Practice 8

Integration by substitution

General substitution : We want to substitute a function whose "derivative can be seen" in the original formula.

Problem 1.

- $\int 3x^2 e^{x^3} dx \overset{dy}{=} \int e^y \, dy = e^y + C = e^{x^3} + C$

  $y = x^3$
  $dy = 3x^2 dx$

- $\int x^2 e^{x^3} dx \overset{\frac{1}{3}dy}{=} \frac{1}{3}\int e^y \, dy = \frac{1}{3} e^y + C = \frac{1}{3} e^{x^3} + C$

  $y = x^3$
  $dy = 3x^2 dx$
  $\frac{1}{3} dy = x^2 dx$

- $\int \frac{x^3}{\sqrt{x^4 - 3}} dx \overset{\frac{1}{4}dy}{=} \frac{1}{4}\int \frac{1}{\sqrt{y}} dy = \frac{1}{4}\int y^{-1/2} dy =$

  $x^4 - 3 = y$
  $4x^3 dx = dy$
  $x^3 dx = \frac{1}{4} dy$

  $= \frac{1}{4} \frac{y^{1/2}}{1/2} = \frac{1}{4} \cdot \frac{(x^4-3)^{1/2}}{1/2} + C$

- $\int x \cos x^2 \, dx \overset{\frac{1}{2}dy}{=} \frac{1}{2}\int \cos y \, dy = \frac{1}{2} \sin y + C = \frac{1}{2} \sin x^2 + C$

  $y = x^2$
  $dy = 2x \, dx$
  $\frac{1}{2} dy = x \, dx$

- $\int \frac{2x^3 - 1}{x^4 - 2x} dx \overset{\frac{1}{2}dy}{=} \frac{1}{2}\int \frac{1}{y} dy = \frac{1}{2} \ln|y| + C = \frac{1}{2} \ln|x^4 - 2x| + C$

  $x^4 - 2x = y$
  $4x^3 - 2 = \frac{dy}{dx}$ $\qquad (2x^3 - 1) dx = \frac{1}{2} dy$
  $(4x^3 - 2) dx = dy$

Partial integration

Formula: $\int fg' = fg - \int f'g$

(1.) We want to integrate a product of a polynomial and an exponential/trigonometric function.

Problem 2.

- $\int x \cdot e^x \, dx = x \cdot e^x - \underbrace{\int 1 \cdot e^x \, dx}_{e^x} = \underline{\underline{x \cdot e^x - e^x + C}}$

$f = x \qquad g' = e^x$

$f' = 1 \qquad g = \int e^x dx = e^x$

- $\int x \cdot \cos x \, dx = \; = x \cdot \sin x - \underbrace{\int 1 \cdot \sin x \, dx}_{-\cos x} = \underline{\underline{x \cdot \sin x + \cos x + C}}$

$f = x \qquad g' = \cos x$

$f' = 1 \qquad g = \int \cos x \, dx = \sin x$

- $\int x^2 \cdot \sin x \, dx = -x^2 \cos x - \int -2x \cos x \, dx =$

$f = x^2 \qquad g' = \sin x$

$f' = 2x \qquad g = \int \sin x \, dx = -\cos x$

$= -x^2 \cdot \cos x + 2 \int x \cos x \, dx = (*)$

Another partial integration that you can see in the previous problem.

$(*) = \underline{\underline{-x^2 \cdot \cos x + 2\left(x \cdot \sin x + \cos x\right) + C}}$

Using this technique we can eliminate the polynomial and we have to integrate a basic function.

(2) We want to integrate a product of a polynomial and a logarithmic function.

Problem 3.

$$\int x \cdot \ln x \, dx = \qquad \ln x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx = \frac{x^2}{2} \ln x - \int \frac{1}{2} x \, dx$$

$$f = \ln x \qquad g' = x$$
$$f' = \frac{1}{x} \qquad g = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \underbrace{\int x \, dx}_{\frac{x^2}{2}}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C$$

We derive the logarithm function and this way, we get a nicer expression.

Problem 4. (Typical problem of the second test)

$$\int x^2 \left(\ln x - \frac{1}{\sqrt[3]{x^3 - 3}} \right) dx = \underbrace{\int x^2 \ln x \, dx}_{I_1} - \underbrace{\int \frac{x^2}{\sqrt[3]{x^3 - 3}} \, dx}_{I_2} = (*)$$

$$I_1 = \int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \underbrace{\int \frac{1}{x} \cdot \frac{x^3}{3} \, dx}_{\frac{1}{3} \int x^2 dx \quad \frac{1}{3} \frac{x^3}{3}} = \frac{x^3}{3} \cdot \ln x - \frac{x^3}{9}$$

$$f = \ln x \qquad g' = x^2$$
$$f' = \frac{1}{x} \qquad g = \frac{x^3}{3}$$

$$I_2 = \int \frac{x^2}{\sqrt[3]{x^3 - 3}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt[3]{y}} \, dy = \frac{1}{3} \int y^{-1/3} \, dy = \frac{1}{3} \cdot \frac{y^{2/3}}{2/3} = \frac{1}{3} \cdot \frac{(x^3 - 3)^{2/3}}{2/3}$$

$$y = x^3 - 3$$
$$dy = 3x^2 \, dx$$
$$\frac{1}{3} dy = x^2 \, dx$$

$$(*) = I_1 - I_2 = \frac{x^3}{3} \ln x - \frac{x^3}{9} - \frac{1}{3} \frac{(x^3 - 3)^{2/3}}{2/3} + C$$