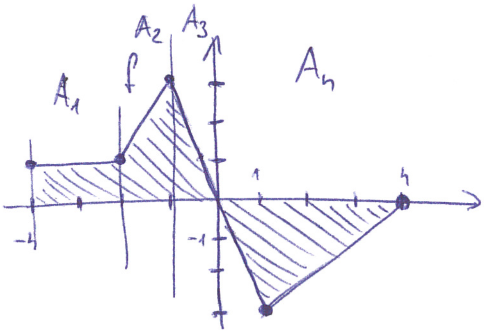


$$\int_{-3}^{5} f(x)\,dx = -\frac{3}{2} + 7 = \underline{\underline{\frac{11}{2}}}$$

$$A_1 = \frac{-3\cdot 1}{2} = -\frac{3}{2}$$

$$A_2 = \frac{(5+2)\cdot 2}{2} = 7$$

Problem 4.



$$\int_{-4}^{4} f(x)\,dx = A_1 + A_2 + A_3 + A_4 =$$

$$= 2 + 2 + \frac{3}{2} - 6$$

$$= \frac{3}{2} - 2 = \underline{\underline{-\frac{1}{2}}}$$

$$A_1 = 1\cdot 2 = 2$$

$$A_2 = \frac{(1+3)\cdot 1}{2} = 2$$

$$A_3 = \frac{3\cdot 1}{2} = \frac{3}{2}$$

$$A_4 = \frac{4\cdot(-3)}{2} = -6$$

Formula of Newton-Leibniz

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$.

Problem 5.

$$\int_{-e}^{-1} \frac{1}{x}\,dx = ?$$

① $\int \frac{1}{x}\,dx = \ln|x|$

② $\int_{-e}^{-1} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-e}^{-1} = \ln|-1| - \ln|-e| = \underbrace{\ln 1}_{0} - \underbrace{\ln e}_{1} = \underline{\underline{-1}}$

Problem 6.

$$\int_{\pi}^{0} x \cdot \cos x \, dx = ?$$

1. $\int x \cos x \, dx = x \sin x - \underbrace{\int 1 \cdot \sin x \, dx}_{-\cos x} = x \sin x + \cos x \; (+C)$

$f = x \qquad g' = \cos x$

$f' = 1 \qquad g = \sin x$

2. $$\int_{\pi}^{0} x \cos x \, dx = \Big[x \cdot \sin x + \cos x \Big]_{\pi}^{0} = \Big(0 \cdot \underbrace{\sin 0}_{0} + \underbrace{\cos 0}_{1}\Big) - \Big(\pi \cdot \underbrace{\sin \pi}_{0} + \underbrace{\cos \pi}_{-1}\Big)$$

$$= 1 - (-1) = \underline{\underline{2}}$$